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PRÆFATIO.

ARS *Analytica in Propoſitionibus demonſtrandis non ſolùm verſatur, ſed etiam methodum docet ipſas inveniendi; huic debetur illa propoſitionum, præcipuè univerſalium, quas invenêre Recentiores, pæne infinita multitudo. De hujuſce Scientiæ Progreſſu, et quæ diverſis temporibus acceperit, incrementis, abs re haud alienum erit pauca præfari.*

Cubicarum Æquationum reſolutionem docuit SCIPIO FERREUS; *Biquadraticas reſolvendi methodos diverſas invenêre* VIETA *et* CARTESIUS. *Coefficientem ſecundi Æquationis termini ſummam eſſe radicum, et ſimiliter omnium coefficientium conſtitutionem docuit, etiamque Radices datæ Æquationis per datas Quantitates auxit et diminuit, multiplicavit et diviſit ille hodiernæ Algebræ Reſtaurator* VIETA.

Originem Æquationum ſuperiorum ex compoſitione laterarum, et exinde Coefficientium Conſtitutionem oſtendit ille magnus Algebraiſta HARRIOTTUS.

Tot Radices affirmativas eſſe, quot ſunt mutationes Signorum a + in —; cæteras vero negativas eſſe, dixit CARTESIUS.

Methodum extrahendi Radicem quadraticam ex quibuſcunque binomiis ſurdis dedit OUGHTREDUS; *generalem methodum extrahendi quamlibet Radicem tradidit* SCHOOTENIUS; *datæ Æquationis ſimplices Diviſores invenire docuit* JACOBUS à WASSENAER; *Quadraticos, Cubicos, &c. inveniendi methodum tradidit* HUDDENIUS: *invenit etiam methodum detegendi, utrum duæ vel plures datæ Æquationis Radices ſunt inter ſe æquales. Duas vel plures Æquationes in unam, ita ut incognitæ Quantitates exterminentur, diverſis modis transformavêre* ERASMUS BARTHOLINUS *et* CRAMER; *etiamque oſtendit* CRAMER *quomodo Quantitates utcunque ſurdæ poſſunt ex Æquationibus tolli.*

Ille

Ille Philoſophorum Princeps, et arte Analyticâ maximè inſignis NEWTONUS, *regulam dedit, ex quâ inveniri poteſt Summa datæ Æquationis Radicum, Summa Quadratorum ex ſingulis Radicibus, Cuborum, &c. invenit etiam Regulam, ex quâ ſæpe cognoſci poteſt, quot datæ Æquationis Radices ſunt poſſibiles, et quot impoſſibiles; et multas Æquationes* (2n) *Dimenſionum in Æquationes* (n) *Dimenſionum reduxit.*

Hic non loquor de ejus eximio binomiali Theoremate, vel infinitis Seriebus: in Fluxionalem Partem hæc transfero.

WALLISIUS *deduxit Methodum, quâ integri Valores datæ Æquationis Radicum, duas incognitas Quantitates habentis, inveniri poſſunt.*

De aliorum Inventionibus ſatis dictum eſt; jam quæ in hoc Tractatu contineantur, breviter ſunt recenſenda.

In primo Capite docetur Methodus inveniendi Æquationem, cujus Radices quamcunque datam Relationem habent ad Radices datæ Æquationis vel datarum Æquationum.

In Secundo agitur de impoſſibilibus Radicibus.

In Tertio dantur Methodi detegendi datæ Æquationis Radicum rationales Valores, etiamque Radices datam inter ſe habentes Relationem.

In Quarto tractatur de Reductione & Reſolutione Æquationum.

In Quinto de duabus vel pluribus Æquationibus duas vel plures incognitas Quantitates habentibus.

Hactenus de prima hujuſce Operis Partis Subjecto, nunc de Curvarum Proprietatibus.

Veteres, inter quos longè eminet APOLLONIUS, *Conicas Sectiones præcipuè tractârunt; Recentiores verò novos aperuerunt Curvarum Campos in infinitum diffuſos;* D[r] NICHOLE *Curvas ſuper Baſes ſibi æquales rotavit;* CARTESIUS *diſtinxit Algebraicas Curvas in ordines ſecundum Dimenſiones Æquationum Relationem inter abſciſſas & earum correſpondentes ordinatas exprimentium.*

Hanc Diſciplinam multum auxit Doctiſſimus BARROW; *docuit Tangentes, Aſymptotos, &c. inveniendi Methodos.*

Ille

Ille præclarus Geometra NEWTONUS *invenit tres ſubſequentes Algebraicarum Curvarum Proprietates*; 1. *Ducantur duæ rectæ inter ſe parallelæ & Curvam ſecantes in tot Punctis, quot ſunt ejus Dimenſiones, & recta, quæ ita ſecat has parallelas, ut Summa partium, inter Curvam & rectam ſecantem jacentium, utrinque ſit inter ſe æqualis, eodem modo ſecabit omnes alias hiſce parallelas.* 2. *Si Curva tot habet Aſymptotos, quot ſunt ejus Dimenſiones, tum Summa partium ordinatæ inter ſingulum crus & ejus aſymptoton interceptarum, utrinque Diametri erit inter ſe æqualis.* 3. *Si Abſciſſa tot habet Vertices, quot ſunt ejus Dimenſiones, tum Contentum ſub omnibus Abſciſſis inter ordinatam & vertices interceptis erit in data ratione ad contentum ſub omnibus ordinatis: enumeravit etiam Lineas tertii Ordinis, & dicit, quod Curvæ* (n) *Dimenſionum projiciunt umbras, quarum Sectiones ſunt Curvæ* (n) *Dimenſionum*; *& rectificavit Epicycloidem, cujus Aream invenit magnus ille Philoſophus* HALLEIUS.

Celeberrimus COTESIUS *invenit generalem Algebraicarum Linearum Proprietatem, quam illo Viro præſtantiſſimo* MAC LAURIN *communicavit Vir Reverendus* ROB. SMITH, *S.T.P. & Sanctæ Trin. Coll. Cantab. Præfectus, ſummus ille Philoſophus, & in Mathematicis Diſciplinis maxime inſignis. Plures Proprietates exinde deduxit* MAC LAURIN: *ille etiam tractavit organicam Curvarum Deſcriptionem, de quâ prius ſcripſit* NEWTONUS.

Horum autem Laudibus, aliorumque, qui Studiis ſuis haſce Diſciplinas promoverunt, non eſt, ut hìc diutiùs immoremur.

Reſtat, ut de iis rebus pauca dicam, quas complectitur hujus Operis Pars Secunda.

In primo Capite agitur de Algebraicarum Curvarum Proprietatibus, de Diametris, Centris, centrali Curvâ, Summâ Subnormalium, & Subtangentium, &c.

In Secundo de Curvoidibus, ſive Curvis ex Rotatione unius Curvæ ſuper alteram generatis.

In Tertio, de Umbris, quas projiciunt vel Solida vel Curvæ in Solida vel Plana.

In Quarto, tractatur de Fluxionalium Curvarum, & Ovalium in ſe Concavarum Proprietatibus.

In

In Quinto de Enumeratione Algebraicarum Linearum Quarti Ordinis.

Nihil reſtat, niſi ut iis agam Gratias, quorum Nominibus hoc opus præcipuè ornatur; eoſque enixè rogem, ut quâ Humanitate Induſtriam meam adjuvârunt, eâdem Fructum ejus, qualiſcunque ſit, tandem accipiant.

Nec tamen ſcribendi Finem prius faciam quam Gratias maximas agam egregio Adoleſcenti, in omni Scientiarum Genere maximè acuto & verſato, JOANNI WILSON, *Div. Pet. Coll. qui multa & præclara à ſe inventa & excogitata habet, cujus cùm Curæ, tum Ingenio hoc Opuſculum quamplurimum debere lubentiſſimè agnoſco.*

Coll. Magd. Cantab.
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MIS-

MISCELLANEA ANALYTICA.

CAPUT I.

LEMMA I.

SINT $\alpha, \beta, \gamma, \delta, \varepsilon$, &c. *radices datæ æquationis*
$x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - vx^{n-5} + wx^{n-6}$. *&c.* $= 0$.
Invenire ſummam $\alpha^m + \beta^m + \gamma^m + \delta^m + \varepsilon^m$, *&c.*

Summa quæſita erit

$$p^m - mqp^{m-2} + mrp^{m-3} - msp^{m-4} + m.\frac{\overline{m-3}}{2} q^2$$

$$\begin{array}{llll}
+ & mtp^{m-5} - & mvp^{m-6} + & mwp^{m-7} \\
- & m.\overline{m-4}\,qr + & m.\overline{m-5}\,qs - & m \times \overline{m-6} \times qt \\
 & & - m \times \frac{\overline{m-5}}{2} \times \frac{\overline{m-4}}{3} \times q^3 & + m \times \overline{m-5} \times \frac{\overline{m-6}}{2} \times q^2 r \\
 & & + m.\frac{\overline{m-5}}{2} r^2 & - m \times \overline{m-6} \times rs
\end{array}$$

Duplex eſt lex, quam obſervat hæc ſeries, altera, quæ producta literalia, altera, quæ coefficientes ſeu uncias reſpicit.

Productorum literalium lex eſt, ut datæ æquationis coefficientium exponentes, in ſingulis productis eidem poteſtati quantitatis p annectendis, eandem conficiunt ſummam, quæ quidem ſumma eſt u, ſi modo p^{m-u} ſit poteſtas quantitatis p, cui annectuntur producta hæc literalia.

Per datæ æquationis coefficientium exponentes intelligo numeros deſignantes earum diſtantias a primo æquationis termino; ſic exponens coefficientis p; eſt 1; coefficientis q, 2; coefficientis r, 3; & ſic deinceps; quo fit, ut exponens cujuſlibet poteſtatis coefficientis p, qualis p^u ſit u; exponens cujuſlibet poteſtatis coefficientis q qualis q^μ, ſit 2μ; exponens cujuſlibet poteſtatis coefficientis r, qualis r^ν, ſit 3ν.

Coefficientium ſeu unciarum hæc eſt lex, cujuſcunque literalis producti $p^{m-u} q^\mu r^\nu s^\pi t^\varrho$, &c. coefficiens erit
$m \times \overline{m-u+1} \times \overline{m-u+2} \times \overline{m-u+3} \times \overline{m-u+4} \times \overline{m-u+5}$, &c. tot factores erunt, quot unitates in ſumma indicum μ, ν, π, ϱ, &c. Dividatur hoc productum per productum ortum ex multiplicatione

 ſeri-

ſerierum $1\times2\times3\times4$, &c. $1\times2\times3\times4$, &c. $1\times2\times3\times4$, &c. quarum prima continet tot terminos, quot ſunt unitates in μ, ſecunda, quot ſunt in ν, tertia, quot ſunt in π, quarta, quot ſunt in ρ, &c.

DEMONSTRATIO.

"Sit $p=a, p^2-2q=b, pb-qa+3r=c, pc-qb+ra-4s=d,$ " $pd-qc+rb-sa+5t=e, pe-qd+rc-sb+ta-6v=f$, & " ſic deinceps. Et erit a ſumma radicum, b ſumma quadratorum ex " ſingulis radicibus, c ſumma cuborum, d ſumma quadrato-quadra- " torum, e ſumma quadrato-cuborum, f ſumma cubo-cuborum, & " ſic in reliquis." *

Et hâc *Newtoni* regulâ ſatis facile colligitur, coefficientem ſeu unciam literalis producti $p^{m-u}q^{\mu}r^{\nu}s^{\pi}t^{\rho}$, &c. in ſummâ $\alpha^m+\beta^m+\gamma^m+\delta^m+\varepsilon^m$, &c. inveniendâ, æqualem eſſe aggregato coefficientium ſeu unciarum literalium productorum $p^{m-u-1}q^{\mu}r^{\nu}s^{\pi}t^{\rho}$, &c. $p^{m-u}q^{\mu-1}r^{\nu}s^{\pi}t^{\rho}$ &c. $p^{m-u}q^{\mu}r^{\nu-1}s^{\pi}t^{\rho}$ &c. $p^{m-u}q^{\mu}r^{\nu}s^{\pi-1}t^{\rho}$ &c. $p^{m-u}q^{\mu}r^{\nu}s^{\pi}t^{\rho-1}$, &c. &c. in ſummis reſpectivis $\alpha^{m-1}+\beta^{m-1}+\gamma^{m-1}+\delta^{m-1}$, &c. $\alpha^{m-2}+\beta^{m-2}+\gamma^{m-2}+\delta^{m-2}$, &c. $\alpha^{m-3}+\beta^{m-3}+\gamma^{m-3}+\delta^{m-3}$, &c. $\alpha^{m-4}+\beta^{m-4}+\gamma^{m-4}+\delta^{m-4}$, &c. $\alpha^{m-5}+\beta^{m-5}+\gamma^{m-5}+\delta^{m-5}$, &c.

Hæ autemunciæ vel coefficientes reſpective ſunt

$$\overline{m-1}\times\overline{m-u}\times\frac{\overline{m-u+1}\times\overline{m-u+2}\times\overline{m-u+3}\ldots\overline{m-u+\mu+\nu+\pi+\rho\ \&c.-2}}{1.2.3\ldots\mu.1.2.3\ldots\nu.1.2.3\ldots\pi.1.2.3\ldots\rho,\ \&c.}$$

$$\mu\times\overline{m-2}\times\frac{\overline{m-u+1}\times\overline{m-u+2}\times\overline{m-u+3}\ldots\overline{m-u+\mu+\nu+\pi+\rho,\ \&c.-2}}{1.2.3\ldots\mu.1.2.3\ldots\nu.1.2.3\ldots\pi.1.2.3\ldots\rho,\ \&c.}$$

$$\nu\times\overline{m-3}\times\frac{\overline{m-u+1}\times\overline{m-u+2}\times\overline{m-u+3}\ldots\overline{m-u+\mu+\nu+\pi+\rho,\ \&c.-2}}{1.2.3\ldots\mu.1.2.3\ldots\nu.1.2.3\ldots\pi.1.2.3\ldots\rho,\ \&c.}$$

$$\pi\times\overline{m-4}\times\frac{\overline{m-u+1}\times\overline{m-u+2}\times\overline{m-u+3}\ldots\overline{m-u+\mu+\nu+\pi+\rho,\ \&c.-2}}{1.2.3\ldots\mu.1.2.3\ldots\nu.1.2.3\ldots\pi.2.3\ldots\rho,\ \&c.}$$

$$\rho\times\overline{m-5}\times\frac{\overline{m-u+1}\times\overline{m-u-2}\times\overline{m-u+3}\ldots\overline{m-u+\mu+\nu+\pi+\rho,\ \&c.-2}}{1.2.3\ldots\mu.1.2.3\ldots\nu.1.2.3\ldots\pi.1.2.3\ldots\rho,\ \&c.}$$

&c.

Aggregatum harum unciarum eſt

$$\left.\begin{array}{ll} m^2-u\times m+ & u \\ \quad -1 & -2\mu \\ \quad +\mu & -3\nu \\ \quad +\nu & -4\pi \\ \quad +\pi & -5\rho \\ \quad +\rho & \&c. \\ \quad \&c. & \end{array}\right\}\times\frac{\overline{m-u+1}\times\overline{m-u+2}\times\overline{m-u+3}\ldots\overline{m-u+\mu+\nu+\pi+\rho,\&c.-2}}{1.2.3\ldots\mu.1.2.3\ldots\nu.1.2.3\ldots\pi.1.2.3\ldots\rho,\&c.}$$

In

* *Newt.* Arith. Univ. pag. 251.

In primo hujus aggregati factore, pro u ſubſtituatur ejus valor $(2\mu+3\nu+4\pi+5\rho$, &c.) & fit aggregatum

$$m\times\overline{m-u+\mu+\nu+\pi+\rho,\&c.-1}\times\frac{\overline{m-u+1}\times\overline{m-u+2}\times\overline{m-u+3}\,..\,\overline{m-u+\mu+\nu+\pi+\rho,\ \&c.-2}}{1\,.\,2\,.\,3\,..\,\mu\,.\,1\,.\,2\,.\,3\,..\,\nu\,.\,1\,.\,2\,.\,3\,..\,\pi\,.\,1\,.\,2\,.\,3\,...\,\rho,\ \&c.}$$

Hæc vero eſt coefficiens ſeu uncia, quæ in datâ ſerie pro uncia literalis producti $p^{m-u}q^{\mu}r^{\nu}s^{\pi}t^{\rho}$, &c. in ſummâ $\alpha^m+\beta^m+\gamma^m+\delta^m+\varepsilon^m$, &c. inveniendâ, ſubſtituitur; & conſequenter ſi ſeries inferiores ſummas $\alpha^{m-1}+\beta^{m-1}+\gamma^{m-1}+\delta^{m-1}$, &c. $\alpha^{m-2}+\beta^{m-2}+\gamma^{m-2}+\delta^{m-2}$, &c. $\alpha^{m-3}+\beta^{m-3}+\gamma^{m-3}+\delta^{m-3}$, &c. $\alpha^{m-4}+\beta^{m-4}+\gamma^{m-4}+\delta^{m-4}$, &c. recte deducit; patet ex iis, quæ jam dicta fuerunt, proxime ſuperiorem ſummam $\alpha^m+\beta^m+\gamma^m+\delta^m+\varepsilon^m$, &c. etiam ex illâ recte aſſignatam eſſe.

EXEMPLUM.

Sit æquatio $x^4-2x^3-7x^2+20x-12=0$. In hoc exemplo $p, q, r, s,$ ſunt reſpective $2, -7, -20, -12$. Quare ſumma quadratorum ex ſingulis radicibus (p^2-2q) eſt $2^2+2\times 7=18$; cuborum ſumma $(p^3-3pq+3r)$ $2^3+3\times 2\times 7-3\times 20=-10$; quadrato-qnadratorum ſumma

$$\left(\begin{matrix}p^4-4p^2q+4pr-4s\\ \qquad\qquad\qquad +2q^2\end{matrix}\right)2^4+4\times 2^2\times 7-4\times 2\times 20+4\times 12+2\times 7^2=114$$

& ſic deinceps.

COROLLARIUM I.

Sit a ſumma radicum, b ſumma quadratorum ex ſingulis radicibus, c ſumma cuborum, d ſumma quadrato-quadratorum, e ſumma quadrato-cuborum, &c.

Invenire ſummam (m) terminorum ab initio hujuſce ſeriei $a+b+c+d+e$, &c.

Summa quæſita eſt

$$\frac{p-p^{m+1}}{1-p}-q\times\frac{2-p-\overline{m+1}p^{m-1}+mp^m}{\overline{1-p}^{\,2}}+r\times\frac{3-2p-\overline{m+1}\,.\,p^{m-2}+mp^{m-1}}{\overline{1-p}^{\,2}}$$

$$-s\times\frac{4-3p-\overline{m+1}\,p^{m-3}+mp^{m-2}}{\overline{1-p}^{\,2}}$$

$$+q^2\times\frac{4-2p-\overline{m+1}\times\overline{m-2}\,p^{m-3}+\overline{2m^2-4m-4}\,p^{m-2}-m\,.\,\overline{m-3}\,.\,p^{m-1}}{2\,.\,\overline{1-p}^{\,3}}$$

$+t\times$

$$+t\times\frac{5-4p-\overline{m+1}.p^{m-4}+mp^{m-3}}{\overline{1-p}^{2}}$$

$$-qr\times\frac{5-3p-\overline{m+1}\times\overline{m-3}.p^{m-4}+\overline{2m^{2}-6m-5}.p^{m-3}-m.\overline{m-4}.p^{m-2}}{\overline{1-p}^{3}},\&c.$$

Literalia producta hujuſce ſeriei eadem fiunt, ac literalia producta præcedentis ſeriei, ſi modo abjiciatur quantitas (p) & ejus poteſtates.

Regula autem, quam obſervant coefficientes ſeu unciæ, ſic poterit enunciari.

Scribatur $u=2\mu+3\nu+4\pi+5\rho$, &c. & $l=1+\mu+\nu+\pi+\rho$, &c. Coefficiens literalis producti $q^{\mu}r^{\nu}s^{\pi}t^{\rho}$, &c. erit æqualis producto $1.2.3.4..l-2$ in fractionem multiplicato, cujus denominator eſt $\overline{1-p}^{l}\times1\times2\times3..\mu\times1\times2\times3..\nu\times1\times2\times3\times4..\pi\times1\times2\times3..\rho$, &c.

Numerator vero $u-\overline{u-l+1}\times p$

$$\left.\begin{array}{l}-u\times\frac{\overline{m-u+2}}{1}\times\frac{\overline{m-u+3}}{2}\times\frac{\overline{m-u+4}}{3}\times\frac{\overline{m-u+5}}{4}\times\frac{\overline{m-u+6}}{5}\ldots\frac{\overline{m-u+l}}{l-1}\\+\overline{u-l+1}\times\frac{\overline{m-u+1}}{1}\times\frac{\overline{m-u+2}}{2}\times\frac{\overline{m-u+3}}{3}\times\frac{\overline{m-u+4}}{4}\times\frac{\overline{m-u+5}}{5}\ldots\frac{\overline{m-u+l-1}}{l-1}\end{array}\right\}p^{m-u+1}$$

$$\left.\begin{array}{l}+u\times\frac{\overline{m-u+1}}{1}\times\frac{\overline{m-u+3}}{1}\times\frac{\overline{m-u+4}}{2}\times\frac{\overline{m-u+5}}{3}\times\frac{\overline{m-u+6}}{4}\ldots\frac{\overline{m-u+l}}{l-2}\\-\overline{u-l+1}\times\frac{\overline{m-u}}{1}\times\frac{\overline{m-u+2}}{1}\times\frac{\overline{m-u+3}}{2}\times\frac{\overline{m-u+4}}{3}\times\frac{\overline{m-u+5}}{3}\ldots\frac{\overline{m-u+l-1}}{l-2}\end{array}\right\}p^{m-u+2}$$

$$\left.\begin{array}{l}-u\times\frac{\overline{m-u+1}}{1}\times\frac{\overline{m-u+2}}{2}\times\frac{\overline{m-u+4}}{1}\times\frac{\overline{m-u+5}}{2}\times\frac{\overline{m-u+6}}{3}\ldots\frac{\overline{m-u+l}}{l-3}\\+\overline{u-l+1}\times\frac{\overline{m-u}}{1}\times\frac{\overline{m-u+1}}{2}\times\frac{\overline{m-u+3}}{1}\times\frac{\overline{m-n+4}}{2}\times\frac{\overline{m-u+5}}{3}\ldots\frac{\overline{m-u+l-1}}{l-3}\end{array}\right\}p^{m-u+3}$$

$$\left.\begin{array}{l}+u\times\frac{\overline{m-u+1}}{1}\times\frac{\overline{m-u+2}}{2}\times\frac{\overline{m-u+3}}{3}\times\frac{\overline{m-u+5}}{1}\times\frac{\overline{m-u+6}}{2}\ldots\frac{\overline{m-u+l}}{l-4}\\-\overline{u-l+1}\times\frac{\overline{m-u}}{1}\times\frac{\overline{m-u+1}}{2}\times\frac{\overline{m-u+2}}{3}\times\frac{\overline{m-u+4}}{1}\times\frac{\overline{m-u+5}}{2}\ldots\frac{\overline{m-u+l-1}}{l-4}\end{array}\right\}p^{m-u+4}$$

$$\left.\begin{array}{l}-u\times\frac{\overline{m-u+1}}{1}\times\frac{\overline{m-u+2}}{2}\times\frac{\overline{m-u+3}}{3}\times\frac{\overline{m-u+4}}{4}\times\frac{\overline{m-u+6}}{1}\ldots\frac{\overline{m-u+l}}{l-5}\\+\overline{u-l+1}\times\frac{\overline{m-u}}{1}\times\frac{\overline{m-u+1}}{2}\times\frac{\overline{m-u+2}}{3}\times\frac{\overline{m-u+3}}{4}\times\frac{\overline{m-u+5}}{1}\ldots\frac{\overline{m-u+l-1}}{l-5}\end{array}\right\}p^{m-u+5}$$

. . . .

$$\left.\begin{array}{l}+m\times\frac{\overline{m-u+1}}{1}\times\frac{\overline{m-u+2}}{2}\times\frac{\overline{m-u+3}}{3}\times\frac{\overline{m-u+4}}{4}\times\frac{\overline{m-u+5}}{5}\ldots\frac{\overline{m-u+l-1}}{l-1}\\+\overline{u-l+1}\times\frac{\overline{m-u}}{1}\times\frac{\overline{m-u+1}}{2}\times\frac{\overline{m-u+2}}{3}\times\frac{\overline{m-u+3}}{4}\times\frac{\overline{m-u+4}}{5}\ldots\times\frac{\overline{m-u+l-2}}{l-1}\end{array}\right\}p^{m-u+l}$$

Ex

Ex methodo inveniendi ſummam $(\overline{m-u+1})$ terminorum ſeriei emergentis ex diviſione quantitatis $u-\overline{u-l+1}\times p$ per $\overline{1-p}^{\,l}$, facile erui & demonſtrari poteſt hoc corollarium.

EXEMPLUM.

Datâ æquatione $x^4-3x^3+x^2+3x-2=0$, invenire ſummam $a+b+c+d$.

Pro p, q, r, s, m ſcribantur reſpective $3, 1, -3, -2, 4$.

$$a+b+c+d=\frac{p-p^5}{1-p}\,(120)-q\times\frac{2-p-5p^3+4p^4}{\overline{1-p}^2}\,(-47)$$

$$+r\times\frac{3-2p-5p^2+4p^3}{\overline{1-p}^2}\,(-45)-s\times\frac{4-3p-5p+4p^8}{\overline{1-p}^2}\,(8)=38.$$

$$+q^2\times\frac{4-2p-10p+12p^2-4p^3}{2.\overline{1-p}^3}\,(2)$$

Si $p=1$, tum $\frac{p-p^{m+1}}{1-p}$ fit (m) & $q\times\frac{2-p-\overline{m+1}.p^{m-1}+mp^m}{\overline{1-p}^2}$ fit $\left(q\times\overline{m+2}\times\frac{\overline{m-1}}{2}\right)$. &c.

Summa quæſita eſt

$$m-\frac{\overline{m+2}}{1}\times\frac{\overline{m-1}}{2}\,q+\frac{\overline{m+3}}{1}\times\frac{\overline{m-2}}{2}\,r-\frac{\overline{m+4}}{1}\times\frac{\overline{m-3}}{2}\times s, \text{ \&c.}$$

$$+\tfrac{1}{2}\times\frac{\overline{2m+4}}{1}\times\frac{\overline{m-3}}{2}\times\frac{\overline{m-2}}{3}\,q^2, \text{ \&c.}$$

Coefficientium ſeu unciarum hujuſce ſeriei hæc eſt lex. Coefficiens cujuſlibet literalis producti $q^\mu r^\nu s^\pi t^\rho$, &c. æqualis eſt fractioni,

$$\overline{\overline{l-1}\times m+u}\times\frac{\overline{m-u+1}\times\overline{m-u+2}\times\overline{m-u+3}\ldots\ldots\overline{m-u+l-1}, \text{ \&c.}}{l\times\overline{l-1}\times 1.2.3\ldots\mu\times 1.2.3\ldots\nu\times 1.2.3\ldots\pi\times 1.2.3\ldots\rho, \text{ \&c.}}$$

COROLLARIUM II.

Invenire ſummam ſeriei $a+b+c+d+e$, &c. in infinitum.

Summa quæſita eſt $\dfrac{p-2q+3r-4s+5t, \text{ \&c.}}{1-p+q-r+s-t, \text{ \&c.}}$

EX-

EXEMPLUM.

Sit æquatio $x^4-x^3+\frac{1}{4}x-\frac{1}{16}=0$, ſcribatur $1, 0, -\frac{1}{4}, -\frac{1}{16}$ pro p, q, r, s, & $\frac{p-2q+3r-4s}{1-p+q-r+s}$ fit $\frac{1-\frac{3}{4}+\frac{1}{4}}{1-1+\frac{1}{4}-\frac{1}{16}}=\frac{\frac{1}{2}}{\frac{3}{16}}=2\frac{2}{3}$.

Quam plurima hujuſce generis problemata deduci poſſunt, datam æquationem in alteram transformando, cujus radices vel ſunt reciprocæ vel quadraticæ, cubicæ, &c. radices ex quibuſlibet complexis functionibus datæ æquationis radicum ($\alpha, \beta, \gamma, \delta$, &c.)

LEMMA II.

Sint $\alpha, \beta, \gamma, \delta, \epsilon$, &c. *radices datæ æquationis* $x^n-px^{n-1}+q^{n-2}-rx^{n-3}$ &c. $=0$ *& ſint* a, b, c, d, e, f, &c. *dati indices, cujus numerus ſit* (m), *&* m *vel æqualis vel minor ſit quam* n,

Invenire ſummam omnium quantitatum hujuſmodi.
$\alpha^a\beta^b\gamma^c\delta^d\epsilon^e$ &c. $+\alpha^b\beta^a\gamma^c\delta^d\epsilon^e$ &c. in quibus numerus factorum eſt (m).

Scribatur

$S^a = \alpha^a+\beta^a+\gamma^a+\delta^a+\epsilon^a$, &c.
$S^b = \alpha^b+\beta^b+\gamma^b+\delta^b+\epsilon^b$, &c.
$S^c = \alpha^c+\beta^c+\gamma^c+\delta^c+\epsilon^c$, &c.
$S^d = \alpha^d+\beta^d+\gamma^d+\delta^d+\epsilon^d$, &c.

$S^{a+b} = \alpha^{a+b}+\beta^{a+b}+\gamma^{a+b}+\delta^{a+b}$, &c.
$S^{a+c} = \alpha^{a+c}+\beta^{a+c}+\gamma^{a+c}+\delta^{a+c}$, &c.
$S^{a+d} = \alpha^{a+d}+\beta^{a+d}+\gamma^{a+d}+\delta^{a+d}$, &c.

$S^{a+b+c} = \alpha^{a+b+c}+\beta^{a+b+c}+\gamma^{a+b+c}+\delta^{a+b+c}$, &c.
$S^{a+b+d} = \alpha^{a+b+d}+\beta^{a+b+d}+\gamma^{a+b+d}+\delta^{a+b+d}$, &c.
$S^{a+c+d} = \alpha^{a+c+d}+\beta^{a+c+d}+\gamma^{a+c+d}+\delta^{a+c+d}$, &c.

$S^{a+b+c+d} = \alpha^{a+b+c+d}+\beta^{a+b+c+d}+\gamma^{a+b+c+d}$, &c.
$S^{a+b+c+e} = \alpha^{a+b+c+e}+\beta^{a+b+c+e}+\gamma^{a+b+c+e}$, &c.

$S^{a+b+c+d+e} = \alpha^{a+b+c+d+e}+\beta^{a+b+c+d+e}+\gamma^{a+b+c+d+e}$, &c.
$S^{a+b+c+d+f} = \alpha^{a+b+c+d+f}+\beta^{a+b+c+d+f}+\gamma^{a+b+c+d+f}$, &c.

& ſic deinceps.

Scribatur

$A = S^a \times S^b \times S^c \times S^d \times S^e$, &c.

$$B = \frac{S^{a+b} \times A}{S^a \times S^b} + \frac{S^{a+c} \times A}{S^a \times S^c} + \frac{S^{a+d} \times A}{S^a \times S^d}, \text{ \&c.}$$

$$C = \frac{S^{a+b+c} \times A}{S^a \times S^b \times S^c} + \frac{S^{a+b+d} \times A}{S^a \times S^b \times S^d} + \frac{S^{a+c+d} \times A}{S^a \times S^c \times S^d}, \text{ \&c.}$$

$$D = \frac{S^{a+b+c+d} \times A}{S^a \times S^b \times S^c \times S^d} + \frac{S^{a+b+c+e} \times A}{S^a \times S^b \times S^c \times S^e}, \text{ \&c.}$$

$$BB = \frac{S^{a+b} \times S^{c+d} \times A}{S^a \times S^b \times S^c \times S^d} + \frac{S^{a+c} \times S^{b+d} \times A}{S^a \times S^b \times S^c \times S^d} + \frac{S^{a+e} \times S^{b+c} \times A}{S^a \times S^e \times S^b \times S^c}, \text{ \&c.}$$

$$E = \frac{S^{a+b+c+d+e} \times A}{S^a \times S^b \times S^c \times S^d \times S^e} + \frac{S^{a+b+c+d+f} \times A}{S^a \times S^b \times S^c \times S^d \times S^f}, \text{ \&c.}$$

$$BC = \frac{S^{a+b} \times S^{c+d+e} \times A}{S^a \times S^b \times S^c \times S^d \times S^e} + \frac{S^{a+c} \times S^{d+e+f} \times A}{S^a \times S^c \times S^d \times S^e \times S^f}, \text{ \&c.}$$

$$F = \frac{S^{a+b+c+d+e+f} \times A}{S^a \times S^b \times S^c \times S^d \times S^e \times S^f}, \text{ \&c.}$$

$$BD = \frac{S^{a+b+c+d} \times S^{e+f} \times A}{S^a \times S^b \times S^c \times S^d \times S^e \times S^f} + \frac{S^{a+b+c+e} \times S^{d+f} \times A}{S^a \times S^b \times S^c \times S^d \times S^f}, \text{ \&c.}$$

$$CC = \frac{S^{a+b+c} \times S^{d+e+f} \times A}{S^a \times S^b \times S^c \times S^d \times S^e \times S^f} + \frac{S^{a+b+d} \times S^{c+e+f} \times A}{S^a \times S^b \times S^c \times S^d \times S^e \times S^f}, \text{ \&c.}$$

$$BBB = \frac{S^{a+b} \times S^{c+d} \times S^{e+f} \times A}{S^a \times S^b \times S^c \times S^d \times S^e \times S^f} + \frac{S^{a+c} \times S^{b+d} \times S^{e+f} \times A}{S^a \times S^b \times S^c \times S^d \times S^e \times S^f}, \text{ \&c.}$$

& ſic deinceps.

Hic obſervandum eſt, quod per S^a, S^b, S^c, &c. S^{a+b}, &c. non intelligo poteſtates (a, b, c, &c. $a+b$, &c.) quantitatis S, eodem modo per BB, BC, &c. non intelligo rectangulum ſub B & B vel C contentum, ſed notationem deſignantem quantitatem, quæ ſibi ipſi per notam æqualitatis adjungitur.

Summa quæſita eſt

$$A - B + 1 \times 2\, C - 1 \times 2 \times 3\, D + 1 \times 2 \times 3 \times 4\, E$$
$$+ 1 \times 1\; BB - 1 \times 1 \times 2\; BC$$

$$- 1 \times 2 \times 3 \times 4 \times 5\, F \quad + 1 \times 2 \times 3 \times 4 \times 5 \times 6\, G$$
$$+ 1 \times 1 \times 2 \times 3 \times BD \quad - 1 \times 1 \times 2 \times 3 \times 4\; BE$$
$$- 1 \times 1 \times 1\; BBB \quad + 1 \times 1 \times 1 \times 2 \times BBC$$
$$+ 1 \times 2 \times 1 \times 2\; CC \quad - 1 \times 2 \times 1 \times 2 \times 3\; CD$$

Literalium quantitatum hæc eſt lex.

Ex-

Exponentes literalium quantitatum terminorum infra ſe poſitorum eandem conficiunt ſummam, quæ quidem ſumma erit ſucceſſive 1, 2, 3, 4, 5, 6, &c.

Per exponentes literarum intelligo numeros deſignantes ſedes illas, quas in ſerie naturali literarum ſingulæ literæ occupant, ſic exponens literæ *A* eſt 1; literæ *B*, 2; literæ *C*, 3; & ſic deinceps, quo fit ut exponens literalis quantitatis *BBB*...*CCC*...*DDD*...*EEE*, &c. (in quâ ſint tot literæ *B*, *C*, *D*, *E*, &c. contentæ, quot unitates in π, ρ, σ, τ, &c. reſpective) ſit $2\pi + 3\rho + 4\sigma + 5\tau$, &c.

Coefficiens literalis quantitatis *BBB*...*CCC*...*DDD*...*EEE*, &c. in quâ tot ſunt literæ *B*, quot unitates in π, tot ſunt literæ *C*, quot unitates in ρ, tot ſunt literæ *D*, *E*, &c. quot unitates in σ, τ, &c. reſpective, æqualis eſt producto $1^{\pi} \times 1^{\rho} \times 2^{\rho} \times 1^{\sigma} \times 2^{\sigma} \times 3^{\sigma} \times 1^{\tau} \times 2^{\tau} \times 3^{\tau} \times 4^{\tau}$, &c.

Si duo vel tres vel quatuor vel plures indices (a, b, c, d, &c.) ſunt inter ſe æquales, tum dividatur ſumma per producta 1×2, $1 \times 2 \times 3$, $1 \times 2 \times 3 \times 4$, &c. reſpective & ſic conjunctim.

EXEMPLUM I.

Sint $\alpha, \beta, \gamma, \delta$, &c. radices datæ æquationis, $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4}$, &c. $= 0$. & inveniantur

$$P^2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^3, \text{ \&c.} = 16.$$
$$P^3 = \alpha^3 + \beta^3 + \gamma^3 + \delta^3, \text{ \&c.} = 30.$$
$$\overline{P^{2+3}} = \alpha^5 + \beta^5 + \gamma^5 + \delta^5, \text{ \&c.} = 265.$$

invenire ſummam,

$$\alpha^2 \times \beta^3 + \alpha^2 \times \gamma^3 + \alpha^2 \times \delta^3 + \text{ \&c.} + \beta^2 \times \alpha^3 + \beta^2 \times \gamma^3 + \beta^2 \times \delta^3 + \text{ \&c.}$$
$$+ \gamma^2 \times \alpha^3 + \gamma^2 \times \beta^3 + \gamma^2 \times \delta^3 + \text{ \&c.} + \delta^2 \times \alpha^3 + \delta^2 \times \beta^3 + \delta^2 \times \gamma^3 + \text{ \&c.}$$

In hoc exemplo eſt,

$$A = P^2 \times P^3 = 16 \times 30, \quad B = P^{2+3} = 265.$$

ſumma quæſita

$$(A - B) \text{ eſt } 16 \times 30 - 265 = 215.$$

EX-

EXEMPLUM II.

Sint,

$$P^{1} = \alpha + \beta + \gamma + \delta, \text{ \&c.} = 7$$
$$P^{2} = \alpha^{2} + \beta^{2} + \gamma^{2} + \delta^{2} + \text{ \&c.} = 12$$
$$P^{3} = \alpha^{3} + \beta^{3} + \gamma^{3} + \delta^{3} + \text{ \&c.} = 18$$
$$P^{1+2} = \alpha^{3} + \beta^{3} + \gamma^{3} + \delta^{3} + \text{ \&c.} = 18.$$
$$P^{1+3} = \alpha^{4} + \beta^{4} + \gamma^{4} + \delta^{4} + \text{ \&c.} = 30.$$
$$P^{2+3} = \alpha^{5} + \beta^{5} + \gamma^{5} + \delta^{5} \text{ \&c.} = 45$$
$$P^{1+2+3} = \alpha^{6} + \beta^{6} + \gamma^{6} + \delta^{6}, \text{ \&c.} = 60.$$

invenire ſummam,

$$\alpha\beta^{2}\gamma^{3} + \alpha\beta^{2}\delta^{3} + \alpha\beta^{3}\gamma^{2} + \alpha\beta^{3}\delta^{2} + \beta\alpha^{2}\gamma^{3} + \beta\alpha^{2}\delta^{3} + \beta\alpha^{3}\gamma^{2} + \beta\alpha^{3}\delta^{2} + \text{ \&c.}$$

Conſtat

$$A = P^{1} \times P^{2} \times P^{3} = 7 \times 12 \times 18 = 1512$$
$$B = P^{1+2} \times P^{3} + P^{1+3} \times P^{2} + P^{2+3} \times P^{1} = \overline{18}^{2} + 30 \times 12 + 45 \times 7 = 999$$
$$C = P^{1+2+3} = 60.$$

Summa quæſita eſt

$$A - B + 2C = 1512 - 999 + 2 \times 60 = 633.$$

COROLLARIUM I.

Sit (a) ſumma radicum, (b) ſumma quadratorum ex ſingulis radicibus, (c) ſumma cuborum, (d) ſumma quadrato-quadratorum, (e) ſumma quadrato-cuborum, f ſumma cubo-cuborum, &c.

Summa contentorum ſub ſingulis (m) radicibus erit æqualis fractioni, cujus numerator eſt

$$a^{m} - m \times \frac{\overline{m-1}}{2} \times a^{m-2} b + m \times \overline{m-1} \times \frac{\overline{m-2}}{3} \times a^{m-3} c - m \times \overline{m-1} \times \overline{m-2} \times \frac{\overline{m-3}}{4} \times a^{m-4} d \quad \text{\&c.}$$
$$+ m \times \overline{m-1} \times \frac{\overline{m-2}}{2} \times \frac{m-3}{2^{2}} a^{m-4} b^{2}$$

Denominator vero $1 \times 2 \times 3 \times 4 \ldots \times m$.

Scribantur pro literis a, b, c, d, e, &c. p, q, r, s, t, &c. reſpective; & hujus ſeriei literalia producta eadem erunt, ac literalia producta ſeriei in lemmate primo.

Coefficiens literalis producti ($a^{m-1}\times b^{\pi}\times c^{\rho}\times d^{\sigma}$, &c.) erit æqualis fractioni, cujus numerator eſt $m\times\overline{m-1}\times\overline{m-2}\times\overline{m-3}$, &c. tot factores erunt, quot unitates in ſummâ $2\pi+3\rho+4\sigma$, &c. Denominator vero æqualis erit producto orto ex continuâ multiplicatione ſerierum $1\times2\times3$, &c. $1\times2\times3$, &c. $1\times2\times3$, &c. $2^{\pi}3^{\rho}4^{\sigma}$, &c. quarum prima continet tot terminos, quot unitates in π, ſecunda, quot ſunt unitates in ρ, tertia, quot ſunt in σ, & ſic deinceps.

COROLLARIUM II.

Sint $\alpha, \beta, \gamma, \delta, \epsilon$, &c. radices datæ æquationis,

$$x^n - px^{n-1} + qx^{n-2} - rx^{n-3}, \text{ \&c.} = 0.$$ & ſit fractio

$$\frac{a - bx + cx^2 - dx^3, \text{ \&c.}}{A - Bx + Cx^2 - Dx^3, \text{ \&c.}}$$ in quâ ſint coefficientes (a, b, c, &c. A, B, C, &c.) datæ,

Invenire ſummam fractionum,

$$\frac{a - b\alpha + c\alpha^2 - d\alpha^3, \text{ \&c.}}{A - B\alpha + C\alpha^2 - D\alpha^3, \text{ \&c.}} + \frac{a - b\beta + c\beta^2 - d\beta^3, \text{ \&c.}}{A - B\beta + C\beta^2 - D\beta^3, \text{ \&c.}} + \frac{a - b\gamma + c\gamma^2 - d\gamma^3, \text{ \&c.}}{A - B\gamma + C\gamma^2 - D\gamma^3, \text{ \&c.}} \text{ \&c.}$$

Reducantur omnes hæ fractiones in communem denominatorem, tum lemmatis primi & ſecundi ope inveniri poteſt & numerator & denominator.

EXEMPLUM.

Sit data æquatio $x^3 - 7x + 6 = 0$, cujus radices ſint α, β, γ, & fractio $\frac{3x^2+5}{4x+1}$.

Invenire ſummam fractionum,

$\frac{3\alpha^2+5}{4\alpha+1}, \frac{3\beta^2+5}{4\beta+1}, \frac{3\gamma^2+5}{4\gamma+1}$, reducantur hæ fractiones in communem denominatorem & fiunt,

$$\frac{48\times\alpha\beta\gamma\times\overline{\alpha+\beta+\gamma}+12\times\overline{\alpha^2\beta+\alpha^2\gamma+\beta^2\alpha+\beta^2\gamma+\gamma^2\alpha+\gamma^2\beta}}{64\times\alpha\beta\gamma+16\times\overline{\alpha\beta+\alpha\gamma+\beta\gamma}+4\times\overline{\alpha+\beta+\gamma}+1} +$$

$$\frac{3\times\overline{\alpha^2+\beta^2+\gamma^2}+80\times\overline{\alpha\beta+\alpha\gamma+\beta\gamma}+40\times\overline{\alpha+\beta+\gamma}+15}{64\times\alpha\beta\gamma+16\times\overline{\alpha\beta+\alpha\gamma+\beta\gamma}+4\times\overline{\alpha+\beta+\gamma}+1}$$

Pro p, q, r, ſcribantur 0, -7, & -6 tum

De-

Denominator eſt

$$64\times\alpha\beta\gamma\,(64\times r=64\times-6)+16\times\overline{\alpha\beta+\alpha\gamma+\beta\gamma}\,(16\times q=16\times-7)+4\times\overline{\alpha+\beta+\gamma}\,(4\times p=0)+1=-495.$$

Numerator vero

$$48\times\alpha\beta\gamma\times\overline{\alpha+\beta+\gamma}\,(48\,rp=0)+12\times\overline{\alpha^2\beta+\alpha^2\gamma+\beta^2\alpha+\beta^2\gamma+\gamma^2\alpha+\gamma^2\beta}\,(12\times\overline{pq-3r}=12\times3\times6)+3\times\overline{\alpha^2+\beta^2+\gamma^2}\,(3\times\overline{p^2-2q}=3\times2\times7)+80\times\overline{\alpha\beta+\alpha\gamma+\beta\gamma}\,(80\times q=80\times-7)+40\times\overline{\alpha+\beta+\gamma}\,(40\times p=40\times0)+15=-287$$

& conſequenter ſumma quæſita eſt $\frac{287}{495}$.

Eodem modo inveniri poteſt aggregatum irrationalium functionum datæ æquationis radicum.

PROB. I.

Datâ unâ, duabus, vel pluribus æquationibus, invenire æquationem, cujus radices, ad radices datarum æquationum quamcunque datam relationem (exponentialibus exceptis) habent.

Scribantur $\alpha, \beta, \gamma, \delta$, &c. pro radicibus unius datæ æquationis, $\mathrm{A}, \mathrm{B}, \Gamma, \Delta$, &c. pro radicibus alterius datæ æquationis & ſic de reliquis.

Ex datâ aſſignabili relatione inveniantur omnes quæſitæ æquationis radices, & lemmatis primi vel ſecundi ope inveniri poteſt ex ſingulis hiſce radicibus aggregatum, quod eſt coefficiens ſecundi quæſitæ æquationis termini.

Ducantur quæque duæ radices quæſitæ æquationis in ſeſe & ex lemmatibus prædictis inveniatur aggregatum rectangulorum reſultantium & erit coefficiens tertii quæſitæ æquationis termini.

Ducantur quæque tres, quatuor, quinque, &c. radices quæſitæ æquationis continuo in ſeſe & ex iiſdem lemmatibus inveniantur aggregata reſultantium productorum & hæc aggregata erunt reſpective coefficientes quarti, quinti, ſexti, &c. quæſitæ æquationis terminorum.

Hinc dilucide conſtat methodus inveniendi quæſitæ æquationis coefficientes, & conſequenter ipſam æquationem.

Hujuſce generis problemata ſæpe ſolvi poſſunt ex reductione duarum vel plurium æquationum in unam, ita ut incognitæ quantitates exterminentur.

EXEMPLUM I.

Sint $\alpha, \beta, \gamma, \delta$ &c. Radices datæ æquationis

$$x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4}, \&c. = o.$$

invenire æquationem

$v^n - Pv^{n-1} + Qv^{n-2} - Rv^{n-3} + Sv^{n-4}$, &c. $= o$. cujus radices ſunt ($\alpha^m, \beta^m, \gamma^m, \delta^m$, &c, reſpective) m poteſtates radicum datæ æquationis.

Scribatur

$$a = \alpha^m + \beta^m + \gamma^m + \delta^m, \&c.$$
$$b = \alpha^{2m} + \beta^{2m} + \gamma^{2m} + \delta^{2m}, \&c.$$
$$c = \alpha^{3m} + \beta^{3m} + \gamma^{3m} + \delta^{3m}, \&c.$$
$$d = \alpha^{4m} + \beta^{4m} + \gamma^{4m} + \delta^{4m}, \&c.$$

Tum ex primo lemmatis ſecundi corollario inveniantur requiſitæ æquationis coefficientes (P, Q, R, S, &c.) & erunt reſpective

$$P = \alpha^m + \beta^m + \gamma^m + \delta^m, \&c. = a$$

$$Q = \alpha^m \times \beta^m + \alpha^m \times \gamma^m + \alpha^m \times \delta^m + \&c. + \beta^m \times \gamma^m + \beta^m \times \delta^m + \&c. + \gamma^m \times \delta^m + \&c. = \frac{a^2 - b}{2}$$

$$R = \alpha^m \times \beta^m \times \gamma^m + \alpha^m \times \beta^m \times \delta^m + \alpha^m \times \gamma^m \times \delta^m + \&c. + \beta^m \times \gamma^m \times \delta^m + \&c. = \frac{a^3 - 3ab + 2c}{6}$$

$$S = \alpha^m \times \beta^m \times \gamma^m \times \delta^m + \&c. = \frac{a^4 - 6a^2b + 8ac - 6d + 3b^2}{24}$$

& ſic de reliquis coefficientibus.

Ex lemmate ſecundo ab aliâ methodo derivatur hujus exempli ſubſequens ſolutio.

Sint b, k, l, o, &c. omnes diverſæ radices m poteſtatis unitatis.

Tum ex hypotheſi $x^m = v$ & conſequenter

$b v^{\frac{1}{m}} = x$, $k v^{\frac{1}{m}} = x$, $l v^{\frac{1}{m}} = x$, $o v^{\frac{1}{m}} = x$, &c.

Quibus valoribus in datâ æquatione pro quantitate (x) ſubſtitutis reſultant æquationes.

$$b^n v^{\frac{n}{m}} - p \times b^{n-1} v^{\frac{n-1}{m}} + q \times b^{n-2} v^{\frac{n-2}{m}} - r \times b^{n-3} \times v^{\frac{n-3}{m}}, \&c. = o.$$

$$k^n v^{\frac{n}{m}} - p \times k^{n-1} \times v^{\frac{n-1}{m}} + q \times k^{n-2} v^{\frac{n-2}{m}} - r \times k^{n-3} \times v^{\frac{n-3}{m}}, \&c. = o.$$

$$l^n v^{\frac{n}{m}} - p \times l^{n-1} v^{\frac{n-1}{m}} + q \times l^{n-2} v^{\frac{n-2}{m}} - r l^{n-3} \times v^{\frac{n-3}{m}}, \&c. = o.$$

$$o^n v^{\frac{n}{m}} - p \times o^{n-1} v^{\frac{n-1}{m}} + q \times o^{n-2} v^{\frac{n-2}{m}} - r o^{n-3} \times v^{\frac{n-3}{m}}, \&c. = o.$$

& ſic deinceps.

Du-

Ducantur hæ æquationes continuo in ſeſe, & æquatio reſultans erit $v^n - Pv^{n-1} + \mathcal{Q}v^{n-2} - Rv^{n-3} + Sv^{n-4}$, &c. $= 0$.
cujus radices ſunt m poteſtates radicum datæ æquationis.

Ex lemmate ſecundo inveſtigari poſſunt coefficientes ($P, \mathcal{Q}, R, S$, &c.) & quo minor eſt numerus (m), eo magis ſimplex eſt lex, quam obſervant quæſitæ æquationis coefficientes.

Ultimus quæſitæ æquationis terminus eſt (m) poteſtas ultimi in datâ æquatione termini.

Ex lemmate primo conſtat etiam ab alia methodo hujus exempli ſolutio.

COROLLARIUM.

Hinc deleri poſſunt quæcunque ſurdæ quantitates ex datâ æquatione.

Sit ſurda quantitas delenda $x^{\frac{1}{n}}$ & data æquatio

$$Ax^{\frac{n-1}{n}} - Bx^{\frac{n-2}{n}} + Cx^{\frac{n-3}{n}}, \text{\&c.} = \mathcal{Q}$$

aſſumatur $v = x^{\frac{1}{n}}$ & invenietur ex exemplo, æquatio, cujus radices ſunt v; & pro v ſubſtituatur x & fit.

EXEMPLUM II.

Invenire æquationem, cujus radices ſunt ($\alpha+\beta, \alpha+\gamma, \alpha+\delta$, &c. $\beta+\gamma, \beta+\delta$, &c. $\gamma+\delta$, &c.) ſummæ ex quibuſcunque duabus datæ æquationis radicibus.

Numerus quæſitæ æquationis radicum & conſequenter ejus dimenſiones erit æqualis fractioni $\frac{n \times \overline{n-1}}{1 \times 2}$.

Summa radicum eſt $\overline{n-1} \times \overline{\alpha+\beta+\gamma+\delta}$, &c. $= \overline{n-1} \times p$ quæ eſt coefficiens ſecundi quæſitæ æquationis termini.

Ducantur quæque duæ radices quæſitæ æquationis in ſeſe & aggregatum reſultantium rectangulorum
($\overline{\alpha+\beta} \times \overline{\alpha+\gamma}$, $\overline{\alpha+\beta} \times \overline{\alpha+\delta}$, $\overline{\alpha+\gamma} \times \overline{\alpha+\delta}$, &c. $\overline{\beta+\gamma} \times \overline{\alpha+\beta}$, $\overline{\beta+\gamma} \times \overline{\alpha+\gamma}$, &c. &c.) erit

$$\overline{n-1} \times \frac{\overline{n-2}}{2} \times \overline{\alpha^2+\beta^2+\gamma^2+\delta^2}, \text{\&c.} + n \times \overline{n-2} \times \overline{\alpha\beta+\alpha\gamma+\alpha\delta, \text{\&c.}}$$
$$\overline{+\beta\gamma+\beta\delta, \text{\&c.} + \gamma\delta, \text{\&c.}}$$ ſed ex lemmate primo conſtat

$\overline{n-1}$

$$\overline{n-1}\times\frac{\overline{n-2}}{2}\overline{\alpha^2+\beta^2+\gamma^2+\delta^2}\ \&c. = \overline{n-1}\times\frac{\overline{n-2}}{2}\times\overline{p^2-2q}$$

$$n\times\overline{n-2}\times\overline{\alpha\beta+\alpha\gamma+\alpha\delta+\&c.+\beta\gamma+\beta\delta+\&c.+\gamma\delta+\&c.} = n\times\overline{n-2}\times q$$

Aggregatum eſt $\overline{n-1}\times\frac{\overline{n-2}}{2}\times p^2+\overline{n-2}\times q$

Sed hoc aggregatum eſt coefficiens tertii datæ æquationis termini.

Ducantur quæque tres radices quæſitæ æquationis in ſeſe

$(\overline{\alpha+\beta}\times\overline{\alpha+\gamma}\times\overline{\alpha+\delta},\ \overline{\alpha+\beta}\times\overline{\alpha+\gamma}\times\overline{\beta+\delta},\ \overline{\alpha+\beta}\times\overline{\alpha+\gamma}\times\overline{\gamma+\delta}$, &c.)

Aggregatum horum productorum eſt

$$\overline{n-1}\times\frac{\overline{n-2}}{2}\times\frac{\overline{n-3}}{3}\times\overline{\alpha^3+\beta^3+\gamma^3+\delta^3}\ \&c. +\overline{n-2}\times\frac{\overline{n^2-2n-1}}{2}\times\overline{\alpha^2\beta+\alpha^2\gamma+\alpha^2\delta+\&c.+\beta^2\alpha+\beta^2\gamma+\beta^2\delta+\&c.+\gamma^2\alpha+\gamma^2\beta+\gamma^2\delta+\&c.}$$
$$+\overline{n-1}\times\overline{n^2-2n-2}\times\overline{\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta+\&c.}$$

Sed ex lemmate primo & ſecundo colligitur

$$\overline{n-1}\times\frac{\overline{n-2}}{2}\times\frac{\overline{n-3}}{3}\times\overline{\alpha^3+\beta^3+\gamma^3+\delta^3}\ \&c. = \overline{n-1}\times\frac{\overline{n-2}}{2}\times\frac{\overline{n-3}}{3}\times\overline{p^3-3pq+3r}$$

$$\overline{n-2}\times\frac{\overline{n^2-2n-1}}{2}\times\overline{\alpha^2\beta+\alpha^2\gamma+\alpha^2\delta+\&c.+\beta^2\alpha+\beta^2\gamma+\beta^2\delta+\&c.+\gamma^2\alpha+\&c.}=\overline{n-2}\times\frac{\overline{n^2-2n-1}}{2}\times\overline{pq-3r}$$

$$\overline{n-1}\times\overline{n^2-2n-2}\times\overline{\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta+\&c.}=\overline{n-1}\times\overline{n^2-2n-2}\times r$$

Aggregatum eſt $= n\times\frac{\overline{n-1}}{2}\times\frac{\overline{n-2}}{3}\times p^3-\overline{n-2}^2\times pq+\overline{n-4}\times r$

Hoc vero aggregatum eſt coefficiens quarti quæſitæ æquationis termini.

Eodem modo inveniri poſſunt reliquæ coefficientes, & conſequenter ſi modo v ſit radix quæſitæ æquationis æquatio quæſita erit

$$v^{n\times\frac{\overline{n-1}}{2}}-\overline{n-1}\times pv^{n.\frac{\overline{n-1}}{2}-1}+\overline{\overline{n-1}\times\frac{\overline{n-2}}{2}p^2+\overline{n-2}\times q}\times v^{n.\frac{\overline{n-1}}{2}-2}-\overline{n\times\frac{\overline{n-1}}{2}\times\frac{\overline{n-2}}{3}\times p^3+\overline{n-2}^2\times pq+\overline{n-4}\times r}\,v^{n\times\frac{\overline{n-1}}{2}-3}\ \&c.=0.$$

EXEMPLUM III.

Sint $\alpha,\beta,\gamma,\delta$, &c. radices datæ æquationis

$$x^n-px^{n-1}+qx^{n-2}-rx^{n-3}+sx^{n-4}\ \&c.=0.$$

& A, B, Γ, Δ radices datæ æquationis

$$z^m-Pz^{m-1}+Qz^{m-2}-Rz^{m-3},\ \&c.=0.$$

invenire æquationem, cujus radices ſint (αA, αB, $\alpha\Gamma$, $\alpha\Delta$, &c. βA,

$\beta A, \beta B, \beta \Gamma, \beta \Delta$, &c. $\gamma A, \gamma B, \gamma \Gamma, \gamma \Delta$, &c. $\delta A, \delta B, \delta \Gamma, \delta \Delta$, &c.) Rectangula ſub ſingulâ unius datæ æquationis radice & ſingulâ alterius radice contenta.

Numerus radicum requiſitæ æquationis & conſequenter ejus dimenſiones æqualis eſt rectangulo ſub indicibus ($n \times m$) datarum æquationum.

Summa radicum & conſequenter ſecundi termini requiſitæ æquationis coefficiens æqualis eſt rectangulo $\overline{\alpha + \beta + \gamma + \delta}$, &c. $\times$ $\overline{A + B + \Gamma + \Delta}$, &c. $= P \times p$.

Ducantur quæque duæ radices datæ æquationis in ſeſe ($\alpha A \times \alpha B$, $\alpha A \times \alpha \Gamma$, $\alpha A \times \alpha \Delta$, &c. $\beta A \times \alpha B$, $\beta A \times \alpha \Gamma$, $\beta A \times \alpha A$, &c. $\gamma A \times \alpha A$, $\gamma A \times \alpha B$, &c.) & rectangulorum reſultantium aggregatum, ſeu coefficiens tertii quæſitæ æquationis termini eſt

$$\overline{\alpha^2 + \beta^2 + \gamma^2 + \delta^2 \text{ \&c.}} \times \overline{AB + A\Gamma + A\Delta + \text{\&c.} + B\Gamma + B\Delta + \text{\&c.}\ \Gamma\Delta +}$$
$$\overline{A^2 + B^2 + \Gamma^2 + \Delta^2 \text{ \&c.}} \times \overline{\alpha\beta + \alpha\gamma + \alpha\delta + \text{\&c.} + \beta\gamma + \beta\delta + \text{\&c.} + \gamma\delta +}$$
$$2 \times \overline{\alpha\beta + \alpha\gamma + \alpha\delta + \text{\&c.} + \beta\gamma + \beta\delta + \text{\&c.} + \gamma\delta + \text{\&c.}} \times \overline{AB + A\Gamma + A\Delta + \text{\&c.} + B\Gamma + B\Delta + \text{\&c.} + \Gamma A}$$

Ex lemmate primo colligitur

$$\overline{\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \text{\&c.}} \times \overline{AB + A\Gamma + A\Delta + \text{\&c.} + B\Gamma + B\Delta + \text{\&c.} + \Gamma\Delta \text{\&c.}} = \overline{p^2 - 2q} \times \mathcal{Q}$$
$$\overline{A^2 + B^2 + \Gamma^2 + \Delta^2 \text{\&c.}} \times \overline{\alpha\beta + \alpha\gamma + \alpha\delta + \text{\&c.} + \beta\gamma + \beta\delta + \text{\&c.} + \gamma\delta \text{\&c.}} = \overline{P^2 - 2\mathcal{Q}} \times q$$
$$2 \times \overline{\alpha\beta + \alpha\gamma + \alpha\delta + \text{\&c.} + \beta\gamma + \beta\delta + \text{\&c.} + \gamma\delta + \text{\&c.}} \times \overline{AB + A\Gamma + A\Delta + \text{\&c.} + B\Gamma + B\Delta + \text{\&c.} + \Gamma\Delta + \text{\&c.}} = 2\mathcal{Q}q$$

Eorum aggregatum eſt $\overline{p^2\mathcal{Q} + P^2 q - 2\mathcal{Q}q}$.

Ex eâdem ratiocinandi methodo coefficiens quarti quæſitæ æquationis termini invenitur

$P^3 r + p^3 R - 3P\mathcal{Q}r - 3pqR + pqP\mathcal{Q} + 3Rr$, & ſic deinceps.

Sit v radix quæſitæ æquationis & quæſita æquatio erit

$$v^{nm} - Ppv^{nm-1} \left.\begin{matrix} + P^2 q - 2\mathcal{Q}q \\ + p^2 \mathcal{Q} \end{matrix}\right| \times v^{nm-2} \left.\begin{matrix} - P^3 r + 3P\mathcal{Q}r \\ - p^3 R + 3pqR - 3Rr \\ - pqP\mathcal{Q} \end{matrix}\right| \times v^{nm-3}, \text{\&c.} = 0.$$

CAP. II.

LEMMA III.

SINT omnes radices datæ æquationis poſſibiles invenire numeros radicibus quam proxime æquales.

REG. 1. Sint $\alpha, \beta, \gamma, \delta$, &c. radices datæ æquationis

$$x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - tx^{n-5}, \text{ \&c.} = 0.$$

& ſit α major quam β, β quam γ, γ quam δ, & ſic deinceps, &

$\overline{\alpha^{2m} + \beta^{2m} + \gamma^{2m} + \delta^{2m} + \text{\&c.}}^{\frac{1}{2m}}$ non major eſt quam $n^{\frac{1}{2m}}\alpha$, &

$$\overline{\frac{\alpha^{2m} \times \beta^{2m} + \alpha^{2m} \times \gamma^{2m} + \alpha^{2m} \times \delta^{2m} + \text{\&c.} + \beta^{2m} \times \gamma^{2m} + \beta^{2m} \times \delta^{2m} + \text{\&c.}}{\alpha^{2m} + \beta^{2m} + \gamma^{2m} + \delta^{2m} + \text{\&c.}}}\Bigg|^{\frac{1}{2m}}$$

non major eſt quam $\overline{n-1}|^{\frac{1}{2m}} \times \beta$, &

$$\overline{\frac{\alpha^{2m} \times \beta^{2m} \times \gamma^{2m} + \alpha^{2m} \times \beta^{2m} \times \delta^{2m} + \alpha^{2m} \times \gamma^{2m} \delta^{2m} + \text{\&c.} + \beta^{2m} \times \gamma^{2m} \times \delta^{2m}}{\alpha^{2m} \times \beta^{2m} + \alpha^{2m} \times \gamma^{2m} + \alpha^{2m} \times \delta^{2m} + \text{\&c.} + \beta^{2m} \times \gamma^{2m} + \beta^{2m} \times \delta^{2m} + \text{\&c.}}}\Bigg|^{\frac{1}{2m}}$$

non major eſt quam $\overline{n-2}|^{\frac{1}{2m}} \times \gamma$, & ſic deinceps obſervatâ progreſſionis lege.

2. Quantitas $\overline{\dfrac{\alpha^{4m} + \beta^{4m} + \gamma^{4m} + \delta^{4m} + \text{\&c.}}{\alpha^{2m} + \beta^{2m} + \gamma^{2m} + \delta^{2m} + \text{\&c.}}}\Big|^{\frac{1}{2m}}$ eſt approximatio ad maximam radicem (α) &

$$\overline{\alpha^{2m} + \beta^{2m} + \gamma^{2m} + \delta^{2m} + \text{\&c.} - \overline{\alpha^{4m} + \beta^{4m} + \gamma^{4m} + \delta^{4m}}^{\frac{1}{2}}}^{\frac{1}{2m}}$$

eſt approximatio ad radicem (β) & ſic deinceps.

3. Datâ æquatione

$$x^n - * + qx^{n-2} - rx^{n-3} + sx^{n-4} - tx^{n-5} + ux^{n-6} - vx^{n-7} + wx^{n+8} - \text{\&c.} = 0.$$

cujus ſecundus terminus deeſt & ſint ejus radices

$-\alpha, +\beta, +\gamma, +\delta, +$ &c. & $-\overline{\alpha^8 + \beta^8 + \gamma^8 + \delta^8 + \text{\&c.}}^{\frac{1}{8}}$ non major eſt quam $-2^{\frac{1}{8}} \times \alpha$, & conſequenter haud differt a radice ($-\alpha$) per decimam ejus partem, ſed per lemma primum

$$\alpha^8 + \beta^8 + \gamma^8 + \delta^8 + \text{\&c.} = 2q^4 - 8qr^2 + 4s^2 - 8q^2s - 8w + 8qu + 8rt;$$

ergo, $\overline{2q^4 - 8qr^2 + 4s^2 - 8q^2s - 8w + 8qu + 8rt}^{\frac{1}{8}}$ negative ſumpta haud differt a radice ($-\alpha$) per decimam ejus partem, & ſic de reliquis.

Ex

Ex methodo ſerierum, erui & demonſtrari poſſunt infinitæ conſimiles propoſitiones.

Et eodem fere modo inveniri poſſunt approximationes ad radices datæ æquationis, quæ habet impoſſibiles radices.

4. Radices æquationis

$$nx^{n-1} - \overline{n-1} \times px^{n-2} + \overline{n-2} \times qx^{n-3} - \&c. = 0.$$

erunt limites inter α & β, β & γ, γ & δ, &c. reſpective, & radices æquationis

$$n \times \overline{n-1}\, x^{n-2} - \overline{n-1} \times \overline{n-2} \times px^{n-2} + \overline{n-2} \times \overline{n-3}\, qx^{n-4} - \&c. = 0.$$

erunt limites inter α & γ, β & δ, γ & ε, &c. & radices æquationis

$$n \times \overline{n-1} \times \overline{n-2}\, x^{n-3} - \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times px^{n-4} + \overline{n-2} \times \overline{n-3} \times \overline{n-4} \times qx^{n-5} - \&c. = 0.$$

erunt limites inter α & δ, β & ε, &c. reſpective.

Sequuntur quædam regulæ, e quibus quot radices impoſſibiles ſunt, cognoſci poteſt.

REG. 1. Data æquatione $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \&c. = 0$, invenire, utrum ullas habet impoſſibiles radices, necne.

Sint $\alpha, \beta, \gamma, \delta$, &c. radices datæ æquationis; per problema primum transformetur data æquatio in alteram, cujus radices (v) erunt reſpective $\alpha^2 + \beta^2 - 2\alpha\beta$, $\alpha^2 + \gamma^2 - 2\alpha\gamma$, $\alpha^2 + \delta^2 - 2\alpha\delta$, &c. $\beta^2 + \gamma^2 - 2\beta\gamma$, $\beta^2 + \delta^2 - 2\beta\delta$, &c. $\gamma^2 + \delta^2 - 2\gamma\delta$, &c.

Æquatio reſultans eſt

$$v^{n \times \frac{n-1}{2}} \overline{- \overline{n-1} \times p^2 - 2nq}\; v^{n \times \frac{n-1}{2} - 1} \overline{+ \overline{n-1} \times \frac{n-2}{2} p^4 - 2n \times \overline{n-2}\, p^2 q - 2 \times \overline{n-3}\, pr + 2ns + \overline{2n+3} \times \overline{n-2}\, q^2} \times v^{n \times \frac{n-1}{2} - 2} - \&c. = 0.$$

Si mutentur hujuſce æquationis reſultantis ſigna alternatim de + in − & − in +; nullam radicem impoſſibilem habet data æquatio, ſin hujuſce æquationis ſigna haud continuo mutantur de + in − & − in +, impoſſibiles radices habet data æquatio.

EXEMPLUM.

Sit data æquatio $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$, & æquatio reſultans erit, $v^6 - 20v^5 + 142v^4 - 460v^3 + 713v^2 - 520v + 144 = 0$.

Hujuſce æquationis ſigna mutantur alternatim de + in — & — in +, & conſequenter nullam habet impoſſibilem radicem data æquatio.

Sit æquatio biquadratica ($x^4+qx^2-rx+s=0$) & ſolutionem paullulum diverſam admittit.

Sint $\alpha,\beta,\gamma,\delta$ ejus radices, inveniatur æquatio cujus radices (v) ſunt $\alpha^2+\beta^2+\gamma^2+\delta^2-2\alpha\gamma-2\beta\delta$, $\alpha^2+\beta^2+\gamma^2+\delta^2-2\alpha\delta-2\beta\gamma$, $\alpha^2+\beta^2+\gamma^2+\delta^2-2\alpha\beta-2\gamma\delta$.

Æquatio reſultans eſt

$$v^3+8qv^2+16\times\overline{q^2-s}\,v+32q^3-24qs-8r^2=0.$$

Sint q & $32q^3-24qs-8r^2$ reſpective negativæ quantitates, q^2-s vero affirmativa & omnes datæ biquadraticæ æquationis radices erunt poſſibiles, ſin aliter, tum vel duæ vel quatuor ſunt impoſſibiles.

Reg. 2. Invenire, utrum data æquatio

$$x^{2n}-px^{2n-1}+qx^{2n-2}-rx^{2n-3}+\&c.=0.$$

poſſibilem habet radicem, necne.

Fingetur $v^{2n-1}-\frac{2n-1}{2n}pv^{2n-2}+\frac{2n-2}{2n}qv^{2n-3}-\frac{2n-3}{2n}rv^{2n-4}\&c.=0$.

& $$v^{2n}-pv^{2n-1}+qv^{2n-2}-rv^{2n-3}+\&c.=w.$$

transformentur hæ duæ æquationes in unam, ita ut exterminetur incognita quantitas (v) & reſultat æquatio, cujus radix eſt (w) quæ ſit

$$w^{2n-1}-Pw^{2n-2}+Qw^{2n-3}-Rw^{2n-4}+\&c.=0;$$

mutentur hujuſce æquationis terminorum ſigna viciſſim de + in — & — in +; i. e. ſint omnes ejus coefficientes ($P,Q,R,S,$&c.) affirmativæ & nullum habet radicem poſſibilem data æquatio, & ſic deinceps.

EXEMPLUM.

Sit data æquatio biquadratica, $x^4-14x^2+24x+37=0$.

Fingetur $v^3-7v+6=0$ & $v^4-14v^2+24v+37=w$ unde æquatio reſultans erit

$$w^3-13w^2-5280w+172800=0.$$

Terminorum hujuſce æquationis ſigna ſeſe ſequuntur hoc ordine + — — +.

Secundi termini ſignum variatur a primi termini ſigno, ſed tertii termini ſignum haud variatur a ſecundi ſigno & conſequenter data æquatio habet poſſibilem radicem.

COROLLARIUM.

Sit æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0$.

Fingetur eodem modo, quo prius docetur, &

$$v^{n-1} - \frac{n-1}{n} pv^{n-2} + \frac{n-2}{n} qv^{n-3} - \&c. = 0;$$

$$v^n - pv^{n-1} + qv^{n-2} - rv^{n-3} + \&c. = w:$$

transformentur hæ duæ æquationes in unam ita ut exterminetur incognita quantitas (v), & æquatio reſultans ſit

$$w^{n-1} - Pw^{n-2} + Qw^{n-3} - Rw^{n-4} + \&c. = 0.$$

Sub primo hujuſce æquationis termino colloca ſignum +, ſub quolibet reliquorum terminorum ſcribe ſignum — vel +, prout ejus ſignum eſt idem vel diverſum a ſigno termini præcedentis, his denique adjiciendum eſt ſignum + vel —, prout (n) eſt par vel impar numerus & fere tot erunt radices poſſibiles, quot ſunt in ſubſcriptorum ſignorum ſerie mutationes de + in —, & — in +.

EXEMPLUM.

Sit æquatio $x^3 - 9x + 20 = 0$.

Fingetur $v^2 - 3 = 0$, & $v^3 - 9v + 20 = w$;

unde æquatio reſultans erit

$$w^2 - 40w + 373 = 0$$

\+ + + —

Sub primo hujuſce æquationis termino (w^2) colloco ſignum +, dein quoniam ſignum (—) ſecundi termini ($-40w$) diverſum eſt a ſigno (+) primi termini ſub termino ($-40w$) colloco ſignum +; & eâ ratione ſub tertio termino colloco ſignum +, his denique adjeci ſignum —, quoniam n eſt impar numerus ſubſcriptorum ſignorum, quæ in hâc ſerie + + + — ſint, una ſolummodo eſt mutatio de + in —; & conſequenter una ſolummodo erit poſſibilis radix.

Hæc regula poſtulat, ut hæ duæ æquationes

$$v^{n-1} - \frac{n-1}{n} pv^{n-2} + \frac{n-2}{n} qv^{n-3} - \frac{n-3}{n} rv^{n-4} + \frac{n-4}{n} sv^{n-5} - \&c. = 0$$

$$v^n - pv^{n-1} + qv^{n-2} - rv^{n-4} + sv^{n-5} - tv^{n-6} + \&c. = w$$

in unam transformentur, ita ut exterminetur incognita quantitas (v.)

Si hoc fit, tum æquatio reſultans erit

$$w^{n-1} + \overline{\frac{\overline{n-1}^{\,n-1}}{n^n}\, p^n - \frac{\overline{n-1}^{\,n-2}}{n^{n-2}} \times qp^{n-2} + \frac{\overline{n-1}^{\,n-3}}{n^{n-3}}\, rp^{n-3} -}$$

$$+$$

$$\left.\begin{array}{l} \dfrac{\overline{n-1}^{\,n-4}}{n^{n-4}}\, sp^{n-4} \qquad\qquad + \dfrac{\overline{n-1}^{\,n-5}}{n^{n-5}}\, tp^{n-5} - \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \&c. \\ \frac{1}{2} n \times \dfrac{\overline{n-2}^{\,2}}{n^2}\, \dfrac{\overline{n-1}^{\,n-4}}{n^{n-4}}\, q^2 - n \times \dfrac{\overline{n-2} \times \overline{n-3}}{n^2} \times \dfrac{\overline{n-1}^{\,n-5}}{n^{n-5}}\, qr + \end{array}\right] \times w^{n-2} + \&c. = 0$$

Lex, quam obſervat hæc ſeries exprimens ſecundi termini reſultantis æquationis coefficientem, vel producta literalia, vel coefficientes reſpicit.

Producta literalia eandem, quam producta literalia ſeriei in lemmate ſecundo traditæ, obſervant legem.

Coefficiens literalis producti $p^{n-m} q^{\alpha} r^{\beta} s^{\gamma} t^{\delta}$, &c. erit æqualis fractioni, cujus numerator eſt

$$n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \ldots \overline{n-m} \times \overline{n-2}^{\,\alpha} \times \overline{n-3}^{\,\beta} \times \overline{n-4}^{\,\gamma} \times \overline{n-5}^{\,\delta} \times \&c. \times \frac{\overline{n-1}^{\,n-m}}{n^{n-m}}$$

Denominator vero eſt

$$1 \times 2 \times 3 \times \ldots \alpha \times 1 \times 2 \times 3 \times \ldots \beta \times 1 \times 2 \times 3 \times 4 \times \ldots \gamma \times 1 \times 2 \times 3 \times 4 \times \ldots \delta, \ \&c.$$

Hic excipiatur primi ſeriei termini coefficiens, quæ ſit $\frac{\overline{n-1}^{\,n-1}}{n^n}$.

Sit Q ultimus datæ æquationis terminus & ejus coefficiens eſt $n-1$.

Coefficientes reliquorum reſultantis æquationis terminorum, legem, quæ exprimi poteſt per ſeriem haud obſervant, ſed hic obſervandum eſt, ut in omnibus litera (p) nunquam habet plures quam (n) dimenſiones.

Reg. 3. Sint $\alpha, \beta, \gamma, \delta$, &c. ſucceſſivæ datæ æquationis ($x^n - px^{n-1} + qx^{n-2} - \&c. = 0$) radices quarum nullæ ſint inter ſe æquales, ſit (π) major quam maxima radix (α,) ρ limes inter α & β, σ inter β & γ & ſic deinceps.

Subſtituantur limites (π, ρ, σ, τ, &c.) pro incognitâ quantitate (x) in datâ æquatione & tot erunt radices poſſibiles, quot ſunt mutationes

ones ſignorum quantitatum reſultantium de + in — & — in +, cæteræ vero impoſſibiles ſunt.

Sed in omni caſu inveniri poſſunt limites (ϱ, σ, τ, &c.) per ſubſequentem methodum: transformetur data æquatio in alteram, cujus radices ſint $\frac{1}{\alpha-\beta}$, $\frac{1}{\alpha-\gamma}$, $\frac{1}{\alpha-\delta}$, &c. $\frac{1}{\beta-\alpha}$, $\frac{1}{\beta-\gamma}$, $\frac{1}{\beta-\delta}$, &c. $\frac{1}{\gamma-\alpha}$, $\frac{1}{\gamma-\beta}$, $\frac{1}{\gamma-\delta}$, &c $\frac{1}{\delta-\alpha}$, &c. reſpective, ſit $\left(\frac{1}{A}\right)$ major quam maxima æquationis reſultantis radix & erit (A) minor vel quam differentia inter α & β, vel β & γ, γ & δ, &c.

Inventis igitur quantitate (π) quæ major eſt quam maxima radix, et quantitate (A) quæ minor eſt quam minima differentia inter duas ſucceſſivas radices; continuo auferantur A, $2A$, $3A$, &c. reſpective a π & tandem reſtabunt limites inter ſucceſſivas radices $\alpha, \beta, \gamma, \delta$, &c.

Reg. 4. Data æquatione, invenire quot ſunt ejus radices impoſſibiles & quot poſſibiles.

Duæ radices primo fiunt æquales, deinde impoſſibiles, & conſequenter regula, quæ invenit, quando duæ radices fiunt æquales, invenit etiam, quando vel duæ, vel ſex vel decem fiunt impoſſibiles.

Duæ radices bis, ter, &c. fiunt æquales, deinde quatuor, ſex, &c. impoſſibiles; & conſequenter regulæ, quæ inveniunt, quando duæ radices bis, ter &c. fiunt æquales, inveniunt etiam quando quatuor &c. ſex &c. fiunt impoſſibiles.

Per ſubſequens caput inveniri poteſt, quando duæ radices, ſemel, bis, ter &c. fiunt æquales & conſequenter quando duæ, quatuor, ſex &c. fiunt impoſſibiles.

EXEMPLUM.

Sit æquatio biquadratica $x^4+qx^2-rx+s=0$ invenire quot ſunt ejus radices impoſſibiles.

Sit $s^3-8q^2s^2+\overline{36r^2q+16q^4}\times s-16r^2q^3-27r^4$ nihilo æqualis & duæ radices erunt æquales, ſit vero negativa quantitas & duæ & non plures radices erunt impoſſibiles, ſit affirmativa quantitas & vel q^2-4s vel $9r^2+8qs-2q^2$ vel $r^4-8qr^2s+16q^2s^2-64s^3$ negativa quantitas & quatuor datæ æquationis radices erunt impoſſibiles.

5.

Reg. 5. Datis quatuor ab initio æquationis n dimenſionum $(x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0)$ terminis $(x^n, px^{n-1}, qx^{n-2}, rx^{n-3})$ invenire, utrum ex iis colligi poteſt æquationem impoſſibiles habere radices.

Sit cubica æquatio $(x^3 - Px^2 + Qx - R = 0)$ & duæ ejus radices erunt impoſſibiles, ſi

$4P^3R - P^2Q^2 - 18PQR + 27R^2 + 4Q^3$ ſit negativa quantitas.

In hac quantitate pro P, Q, R, ſcribe

$\frac{np}{3}$, $\frac{n}{2} \times \frac{n-1}{3} q$, $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} r$, ſi quantitas reſultans ſit negativa, exinde colligi poteſt æquationem impoſſibiles habere radices.

Datâ multarum dimenſionum æquatione & per præcedentes regulas invenire, quot ſunt radices impoſſibiles, nonnunquam requiritur calculus maxime laborioſus; ſunt quidem alii modi, quibus hoc problema ſolvi poteſt & contractiores, at ſæpenumero ne unam radicem quidem impoſſibilem inveniant, quamvis plurimæ ſint, & in altis æquationibus perraro omnes impoſſibiles radices detegant.

Reg. 1. Sit æquatio $x^n - px^{n-1} + qx^{n-2}$ &c. $= 0$, transformetur hæc æquatio in alteram, cujus radices ſint quadrata hujuſce æquationis radicum & tot erunt radices impoſſibiles, quot æquationis reſultantis ſignorum in continuâ ſerie ſunt progreſſiones de + in + vel — in —.

Reg. 2. Datâ æquatione $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0$. Sub primo & ultimo hujuſce æquationis terminis colloca ſignum +, ſub quolibet intermedio termino (x^{n-m}) ſi quadratum ejus ductum in fractionem $\frac{\overline{n-m} \times m}{\overline{n-m+1} \times \overline{m+1}}$ majus eſt quam rectangulum terminorum utrinque conſiſtentium colloca ſignum +, ſin minus, ſignum —; & tot erunt radices impoſſibiles, quot ſunt in ſubſcriptorum ſignorum ſerie mutationes de + in — & — in +.

Reg. 3. Datâ æquatione $x^n + qx^{n-2} - rx^{n-3} + \&c. = 0$, in quâ deeſt ſecundus terminus.

Sub primo & ultimo hujuſce æquationis termino colloca ſignum +, ſub quolibet reliquorum terminorum (x^{n-m}) ſi quadratum ejus ductum

ductum in fractionem $\frac{\overline{n-m}\times\overline{m+1}^2\times\overline{m-1}}{\overline{n-m+1}\times\overline{m+2}\times m^2}$ majus eſt quam rectangulum terminorum utrinque conſiſtentium, colloca ſignum + ſin minus ſignum —, & fere tot erunt radices poſſibiles, quot ſunt in ſubſcriptorum ſignorum ſerie mutationes de + in — & — in +.

Modus demonſtrandi maxime accommodatus huic generi, in quo quæritur, quot radices impoſſibiles continentur in data æquatione; eſt multiplicare datam æquationem 1mo. Per $x-a=0$; deinde per $x^2-2ax+a^2+b^2=0$, cum enim prior æquatio unicam radicem habeat, eamque poſſibilem; poſterior autem duas impoſſibiles; manifeſtum eſt eundem futurum eſſe numerum radicum impoſſibilium in æquatione, quæ oritur ex priori multiplicatione ac in data æquatione: æquatio autem, quæ oritur ex ſecunda multiplicatione duas habebit radices impoſſibiles, quæ in datâ æquatione non continentur.

EXEMPLUM.

Sit data æquatio $x^2-2x+2=0$; quæ ſecundum *Newtoni* regulam duas habet radices impoſſibiles, multiplicatur per $x+2=0$ & fit $x^3-2x+4=0$, quæ ſecundum eandem regulam nullas habet radices impoſſibiles, debet autem habere duas, regula, igitur non ſemper vera eſt.

Modus autem demonſtrationis de quo loquor, non huic quæſtioni ſolummodo proprius eſt; ſed commode adhiberi poteſt, ubicunque quæritur, quot quantitates dati cujuſvis generis in datâ æquatione contineantur.

Exemplum ſit notiſſima *Carteſii* regula "tot eſſe radices affirmati-
"vas, quot ſignorum in continua ſerie mutationes de + in — & —
"in +; ſi modo nullæ æquationis radices ſunt impoſſibiles."

Si enim data quævis æquatio, multiplicetur per $x-a=0$: mutationum ſignorum numerus augebitur per unitatem, ſi autem eadem æquatio multiplicetur per $x+a=0$, eædem erunt ſignorum mutationes in novâ, ac in datâ æquatione.

Sit enim data æquatio

$$x^n - Ax^{n-1} - Bx^{n-2} + Cx^{n-3} + Dx^{n-4} + Ex^{n-5} - Fx^{n-6} - \&c. = 0$$

quæ ducta in $x - a$

dat

$$\begin{array}{llllllll} x^{n+1} - Ax^n & - Bx^{n-1} & + Cx^{n-2} & + Dx^{n-3} & + Ex^{n-4} & - Fx^{n-5} & - \&c. = 0 \\ \quad\; - a & + Aa & + aB & - aC & - aD & - aE & + \&c. = 0 \end{array}$$

Jam vero quod ad ſigna hujus novæ æquationis attinet, manifeſtum eſt,

1. Terminum quemcunque novæ æquationis idem habere ſignum cum termino datæ æquationis, cui ſubſcribitur, ſi modo coefficiens iſtius termini major ſit quam coefficiens termini antecedentis multiplicata per (a.)

Ex. gr. Signum quinti termini in data æquatione eſt affirmativum ſcil ($+D$,) ſignum autem termini quinti in novâ æquatione eſt ($D - aC$) quod etiam affirmativum eſt, ſi modo D major eſt quam aC.

2. Si vero coefficiens alicujus termini in datâ æquatione minor ſit quam coefficiens termini antecedentis multiplicata in (a); Tum terminus novæ æquationis, qui termino propoſito ſubſcribitur diverſum habet ſignum a termino antecedente in datâ æquatione.

Sit, ex. gr. coefficiens termini quinti in datâ æquatione minor quam (aC), coefficiens termini antecedentis multiplicata per (a), & ($D - aC$) coefficiens termini quinti in novâ æquatione eſt negativa, coefficiens autem termini antecedentis in data æquatione eſt affirmativa $+C$.

Ex. gr. Sit igitur B major quam Aa, C quam aB: D autem minor quam aC. Jam vero quia D minor eſt quam aC, erit etiam E minor quam aD, & F quam aE, &c. uſque ad finem æquationis, cum enim omnes radices datæ æquationis ſint poſſibiles, notum eſt

	EC	minorem eſſe quam	D^2
Eſt etiam	D	minor quam	aC
Ergo	E	minor eſt quam	aD.

Et duæ æquationes hoc modo ſcribi poſſunt.

$$x^n - Ax^{n-1} - Bx^{n-2} + Cx^{n-3} + Dx^{n-4} + Ex^{n-5} - Fx^{n-6} \&c. = 0.$$

$$x^{n+1} - \overline{A+a}\,x^n - \overline{B-aA}\,x^{n-1} + \overline{C+aB}\,x^{n-2} - \overline{aC-D}\,x^{n-3} - \overline{aD-E}\,x^{n-4} - \overline{aE+F}\,x^{n-5} \&c. = 0.$$

Patet

Patet igitur a termino primo ad terminum quartum in utrâque æquatione ſigna eſſe eadem.

A termino autem quarto in datâ æquatione, & a termino quinto in novâ, ſigna eſſe ſemper diverſa.

In utroque caſu igitur eædem ſunt mutationes ſignorum; eſt autem in novâ æquatione a termino quarto ad quintum unica mutatio ſignorum, quæ non eſt in data æquatione.

Eodem modo pateret numerum mutationum ſignorum nec augeri nec minui multiplicatione per $(x+a.)$

Unde ſequitur tot eſſe radices affirmativas, quot ſunt mutationes, ſignorum de + in — & — in +, cæteras vero negativas eſſe.

CAP. III.

PROB. II.

INVENIRE illas datæ æquationis

$$x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \&c. = 0.$$

Radices $(\alpha, \beta, \gamma, \delta, \&c.)$ *quæ datam inter ſe habent relationem.*

Scribatur $x-\alpha=0$, $x-\beta=0$, $x-\gamma=0$, $x-\delta=0$, &c. & per æquationem ex horum continuâ multiplicatione $\overline{x-\alpha}\times\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\delta}$, &c. $=0$. generatam dividatur data æquatio.

Singulus reſidui terminus nihilo æqualis fiat; & æquationum reſultantium quicunque communis diviſor erit radix quæſita.

EXEMPLUM I.

Sint $(\alpha$ & $\beta)$ duæ radices datæ æquationis inter ſe æquales: & ſit α vel $\beta=a$; ſcribatur $x-a=0$ & $x-a=0$, & rectangulum $\overline{x-a}\times\overline{x-a}=0$ fit $x^2-2ax+a^2=0$; per hoc rectangulum data dividatur æquatio; & operatio eſt hujuſmodi,

$$x^2-2ax+a^2)\,x^n-px^{n-1}+qx^{n-2}-rx^{n-3}+\&c.=0\,(x^{n-2}+\overline{2a-p}\,x^{n-3}\ \&c.$$

$$\underline{x^n-2ax^{n-1}+a^2x^{n-2}}$$

$$\overline{2a-p}\,x^{n-1}+\overline{q-a^2}\,x^{n-2}$$

$$\underline{\overline{2a-p}\,x^{n-1}-\overline{4a^2-2ap}\,x^{n-2}+\overline{2a^3-a^2p}\,x^{n-3}}$$

$$\overline{3a^2-2ap+q}\,x^{n-2}-\overline{2a^3-a^2p+r}\,x^{n-3},\ \&c.$$

Patet, prioris reſidui termini coefficientem eſſe

$$na^{n-1}-\overline{n-1}\,pa^{n-2}+\overline{n-2}\,qa^{n-3}-\overline{n-3}\,ra^{n-4}+\&c.$$

poſteriorem vero reſidui terminum eſſe

$$\overline{n-1}\,a^n-\overline{n-2}\,pa^{n-1}+\overline{n-3}\,qa^{n-2}-\overline{n-4}\,ra^{n-3}\ \&c.$$

Fiant hæ duæ quantitates nihilo reſpective æquales, & duarum æquationum reſultantium quicunque communis diviſor erit radix quæſita.

$-p x^{n-m-1} + \&c.$

&c.

$1 \times \frac{m}{2} \times \frac{m+1}{3} \ldots \frac{n-2}{n-m} q a^{n-m} + \&c.$

$^{n-m+2} + \overline{n-m-1} \times \frac{n-m}{2} \cdot \frac{\overline{m-2}}{1} \ldots \frac{n-2}{n-m+1} q a^{n-m+2}$

$\frac{n-1}{n-m+3} p a^{n-m+3} + \overline{n-m+1} \times \frac{\overline{n-m}}{2} \times \frac{n-m+1}{3} \times m-2 \ldots \frac{n-2}{n-m+2} q a^{n-m+1}$ &c.

EXEMPLUM III.

Sint radices ($\alpha, \beta, \gamma, \delta$, &c.) reſpective arithmeticæ progreſſionis (m) termini ($a, a+b, a+2b, a+3b, \ldots a+\overline{m-1}b$.)

Dicantur ſumma numerorum ($1, 2, 3, 4, 5 \ldots m-1$,) & aggregata contentorum ſub ſingulis binis, ternis, quaternis, &c. reſpective P, Q, R, S, T, & multiplicentur radices

$$\overline{x-a} \times \overline{x-a-b} \times \overline{x-a-2b} \ldots \overline{x-a-\overline{m-1}\,b}$$

continuo in ſeſe, & contentum erit

$$\begin{array}{llll} x^m - max^{m-1} & + m \times \frac{m-1}{2} a^2 x^{m-2} & - m \times \frac{m-1}{2} \times \frac{m-2}{3} a^3 x^{m-3} \\ \quad - Pb & + \overline{m-1}\,Pba & - \overline{m-1} \times \frac{m-2}{2} Pba^2 \\ & + \quad Qb^2 & - \quad \overline{m-2}\,Qb^2 a \\ & & - \quad Rb^3 \end{array}$$

$$\begin{array}{ll} + & m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} a^4 x^{m-4} \text{ \&c.} = 0, \\ + & \overline{m-1} \times \frac{m-2}{2} \times \frac{m-3}{3} Pba^3 \\ + & \overline{m-2} \times \frac{m-3}{2} Qb^2 a^2 \\ + & \overline{m-3} \times Rb^3 a \\ & Sb^4 \end{array}$$

Per hoc contentum data dividatur æquatio, & reſidui terminorum quicunque communis diviſor erit radix quæſita.

EXEMPLUM IV.

Sint radices ($\alpha, \beta, \gamma, \delta$, &c.) geometricæ progreſſionis ($a, ra, r^2 a \ldots r^{m-1} a$) ($m$) termini.

Multiplicentur factores $\overline{x-a} \times \overline{x-ar} \times \overline{x-ar^2} \ldots \overline{x-ar^{m-1}}$ continuo in ſeſe, & contentum erit

$$x^m - \frac{a - ar^m}{1-r} x^{m-1} + \frac{a^2 \times \overline{r - r^m - r^{m+1} + r^{2m}}}{\overline{1-r} \times \overline{1-r^2}} x^{m-2} - \text{\&c.} = 0.$$

Coefficiens termini (x^{m-s}) hujus æquationis erit æqualis fractioni, cujus numerator eſt

$$a^s r^{s \times \frac{s-1}{2}} \times \overline{1-r^{m-s+1}} \times \overline{1-r^{m-s+2}} \times \overline{1-r^{m-s+3}} \ldots \times \overline{1-r^m}$$

Denominator vero $\overline{1-r} \times \overline{1-r^2} \times \overline{1-r^3} \ldots \overline{1-r^s}$.

Per hoc contentum data dividatur æquatio, & terminorum reſidui communis diviſor erit radix quæſita.

Eſt & alia methodus & ſæpe multo contractior inveniendi radices datæ æquationis datam inter ſe habentes relationem.

1. Sint $\alpha, \beta, \gamma, \delta$, &c. radices datæ æquationes, datam inter ſe habentes relationem, ſcribantur pro incognitâ datæ æquationis quantitate (x) reſpective $\alpha, \beta, \gamma, \delta$, &c. & æquationum reſultantium communis diviſor erit una radicum, quæ datam habent relationem ad ſe invicem.

EXEMPLUM I.

Sint $\alpha, \beta, \gamma, \delta$, &c. reſpective, $a, a+b, a+2b$, arithmeticæ progreſſionis termini & ſcribantur pro incognitâ quantitate (x) reſpective $a, a+b, a+2b$, &c. & reſultant æquationes.

$$a^n - pa^{n-1} + qa^{n-2} - ra^{n-3} + \text{\&c.} = 0.$$

$$\overline{a+b}^n - p \times \overline{a+b}^{n-1} + q\overline{a+b}^{n-2} - r\overline{a+b}^{n-3} + \text{\&c.} = 0.$$

$$\overline{a+2b}^n - p\overline{a+2b}^{n-1} + q\overline{a+2b}^{n-2} - r\overline{a+2b}^{n-3} + \text{\&c.} = 0.$$

Harum æquationum communis diviſor, erit radix quæſita.

Hinc patet tertium & quartum præcedentis methodi exemplum ex hâc quam priori methodo multo contractiorem habere ſolutionem.

COROLLARIUM.

Omnis impoſſibilis radix datæ æquationis hujuſce formulæ $a+\sqrt{-b^2}$, alteram habet ſibi ipſi correſpondentem $a-\sqrt{-b^2}$, quibus compoſitis reſultat quadraticus diviſor $x^2 - 2ax + a^2 + b = 0$.

Subſtituatur enim in data æquatione pro incognita quantitate (x) & $a+\sqrt{-b^2}$ & $a-\sqrt{-b^2}$; & conſtabit ſi modo $a+\sqrt{-b^2}$ ſit datæ æquationis radix, $a-\sqrt{-b^2}$ etiam eſſe datæ æquationis radicem.

2. Si data relatio inter m radices $(\alpha, \beta, \gamma, \delta$, &c.) datæ æquationis,

$$(x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \text{\&c.} = 0)$$

ſolummodo exprimi poteſt per æquationem hujuſmodi,

$$x^m - Px^{m-1} + Qx^{m-2} - Rx^{m-3}, \text{\&c.} = 0,$$

tum

tum aſſumatur data æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = w$, & problematis primi ope, inveniatur æquatio, cujus radix eſt w, quæ ſit $w^m - aw^{m-1} + bw^{m-2} - cw^{m-3} + \&c. = 0$.

Fiant coefficientes a, b, c, &c. nihilo reſpective æquales & æquationum reſultantium communis diviſor eſt radix quæſita.

Eſt etiam alia methodus inveniendi radices datæ æquationis datam inter ſe habentes relationem.

Sint (m) radices $\alpha, \beta, \gamma, \delta$, &c. datam inter ſe habentes relationem, ducantur continuo in ſeſe factores $\overline{x-\alpha} \times \overline{x-\beta} \times \overline{x-\gamma} \times \overline{x-\delta}$, &c. & contentum multiplicetur in quantitatem hujus generis

$$x^{n-m} - Ax^{n-m-1} + Bx^{n-m-2} - \&c. = 0$$

& reſultantis producti coefficientes fiant æquales datæ æquationis coefficientibus correſpondentibus, æquationum reſultantium communis diviſor erit radix quæſita.

EXEMPLUM

Sint (m) radices $\alpha, \beta, \gamma, \delta$, &c. inter ſe æquales & ſit

$$a = \alpha = \beta = \gamma = \delta, \&c.$$

tum contentum ex continuâ multiplicatione factorum

$$\overline{x-\alpha} \times \overline{x-\beta} \times \overline{x-\gamma}, \&c.$$

erit $\overline{x-a}^m = x^m - max^{m-1} + m.\frac{m-1}{2}a^2x^{m-2} - \&c. = 0$, & eorum contentum multiplicetur in aſſumptam quantitatem

$x^{n-m} - Ax^{n-m-1} + Bx^{n-m-2} - \&c. = 0$, & productum erit

$$x^n - \overline{A+ma}\,x^{n-1} + \overline{B+m.\frac{m-1}{2}a^2}\,x^{n-2} - \overline{C+m.\frac{m-1}{2}.\frac{m-2}{3}a^3}\,x^{n-3}\,\&c. = 0.$$

Fiant coefficientes hujuſce æquationis æquales datæ æquationis correſpondentibus coefficientibus.

$$A+ma=p,\ B+m.\frac{m-1}{2}a^2=q,\ C+m.\frac{m-1}{2}.\frac{m-2}{3}a^3=r$$

& harum æquationum communis diviſor erit radix quæſita.

PROB. III.

Datâ æquatione

$$x^n - px^{n-1} + qx^{n-2} - rx^{n-3} - \&c. = 0,$$

invenire ejus rationales diviſores.

1. Sint

1. Sint p, q, r, s, &c. integri numeri, & pro incognitâ quantitate (x) ſubſtitue tres vel plures hujus arithmeticæ progreſſionis terminos $-1, 0, 1, 2$, &c. & numerorum reſultantium ſint $z+1$, z, $z-1$, &c. reſpective diviſores, quorum ſit (z) diviſor quantitatis reſultantis e ſubſtitutione termini (o) & $x-z$ erit quæſitus diviſor.

2. Eodem modo transformetur data æquatio in aliam, cujus radices ſunt quæcunque rationalis functio datæ æquationis radicum, ultimi & datæ & reſultantis æquationis termini ſint α & π diviſores reſpective, & (π) erit ea rationalis functio diviſoris (α), ſimodo α ſit rationalis radix quæſita.

Eſt alia methodus rationales datæ æquationis

$$S+Rx+Qx^2+Px^3+Ox^4 \ldots x^n=0$$

in quâ S, R, Q, P, &c. integri ſunt numeri, radices inveniendi.

1. Sit coefficiens termini (x^n) in quo dimenſiones incognitæ quantitatis (x) inveniuntur maximæ, unitas, & rationalis radix quæſita (a): tum $\frac{S}{a}$, $\frac{Ra+S}{a^2}$, $\frac{Qa^2+Ra+S}{a^3}$, $\frac{Pa^3+Qa^2+Ra+S}{a^4}$, $\frac{Oa^4+Pa^3+Qa^2+Ra+S}{a^5}$, &c. integri ſunt numeri.

EXEMPLUM.

Sit data æquatio $-35-44x+7x^2-7x^3+x^4=0$.

Pro S, R, Q, P, O, ſcribantur reſpective $-35, -44, 7, -7, 1$.

Coefficientis ($S, -35$) diviſores ſunt $1, 5, 7, -1, -5, -7$, pro rationali radice quæſita (a) aſſumatur diviſor (7), & $\frac{S}{a}=-\frac{35}{7}=-5$, $\frac{Ra+S}{a^2}=-\frac{5-44}{7}=-7$, $\frac{Qa^2+Ra+S}{a^3}=\frac{7-7}{7}=0$. $\frac{Pa^3+Qa^2+Ra+S}{a^4}=\frac{-7+0}{7}=-1$; unde ſequitur numerum (7) eſſe rationalem datæ æquationis radicem.

2. Sit coefficiens termini (x^n) quicunque integer numerus, cujus ſit (m) diviſor & $\left(\frac{a}{m}\right)$ quæſita radix; tum $\frac{S}{a}$, $\frac{Ra+mS}{a^2}$, $\frac{Qa^2+mRa+m^2S}{a^3}$, $\frac{Pa^3+mQa^2+m^2Ra+m^3S}{a^4}$ & ſic deinceps, integri erunt numeri.

Si

Si nullus hac methodo occurrit diviſor, concludendum eſt æquationem non admittere ſimplicem diviſorem, poteſt tamen fortaſſe habere quadraticum diviſorem, qui ſit $mx^2 - bx + a$, in quo (m) eſt diviſor coefficientis datæ æquationis termini (x^n); & (a) diviſor ultimi datæ æquationis termini (S); & $m - b + a$ diviſor quantitatis $S + R + Q + P + C$, &c.

tum $\dfrac{Ra + bS}{a^2}$, $\dfrac{Qa^2 - mSa + bRa + b^2 S}{a^3}$

$\dfrac{Pa^3 + bQa^2 - mRa^2 - 2mbSa + b^2 Ra + b^3 Sa}{a^4}$ &c. integri erunt numeri, & ſic deinceps.

PROB. IV.

Invenire datæ æquationis duas incognitas quantitates (x & y) *habentis, rationales radices ſibi ipſis correſpondentes.*

1. Sit data æquatio $\alpha x - \beta y = P$ in quâ incognitæ quantitates (x & y) unam ſolummodo habent dimenſionem, invenire earum correſpondentes radices, quæ ſunt integri numeri.

Sit (α) minor coefficiens, & dividantur duo reliqui termini ($-\beta y$ & P) per minorem coefficientem (α), & ſit reſiduum $\Gamma y = Q$: ſupponatur $\alpha v - \Gamma y = Q$ & in hâc novâ æquatione reſultante ſit operatio eadem ac in datâ; & ſic deinceps, uſque dum reſiduum nihilo fiat æquale.

Deinde pro ultimæ æquationis quantitatibus aſſumantur quicunque integri numeri & quod prius ſubductum fuit, addatur, &c. & inveniantur incognitarum quantitatum (x & y) correſpondentes integri valores.

EXEMPLUM.

Sit $7x = 16y + 20$ dividatur ($16y + 20$) per (7) & reſiduum eſt $2y + 6 = 7v$: dividatur hæc æquatio per (2) minorem coefficientem & reſiduum eſt $v = 2w$. Pro (w) aſſumatur quicunque integer numerus & $2w = v$ & $2y = 7v - 6 = 14w - 6$

unde $y = 7w - 3$ &

$$x = \frac{16y + 20}{7} = \frac{112w - 28}{7} = 16w - 4.$$

Et ſic de æquatione plures incognitas quantitates habente.

2. Sit data æquatio duas incognitas quantitates (x & y) continens, quæ habent duas vel plures dimenſiones.

Invenire rationales incognitarum quantitatum (x & y) radices.

1mo. Sit data æquatio (n) dimenſionum, pro incognitis quantitatibus x & y in data æquatione ſubſtituantur reſpective

$$(x=)Az^n+Bz^{n-1}+Cz^{n-2}+ \&c. \ \&$$
$$(y=)az^n+bz^{n-1}+cz^{n-2}+ \&c.$$

vel

$$(x=)\frac{Az^n+Bz^{n-1}+Cz^{n-2}+ \&c.}{Pz^n+Qz^{n-1}+Rz^{n-2}+ \&c.}$$
$$(y=)\frac{az^n+bz^{n-1}+cz^{n-2}+ \&c.}{Pz^n+Qz^{n-1}+Rz^{n-2}+ \&c.}$$

Æquationis reſultantis ſupponantur ſinguli termini nihilo æquales, & exinde inveniri poſſunt rationales coefficientium ($A,B,C,$ &c. $a,b,c,$ &c.) valores.

EXEMPLUM.

Sit æquatio $4y^2-4xy+x^2+5x-11y+8=0$. Scribantur pro incognitis quantitatibus (x & y) reſpective

$$(y=)\ z^2+Bz+C \ \&$$
$$(x=)\ az^2+bz+c$$ & reſultat æquatio

$$\overline{4-4a+a^2}\times z^4+\overline{8B-4aB-4b+2ab}\,x^3+\overline{8C-4c-4aC+2ac+5a+4B^2-4Bb+b^2-11}\,z^2+\overline{8BC-4bC-4Bc+2cb+5b-11B}\,z+4C^2-4Cc+c^2+5c-11C+8=0.$$

Fiant hujus æquationis reſultantis coefficientes nihilo reſpective æquales: & invenientur coefficientium (a,b,c,B,C) valores reſpective (2,1,1,1,2): ſubſtituantur hi numeri pro ſuis valoribus, & fit

$$y=z^2+z+2 \ \&$$
$$x=2z^2+z+1.$$

COROLLARIUM.

Sint x & y incognitæ datæ æquationis quantitates &

$$x=Az^n+Bz^{n-1}+Cz^{n-2}+Dz^{n-3}+$$
$$y=az^n+bz^{n-1}+cz^{n-2}+dz^{n-3}+ \&c.$$

Aſſumatur æquatio

$$Ay^m+\overline{a+bx}\,y^{m-1}+\overline{c+dx+ex^2}\,y^{m-2}+ \&c.=0.$$

ita

ita ut ſit productum $\left(m \times \overline{\frac{m+3}{2}}\right)$ æquale vel majus quam $(nm+1)$: ſubſtituantur pro incognitis quantitatibus (x & y) earum reſpectivi valores, æquationis reſultantis omnium terminorum ſingulæ coeffici-entes nihilo æquales fiant, & ſi $\left(m \times \overline{\frac{m+3}{2}}\right)$ majus eſt quam $(nm+1,)$ tum infinitæ, ſin æquales, tum una ſolummodo deduci poteſt æquatio, in quâ incognitæ quantitates (x & y) ſunt reſpective

$$Az^n + Bz^{n-1} + Cz^{n-2} + \&c.$$
$$\& \; az^n + bz^{n-1} + cz^{n-2} + \&c.$$

Eodem modo, ſæpe dato uno, duobus vel pluribus correſpondentibus valoribus incognitarum quantitatum (x & y) inveniri poſſunt alii innumeri.

Si nullæ rationales radices hâc methodo occurrunt, concludendum eſt datam æquationem haud admittere generalem regulam, quæ inveniat omnes incognitarum quantitatum rationales radices: ſed fortaſſe habent incognitæ quantitates innumeras rationales radices ſibi ipſis correſpondentes, & juxta *Walliſii* approximationis methodum ſemper inveniri poſſunt.

Sæpe vero in hujuſmodi problematibus ſolvendis tentando vel conjecturâ ſolummodo procedere poſſimus; vix enim ullam aliam memethodum præbet natura.

Sunt etiam hujuſmodi theoremata, quorum demonſtrationes invenire nunquam valebit humani ingenii vis; ſi vero demonſtrationem admittunt quævis propoſitiones, ea ex præcedentibus principiis petenda eſt.

CAP. IV.

PROB. V.

INVENIRE æquationis dimenſiones, cujus ope altera ($x^n - px^{n-1} + qx^{n-2}$, &c. $= o$) n *dimenſionum in alteram* ($x^m - ax^{m-1} + bx^{m-2}$, &c. $= o$) m *dimenſionum reduci poteſt, poſitâ* n *majore quam* m, *& radicibus reductæ æquationis* $x^m - ax^{m-1} + bx^{m-2}$ &c. $= o$, *etiam eſſe radices æquationis* $x^n - px^{n-1} + qx^{n-2}$, &c. $= o$, *ex quâ reducta fuit.*

Dimenſiones requiſitæ erunt æquales fractioni

$$n . \frac{n-1}{2} . \frac{n-2}{3} \ldots \frac{n-m+1}{m} .$$

DEMONSTRATIO.

Æquatio $x^n - px^{n-1} + qx^{n-2}$, &c. $= o$, habet (n) radices, & ex quâvis combinatione (m) radicum hujus æquationis formari poteſt æquatio m dimenſionum, quæ reſpondebit problemati; ergo quot ſunt combinationes m radicum, in majore multitudine (n) radicum, tot erunt problematis ſolutiones & conſequenter tot erunt radices æquationis reducentis.

Numerus autem combinationum, quibus quantitatum vel radicum multitudo (m) in majori multitudine (n) eſſe poſſit; æqualis erit fractioni $n . \frac{n-1}{2} . \frac{n-2}{3} \ldots \frac{n-m+1}{m}$, † ergo numerus radicum & conſequenter dimenſiones æquationis reducentis æqualis erit fractioni $n . \frac{n-1}{2} . \frac{n-2}{3} \ldots \frac{n-m+1}{m}$. Q. E. D.

COR. I.

Biquadratica ($x^4 * + qx^2 - rx + s = 0$) cujus ſecundus terminus deeſt, reduci poteſt in quadraticam ($x^2 - Px + Q = 0$,) æquationis ($v^6 - av^4 + bv^2 - c = 0$) ſex dimenſionum ope, cujus formula eſt cubica.

COR. II.

Hac vero methodo æquatio reducens plures habet dimenſiones quam data æquatio; fruſtra eſſet igitur, per talem reductionem æquationum reſolutionem generalem quærere. *

† *De Moiv.* Miſc. et Anal. p. 180. * *Newt.* Arith. Univ. p. 237.

PROB.

PROB. VI.

Nonnullas æquationes reducere, eas in duas æquales partes ita dividendo, ut ex utrâque parte eadem radix extrabatur.

EXEMPLUM I.

Biquadraticam æquationem $x^4+2px^3=qx^2+rx+s$ in quadraticam reducere.

Addatur ad utramque æquationis partem

$$\overline{p^2+2n}\times x^2+2pnx+n^2 \text{ \& prodit}$$

$$x^4+2px^3+\overline{p^2+2n}\times x^2+2pnx+n^2=\overline{p^2+2n+q}\times x^2+\overline{2pn+r}\times x+\overline{s+n^2}.$$

Supponatur $4\times\overline{s+n^2}\times\overline{p^2+2n+q}=\overline{2pn+r}^2$ reducatur hæc æquatio & tranſponantur termini, & fit cubica

$$8n^3+4qn^2+\overline{8s-4rp}\times n+4qs+4p^2s-r^2=0.$$

Hinc inveniatur incognita quantitas (n), quâ pro ſuo valore (n) in æquatione

$$x^4+2px^3+\overline{p^2+2n}\times x^2+2pnx+n^2=\overline{p^2+2n+q}\times x^2+\overline{2pn+r}\times x+s+n^2$$

ſubſtitutâ, & extractâ utrobique radice quadraticâ, reſultat æquatio quadratica

$$x^2+px+n=\sqrt{p^2+2n+q}\times x+\sqrt{s+n^2}.$$

Ex. Gr. Sit biquadratica æquatio $x^4-25x^2+60x-36=0$, pro p, q, r, s, ſcribantur $0, 25, -60, 36$ reſpective in reducenti æquatione

$$8n^3+4qn^2+\overline{8s-4rp}\times n+4qs+4p^2s-r^2=0 \text{ \& fit}$$

$8n^3+100n^2+288n=0$, cujus radices ſunt reſpective $0, -8, -\frac{9}{2}$, quibus pro ſuo valore (n) in æquatione

$$x^2+2px+n=\sqrt{q+p^2+2n}\times x+\sqrt{s+n^2}$$

ſubſtitutis, reſultant tres æquationes

$$x^2 \quad =\sqrt{25}\,x+\sqrt{36}$$

$$x^2-8=\sqrt{9}\,x+\sqrt{100}$$

$$x^2-\tfrac{9}{2}=\sqrt{16}\,x+\sqrt{\tfrac{225}{4}}$$

Ex ſingulis his tribus æquationibus, radices vel ſunt $1, 2, 3, -6$ (qui numeri ſunt radices biquadraticæ æquationis $x^4-25x^2+60x-36=0$) vel earum negativæ $-1, -2, -3, +6$.

EXEMPLUM II.

Æquationem quadrato-cubicam ($x^5+qx^3+rx^2+sx+t=0$) in cubicam reducere.

Multiplicetur data æquatio per (x) & reſultat

$$x^6+qx^4+rx^3+sx^2+tx=0,$$

addatur ad utramque æquationis partem $\overline{kx^3+dx+b}^2$, & prodit

$$\overline{kx^3+dx+b}^2=\overline{k^2+1}x^6+\overline{2kd+q}x^4+\overline{2kb+r}x^3+\overline{d^2+s}x^2+\overline{2db+t}x+b^2.$$

Supponantur $2\sqrt{k^2+1}\times b=2kb+r$, $2\sqrt{k^2+1}\times\sqrt{d^2+s}=2kd+q$, $2\sqrt{d^2+s}\times b=2db+t$ quibus reductis reſultant tres æquationes

$$4b^2=4rkb+r^2$$

$$4sk^2+4d^2+4s=4qkd+q^2$$

$$4sb^2=4tdb+t^2$$ e primâ æquatione ſequitur $k=\frac{4b^2-r^2}{4rb}$, e tertiâ vero $d=\frac{4sb^2-t^2}{4tb}$, quibus quantitatibus pro ſuis valoribus in ſecundâ æquatione ſubſtitutis, reſultat æquatio biquadratica, formulâ vero quadraticæ

$$4\times\frac{\overline{4sb^2-t^2}^2}{\overline{4tb}^2}+4s\times\frac{\overline{4b^2-r^2}^2}{\overline{4rb}^2}+4s=4q\times\frac{4b^2-r^2}{4rb}\times\frac{4sb^2-t^2}{4tb}+q^2$$

Reducatur hæc æquatio & fit

$$16sb^4+4qrtb^2+r^2t^2=0.$$

Ex hâc æquatione inveniri poteſt incognita quantitas (b) & conſequenter $k=\frac{4b^2-r^2}{4rb}$ & $d=\frac{4sb^2-t^2}{4tb}$, quibus quantitatibus pro ſuis valoribus ſubſtitutis, reſultat cubica æquatio

$$Ax^3+Bx+C=0.$$

Ex. Gr. Sit data quadrato-cubica æquatio,

$$x^5+35x^3+30x^2+304x-480=0.$$

Pro (q,r,s,t) ſcribantur in reducenti æquatione reſpective 35, 30, 304, —480, & fit

$$16 \times 304 b^4 - 4 \times 35 \times 30 \times 480 b^2 + 30^2 \times 480^2 = 0:$$

dividatur hæc æquatio per (16×16) & reſultat

$$19 b^4 - 35 \times 15^2 b^2 + 30^4 = 0:$$

hujuſce æquationis radices erunt $b = \pm \frac{\sqrt{\pm 675 + 7875}}{\sqrt{38}}$; ſubſtituatur hæc quantitas pro ejus valore in æquationibus

$$k = \frac{4b^2 - r^2}{4rb} \ \& \ d = \frac{4sb^2 - t^2}{4tb},$$

& reſultant quantitates

$$k = \frac{\pm 45 - 45}{2\sqrt{38}\sqrt{\pm 675 + 7875}}$$

$$d = \frac{\pm 855 - 855}{2\sqrt{38}\sqrt{\pm 675 + 7875}}$$

ſubſtituantur hæ quantitates pro ſuis valoribus in cubicâ æquatione,

$$kx^3 + dx + b = \pm\sqrt{k^2+1}\, x^3 \pm \sqrt{d^2+s}\, x \pm q$$

& reſultant ex ſingulâ ſubſtitutione æquationes, quarum ſex radices erunt $0, 2, 3, 4, -4, -5$ datæ æquationis radices.

EXEMPLUM III.

Æquationem cubo-cubicam $(x^6 * + px^4 + qx^3 + rx^2 + sx + t = 0)$ in cubicam reducere.

Addatur ad utramque æquationis partem $\overline{kx^3 + dx + b}^2$ & prodit

$$\overline{kx^3 + dx + b}^2 = \overline{k^2+1} \times x^6 + \overline{2kd+q} \times x^4 + \overline{2kb+r} \times x^3 + \overline{d^2+s} \times x^2 + \overline{2dq+t} \times x + b^2 + v.$$

Supponantur $2\sqrt{k^2+1} \times \sqrt{r+d^2} = p + 2kd$,

$2\sqrt{k^2+1} \times \sqrt{t+b^2} = 2kb + q$ & $2\sqrt{r+d^2} \times \sqrt{t+b^2} = s + 2db$

quibus reductis reſultant tres æquationes

$$4tk^2 + 4b^2 + 4t = 4qkb + q^2$$

$$4rk^2 + 4d^2 + 4r = 4pkd + p^2$$

$$4tr + 4td^2 + 4rb^2 = 4sdb + s^2$$

e primâ æquatione ſequitur $k = \sqrt{\frac{q^2 - 4t}{4t} - \frac{4t - q^2}{4t^2} b^2} + \frac{qb}{2t}$

e tertiâ

e tertiâ $d=\sqrt{\frac{s^2-4tr}{4t}-\frac{\overline{4rt-s^2}}{4t^2}\times b^2}+\frac{sb}{2t}$ quibus quantitatibus in ſecundâ æquatione pro ſuis valoribus ſubſtitutis, prodit

$$4r\times\overline{\sqrt{\frac{q^2-4t}{4t}-\frac{\overline{4t-q^2}}{4t^2}\times b^2}+\frac{qb}{2t}}^{\,2}+4\times\overline{\sqrt{\frac{s^2-4tr}{4t}-\frac{\overline{4rt-s^2}}{4t^2}\times b^2}+\frac{sb}{2t}}^{\,2}$$

$$+4r=4p\times\sqrt{\frac{q^2-4t}{4t}-\frac{4t-q^2}{4t^2}b^2}+\frac{qb}{2t}\times\sqrt{\frac{s^2-4tr}{4t}-\frac{4rt-s^2}{4t^2}b^2}+\frac{sb}{2t}$$

$+p^2$.

Reducatur hæc æquatio, ita ut exterminentur ſurdæ quantitates & reſultat æquatio, quæ octo habet dimenſiones, ſed ejus formula eſt biquadratica.

Ex hâc æquatione inveniri poteſt ejus incognita quantitas (b) & conſequenter $\sqrt{\frac{q^2-4t}{4t}-\frac{4t-q^2}{4t^2}b^2}+\frac{qb}{2t}=k$ & $\sqrt{\frac{s^2-4tr}{4t}-\frac{4rt-s^2}{4t^2}b^2}+\frac{sb}{2t}=d$ quibus quantitatibus in æquatione

$$\overline{kx^3+dx+b}^{\,2}=\overline{k^2+1}x^6+\overline{2kd+p}x^4+\overline{2kb+q}x^3+\overline{d^2+r}\times x^2+\overline{2db+s}xx+b^2+t$$

pro ſuis valoribus (b, k, d) ſubſtitutis, & æquationis reſultantis extracta utrobique radice; prodit cubica æquatio

$$kx^3+dx+b=\sqrt{k^2+1}\,x^3+\sqrt{d^2+r}\,x+\sqrt{b^2+t}.$$

Infinitæ ſunt conſimiles æquationes reducendi methodi; ſed facile patet ex hoc fonte haud generalem æquationum reductionem oriri poſſe.

PROB. VII.

Nonnullas æquationes reſolvere, eas transformando in alias, quarum radices ad radices datæ æquationis datam habent relationem.

EXEMPLUM I.

Sit æquatio $y^3+Ay-B=0$, aſſumatur æquatio $v-\frac{A}{3v}=y$, quâ quantitate pro y ſubſtitutâ, transformatur data æquatio in æquationem $v^6-Bv^3-\frac{A^3}{27}=0$, cujus ope invenitur v & conſequenter $v-\frac{A}{3v}=y$.

EX-

EXEMPLUM. II.

Sit æquatio $y^4+py^2+qy+r=0$. aſſumatur æquatio $y^2+ay+b=v$, reducantur hæ duæ æquationes in unam, ita ut exterminetur y, & fit æquatio reſultans

$$\begin{array}{llll} v^4-4b\,v^3 & +6b^2\;\;v^2 & +2pr\;\;\;\;v & +\left.\begin{array}{r}\overline{\tfrac{1}{4}a^2-b}^{\,2}+a^2.\,\overline{\tfrac{1}{4}a^2-b}\\ +p\,\overline{\tfrac{1}{2}\,a^2-b}\\ -\tfrac{1}{2}.qa\\ r\end{array}\right]^2 \\ \;\;\;\;\;\;+2p & +2r & -q^2 & \\ & +3aq & +apq & \\ & -6pb & +4a^2r & \\ & +p^2 & -4br & \\ & +a^2p & -2bp^2 & =0 \\ & & +a^3q & \\ & & -6abq & -\overline{\tfrac{1}{4}a^2-b}.\left.\begin{array}{r}\overline{2a.\tfrac{1}{2}a^2-b}\\ -pa\\ +q\end{array}\right]^2 \\ & & +6b^2p & \\ & & -2a^2bp & \\ & & -4b^3 & \end{array}$$

nihilo æqualis fiat ſecundus æquationis terminus $\left(\begin{array}{l}-4b\times v^3\\+2p\end{array}\right)$ & fit $b=\frac{p}{2}$, qua quantitate pro ſuo valore in quarto termino ſubſtituta, erit

$$\begin{array}{l}qa^3-p^2a^2-2pqa-q^2\\ \;\;\;\;\;+4r\end{array}$$ quod nihilo fiat æquale & reſultat cubica,

cujus ope invenitur (a,) in æquatione reſultanti pro a & b ſubſtituantur earum valores, & reſultat æquatio biquadratica ($v^4-Pv^2+Q=0$) formulâ vero quadraticæ, cujus ope invenitur (v) & conſequenter ex aſſumptâ æquatione $y^2+ay+b=v$ inveniri poteſt datæ æquationis radix (y).

COROLLARIUM.

Sit data æquatio

$$x^n-px^{n-1}+qx^{n-2}, \text{ \&c.}=0.$$

aſſumatur $ax^2+bx+c=v$, & transformentur hæ duæ æquationes in unam, cujus radix eſt (v); tum ita aſſumi poſſunt coefficientes a, b, c, ut quilibet duo reſultantis æquationis termini nihilo æquales fiant.

Et ſic de transformandis æquationibus, ita ut plures termini e reſultanti æquatione tollantur.

Sed haud tolli poſſunt e reſultanti æquatione generaliter plures termini, quam incognitæ ſunt in aſſumptâ quantitate coefficientes.

COROLLARIUM.

In æquationibus multarum dimenſionum, reſolvens plures habet quam data æquatio dimenſiones: fruſtra eſſet igitur per talem methodum generalem æquationum reſolutionem quærere.

Sint incognitæ in aſſumptâ quantitate coefficientes plures, quam termini tollendi.

Inveniantur termini tollendi, nihilo reſpective fiant æquales, & transformentur reſultantes æquationes in unam ita ut exterminentur quædam incognitæ coefficientes, & reſultat æquatio duas vel plures incognitas quantitates habens; inveniantur ex ſingulis his integræ vel rationales radices ſibi ipſis correſpondentes, quibus ſubſtitutis, tolluntur termini requiſiti.

COROLLARIUM.

Per hoc problema nonnullæ æquationes $n \times \frac{n-1}{2}$, $n \times \frac{n-1}{2} \times \frac{n-2}{3}$, &c. dimenſionum deprimi poſſunt in æquationes (n) dimenſionum.

Aſſumatur æquatio (n) dimenſionum, & per problema primum inveniantur aliæ, quarum radices ſunt ſummæ ex quibuſque duabus, tribus, &c. & earum dimenſiones erunt $n \times \frac{n-1}{2}$, $n \times \frac{n-1}{2} \times \frac{n-2}{3}$, &c. hæ æquationes reſultantes iterum deprimi poſſunt in (n) dimenſionum æquationem.

Et ſic inveniri poſſunt innumeræ æquationes, quas deprimere licet.

LEMMA.

Æquatio

$$x^{2n} \pm x^{2n-1} + x^{2n-2} \pm x^{2n-3} + x^{2n-4} \pm \text{\&c.} = 0,$$

in quadraticam $x^2 + vx + 1 = 0$ reduci poteſt æquationis ope

$$v^n \mp v^{n-1} - \overline{n-1}\, v^{n-2} \pm \overline{n-2}\, v^{n-3} + \overline{n-2} \times \frac{\overline{n-3}}{2} v^{n-4} \mp \overline{n-3} \times \frac{\overline{n-4}}{2} v^{n-5} - \overline{n-3} \times \frac{\overline{n-4}}{2} \times \frac{n-5}{3} v^{n-6} \pm \overline{n-4} \times \frac{\overline{n-5}}{2} \times \frac{\overline{n-6}}{3} \times v^{n-7} + \overline{n-4} \times \frac{\overline{n-5}}{2} \times \frac{\overline{n-6}}{3} \times \frac{\overline{n-7}}{4} v^{n-8} \mp \text{\&c.} = 0.$$

CO-

COROLLARIUM.

Quatuor radices æquationis ($x^4+1=0$) erunt $\pm\sqrt{\tfrac{1}{2}}\pm\sqrt{-\tfrac{1}{2}}$, quinque vero radices æquationis ($x^5+1=0$) ſunt

-1, $\dfrac{\pm\sqrt{1\frac{1}{4}}+\frac{1}{2}}{2}\pm\dfrac{\sqrt{-2\frac{1}{2}\pm\sqrt{1\frac{1}{4}}}}{2}$, ſex vero radices æquationis ($x^6+1=0$) ſunt

$$\pm\sqrt{-1},\ \frac{\pm\sqrt{3}\pm\sqrt{-1}}{2}$$

LEMMA V.

Datâ quantitate utcunque irrationali, invenire alteram, quæ in datam ducta, rationale productum facit.

Datæ irrationalis quantitatis inveniantur ſinguli diverſi valores, quibus in ſeſe continuo ductis, contentum erit rationale productum; dividatur hoc productum per datam quantitatem, & quotiens erit irrationalis quantitas quæſita.

EXEMPLUM.

Invenire irrationalem quantitatem, quæ in $\sqrt[2]{a+bx}+\sqrt[3]{c+dx}$ ducta, rationale productum facit.

Singuli diverſi valores datæ irrationalis quantitatis $\sqrt[2]{a+bx}+\sqrt[3]{c+dx}$ erunt reſpective

$\sqrt{a+bx}+\sqrt[3]{c+dx}$, $-\sqrt{a+bx}+\sqrt[3]{c+dx}$, $\sqrt{a+bx}-\dfrac{1+\sqrt{-3}}{2}\sqrt[3]{c+dx}$, $-\sqrt{a+bx}-\dfrac{1+\sqrt{-3}}{2}\sqrt[3]{c+dx}$, $\sqrt{a+bx}-\dfrac{1-\sqrt{-3}}{2}\sqrt[3]{c+dx}$, $-\sqrt{a+bx}-\dfrac{1-\sqrt{-3}}{2}\times\sqrt[3]{c+dx}$. Contentum $\left(\overline{a+bx}^{3}-\overline{c+dx}^{2}\right)$ ex his omnibus in ſeſe ductis erit rationale productum, & contentum

$$\left(\overline{a+bx}^{\frac{5}{2}}-\overline{a+bx}^{\frac{4}{2}}\times\overline{c+dx}^{\frac{1}{3}}+\overline{a+bx}^{\frac{3}{2}}\times\overline{c+dx}^{\frac{2}{3}}-\overline{a+bx}^{\frac{2}{2}}\times\overline{c+dx}^{\frac{3}{3}}+\overline{a+bx}^{\frac{1}{2}}\times\overline{c+dx}^{\frac{4}{3}}-\overline{c+dx}^{\frac{5}{3}}\right)$$

ex omnibus niſi prima (quæ ſit data) in ſeſe ductis, erit irrationalis quantitas quæſita.

COR.

COR. I.

Hinc colligi poteſt, e quâ multiplicatione vel conſtructione orientur quæcunque datæ quantitates.

COR. II.

Datâ æquatione ſurdas involvente, ex hoc lemmate manifeſto conſtant dimenſiones, ad quas reſultans æquatio aſſurgit.

COR. III.

Hoc lemmate irrationalis quantitas (A) a denominatore in numeratorem & vice verſâ, transferri poſſit.

Inveniatur rationalis quantitas (B), cujus data irrationalis eſt diviſor, ſit vero $C = \frac{B}{A}$; & pro A ſubſtituatur $\frac{B}{C}$, & fit.

COR. IV.

Quantitate datâ, quæ ſimplicium terminorum juxta dimenſiones cujuſlibet literæ (x) progredientium ſeriem $(a+b+c+d+e+f+g$, &c.$)$ exprimit, invenire quantitatem, alternis ſeriei terminis $(a+c+e+g$, &c.$)$ æqualem, vel terminis ſeriei propoſitæ binis vel ternis vel quaternis vel denique (n) intervallis a ſe invicem diſtantibus.

Sint $\alpha, \beta, \gamma, \delta$, &c. reſpectivæ radices $(n+1)$ poteſtatis numeri (1).

In datâ quantitate pro literâ (x) ſubſtituantur reſpective $\alpha x, \beta x, \gamma x, \delta x$, &c. quantitatum reſultantium ſumma per numerum $(n+1)$ diviſa, erit quæſita quantitas.

EXEMPLUM. I.

Sit quantitas $\overline{a+x}^m$, quæ exprimit ſeriem ſequentem ſimplicium terminorum juxta dimenſiones literæ (x) progredientium,

$$a^m + m a^{m-1} x + m \frac{m-1}{2} a^{m-2} x^2 + \text{\&c.}$$

invenire quantitatem alternis ſeriei terminis æqualem.

Pro $n+1$ ſcribatur numerus $(2,)$ & quoniam radices quadraticæ unitatis ſunt 1 & -1, pro (x) in data quantitate ſcribantur reſpective $+x$ & $-x$; & reſultant quantitates $\overline{a+x}^m$ & $\overline{a-x}^m$, quarum ſumma per numerum $(n+1=2)$ diviſa, erit quæſita quantitas

$$i.e. \frac{\overline{a+b}^m + \overline{a-b}^m}{2} = a^m + m \times \frac{m-1}{2} a^{m-2} b^2 + m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} a^{m-4} b^4 + \text{\&c.}$$

EX

EXEMPLUM II.

Eâdem quantitate datâ, ſit vero ($n+1=3$;) cubicæ radices numeri (1) ſunt 1, $\frac{-1+\sqrt{-3}}{2}$, $\frac{-1-\sqrt{-3}}{2}$; in datâ igitur quantitate pro x ſubſtituantur x, $\frac{-1+\sqrt{-3}}{2}x$, $\frac{-1-\sqrt{-3}}{2}x$, reſpective, & reſultant quantitates $\overline{a+x}|^m$, $\overline{a-\frac{1+\sqrt{-3}}{2}x}|^m$, $\overline{a-\frac{1-\sqrt{-3}}{2}x}|^m$, quarum ſumma per numerum ($n+1=3$) diviſa, erit quæſita quantitas *i. e.* $\frac{\overline{a+x}|^m + \overline{a-\frac{1+\sqrt{-3}}{2}x}|^m + \overline{a-\frac{1-\sqrt{-3}}{2}x}|^m}{3} =$

$$= a^m + m\times\frac{m-1}{2}\times\frac{m-2}{3}a^{m-3}x^3 + m\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}\times\frac{n-4}{5}\times\frac{m-5}{6}\times a^{m-6}x^6 \text{ \&c.}$$

EXEMPLUM III.

Datâ quantitate $\overline{a+bx+\overline{c+dx+ex^2}|^l}|^m$ exprimente ſeriem terminorum juxta dimenſiones literæ (x) progredientium, invenire quantitatem alternis hujus ſeriei terminis æqualem. Hoc caſu $\overline{n+1}=2$, ergo pro x in data quantitate ſcribantur reſpective x & $-x$; & reſultant quantitates $\overline{a+bx+\overline{c+dx+ex^2}|^l}|^m$ & $\overline{a-bx+\overline{c-dx+ex^2}|^l}|^m$: quarum ſumma per numerum ($n+1=2$) diviſa, eſt quæſita quantitas

$$\left(\frac{\overline{a+bx+\overline{c+dx+ex^2}|^l}|^m + \overline{a-bx+\overline{c-dx+ex^2}|^l}|^m}{2}.\right)$$

PROB. VIII.

Datâ reſolutione, vel reductione, æquationem invenire.

Præcedentis lemmatis vel problematis primi ope, ita reducatur data æquatio, ut ex eâ exterminentur irrationales quantitates, & ſit problema.

Exemplum I.

Sit (x) incognita quantitas, & data reſolutio

$x = \sqrt[3]{-\frac{1}{2}b + \sqrt{\frac{1}{4}b^2 + \frac{1}{27}a^3}} + \sqrt[3]{-\frac{1}{2}b - \sqrt{\frac{1}{4}b^2 + \frac{1}{27}a^3}}$; ita reducatur data æquatio, ut exterminentur irrationales quantitates, & reſultat æquatio $x^9 + 3bx^6 + \overline{3b^2 + a}^3 x^3 + b^3 = 0$, cujus, cubicæ æquationes $x^3 + ax + b = 0$, $x^3 - \frac{1 - \sqrt{-3}}{2} ax + b = 0$, $x^3 - \frac{1 + \sqrt{-3}}{2} ax + b = 0$, ſunt diviſores.

Exemplum II.

Sit reſolutio $x = \sqrt[3]{-\frac{1}{2}b + \sqrt{\frac{1}{4}b^2 + \frac{1}{27}a^3}} - \frac{\frac{1}{3}a}{\sqrt[3]{-\frac{1}{2}b + \sqrt{\frac{1}{4}b^2 + \frac{1}{27}a^3}}}$; exterminentur irrationales quantitates, & æquatio reſultans erit $x^6 + 2ax^4 + bx^3 + a^2x^2 + abx = 0$, cujus, æquationes cubicæ $x^3 + ax + b = 0$, & $x^3 + ax = 0$, ſunt diviſores.

Exemplum III.

In genere ſit reſolutio $x = \sqrt[n]{a + \sqrt{-b^2}} + \sqrt[n]{a - \sqrt{-b^2}}$, exterminentur irrationales quantitates & reſultat æquatio ($n \times n$) dimenſionum, cujus æquatio

$x^n - n \times \overline{a^2 - b^2}^{\frac{1}{n}} x^{n-2} + n \times \frac{n-3}{2} \overline{a^2 - b}^{\frac{2}{n}} x^{n-4} - n \times \frac{n-4}{2} \times \frac{n-5}{3} \overline{a^2 - b}^{\frac{3}{n}} x^{n-6} + n \times \frac{n-5}{2} \times \frac{n-6}{3} \times \frac{n+7}{4} \overline{a^2 - b}^{\frac{4}{n}} x^{n-8} -$ &c. $= 0$, eſt diviſor.

Exemplum IV.

Sit reſolutio $x = a\sqrt[n]{p} + b\sqrt[n]{p^2} + c\sqrt[n]{p^3} + d\sqrt[n]{p^4} \ldots\ldots A\sqrt[n]{p^{n-4}} + B\sqrt[n]{p^{n-3}} + C\sqrt[n]{p^{n-2}} + D\sqrt[n]{p^{n-1}}$. exterminentur irrationales quantitates & reſultat æquatio

$$x^n - n \times \overline{aD + bC + cB + dA}, \text{ \&c. } x^{n-2} + \text{ \&c. } = 0.$$

Hinc deduci poteſt cubicæ æquationis reſolutio.

Sit reſolutio

$x = a\sqrt[3]{p} + \sqrt[3]{p^2}$, reducatur, ita ut exterminentur irrationales quantitates, & reſultat æquatio $x^3 - 3apx - p^2 - a^2p = 0$, ſit cubica æquatio

tio $x^3+Ax-B=o$, cujus reſolutio requiritur; aſſumatur $A=-3ap$ & $B=p^2+a^3p$ & conſequenter

$$a=\frac{-A}{3p},\ p^2+a^3p=p^2-\frac{A^3}{27p^3}p=B \text{ vel } p^4-Bp^2=\frac{A^3}{27} \ \&$$

$$p^2=\pm\sqrt{\frac{A^3}{27}+\frac{B^2}{4}}+\tfrac{1}{2}B,$$ exinde datur p & ſolutio eadem fit ac Cardani.

Aliquantulum diverſa eſt reductio ſi modo reſolutio ſit $x=\sqrt[3]{p}+a\sqrt[3]{p^2}$; exterminentur irrationales quantitates & reſultat $x^3-3apx-p-a^3p^2=o$; fiat hæc eadem ac cubica æquatio $x^3+Ax-B=o$ & fit $p^2-Bp-\frac{A^3}{27}=o$, æquatio quadratica.

COROLLARIUM I.

Ex hoc problemate inveniri poſſunt quilibet æquationum numerus quarum dantur reſolutiones.

Aſſumatur reſolutio, & transformetur hæc æquatio ita ut deleantur irrationales quantitates & reſultat æquatio, cujus datur reſolutio. Et ſic in quamplurimis caſibus reſolvi poteſt data æquatio.

EXEMPLUM I.

Sit data æquatio

$$x^3+Ax+B=o.$$

Aſſumatur $x=\sqrt[3]{\alpha}+\sqrt[3]{\beta}$ & conſequenter

$$x^3=\alpha+\beta+3\sqrt[3]{\alpha\beta}\times\overline{\sqrt[3]{\alpha}+\sqrt[3]{\beta}}$$

$$+Ax= \qquad A\times\overline{\sqrt[3]{\alpha}+\sqrt[3]{\beta}}$$

$$B= \quad B.$$

Supponatur $\alpha+\beta+B=o$, & $3\sqrt[3]{\alpha\beta}+A=o$, unde $\alpha+\beta=-B$ & $\alpha\beta=\frac{-A^3}{27}$. & α & β erunt duæ radices quadraticæ æquationis

$$z^2+Bz-\frac{A^3}{27}=o.$$

EXEMPLUM II.

Sit data æquatio

$$x^4+qx^2+rx+s=o.$$

aſſumatur pro eius reſolutione

$$x=\sqrt[4]{\alpha}+\sqrt[4]{\beta}+\sqrt[4]{\gamma}.$$

&

&

$$x^4 = \alpha+\beta+\gamma+6\overline{\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}+\alpha^{\frac{1}{2}}\gamma^{\frac{1}{2}}+\beta^{\frac{1}{2}}\gamma^{\frac{1}{2}}}+4\times\overline{\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}}+\gamma^{\frac{1}{2}}}\times\overline{\alpha^{\frac{1}{4}}\beta^{\frac{1}{4}}+\alpha^{\frac{1}{4}}\gamma^{\frac{1}{4}}+\beta^{\frac{1}{4}}\gamma^{\frac{1}{4}}}$$

$$8\alpha^{\frac{1}{4}}\beta^{\frac{1}{4}}\gamma^{\frac{1}{4}}\times\overline{\alpha^{\frac{1}{4}}+\beta^{\frac{1}{4}}+\gamma^{\frac{1}{4}}}$$

$$qx^2 = \quad q\times\overline{\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}}+\gamma^{\frac{1}{2}}}+ \quad 2q\times\overline{\alpha^{\frac{1}{4}}\beta^{\frac{1}{4}}+\alpha^{\frac{1}{4}}\gamma^{\frac{1}{4}}+\beta^{\frac{1}{4}}\gamma^{\frac{1}{4}}}$$

$$rx = \quad r\times\overline{\alpha^{\frac{1}{4}}+\beta^{\frac{1}{4}}+\gamma^{\frac{1}{4}}}$$

$$s = \quad s.$$

Supponantur $4\times\overline{\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}}+\gamma^{\frac{1}{2}}}+2q=0$,

$\alpha+\beta+\gamma+6\times\overline{\alpha^{\frac{1}{2}}\beta^{\frac{1}{2}}+\alpha^{\frac{1}{2}}\gamma^{\frac{1}{2}}+\beta^{\frac{1}{2}}\gamma^{\frac{1}{2}}}+q\times\overline{\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}}+\gamma^{\frac{1}{2}}}+s=0$

& $8\,\alpha^{\frac{1}{4}}\beta^{\frac{1}{4}}\gamma^{\frac{1}{4}}+r=0$. e primâ æquatione ſequitur

$\alpha^{\frac{1}{2}}+\beta^{\frac{1}{2}}+\gamma^{\frac{1}{2}}=-\frac{q}{2}$, quo valore in ſecundâ ſubſtituto, reſultat

$\alpha+\beta+\gamma+6\times\dfrac{\frac{q^2}{4}-\overline{\alpha+\beta+\gamma}}{2}-\dfrac{q^2}{2}+s=0$, unde ſequitur

$\alpha+\beta+\gamma=\frac{s}{2}+\frac{q^2}{8}$. Per lemma ſecundum capitis primi inveniri poteſt

$\alpha\beta+\alpha\gamma+\beta\gamma=+\frac{qr^2}{64}+\frac{s^2}{16}+\frac{q^2s}{32}+\frac{q^4}{256}$ Sequitur & e tertiâ æquatione

$\alpha^{\frac{1}{4}}\beta^{\frac{1}{4}}\gamma^{\frac{1}{4}}=\frac{-r}{8}$; quâ reductâ, fit $\alpha\beta\gamma=\frac{r^4}{4096}$: & conſequenter æquatio, cujus radices (z) ſunt α, β, γ erit

$$z^3-\overline{\frac{s}{2}+\frac{q^2}{8}}\,z^2+\overline{\frac{qr^2}{64}+\frac{s^2}{16}+\frac{q^2s}{32}+\frac{q^4}{256}}\,z-\frac{r^4}{4096}=0.$$

Datæ æquationis quatuor radices erunt reſpective

$$\sqrt[4]{\alpha}\qquad+\sqrt[4]{\beta}\qquad+\sqrt[4]{\gamma},$$
$$-\sqrt[4]{\alpha}-\sqrt{-1}\sqrt[4]{\beta}+\sqrt{-1}\sqrt[4]{\gamma},$$
$$\sqrt{-1}\sqrt[4]{\alpha}-\qquad\sqrt[4]{\beta}-\sqrt{-1}\sqrt[4]{\gamma},$$
$$-\sqrt{-1}\sqrt[4]{\alpha}+\sqrt{-1}\sqrt[4]{\beta}\qquad-\sqrt[4]{\gamma}.$$

EXEMPLUM III.

Sit data æquatio

$$x^5+qx^3+rx^2+sx+t=0$$

aſſumatur pro ejus reſolutione $x=\sqrt[5]{\alpha}+\sqrt[5]{\beta}+\sqrt[5]{\gamma}+\sqrt[5]{\delta}$; ex calculo inveni æquationem biquadraticam, cujus radices ſunt α, β, γ, δ, irrationales quantitates neceſſario habere & conſequenter per hanc methodum haud reſolvi poteſt data æquatio.

EXEMPLUM IV.

Sit reſolutio

$$x=\left(a+\left(b+\left(c+\left(d+e^{\frac{1}{m}}\right)^{\frac{1}{n}}\right)^{\frac{1}{o}}\right)^{\frac{1}{p}}\right)^{\frac{1}{q}}+f$$

ita transformetur hæc æquatio, ut exterminentur irrationales quantitates & reſultat æquatio

$$\left(\left(\left(\left(\left(x-f\right)^{m}-a\right)^{n}-b\right)^{o}-c\right)^{p}-d\right)^{q}-e=0$$

Et ſic de infinitis hujuſmodi exemplis.

Ab exemplo primo & ſecundo patet cubicarum æquationum generalem reſolutionem eſſe reſolutionem ſex vel novem dimenſionum æquationum.

Ab exemplo ſecundo corollarii patet eam biquadraticæ æquationis reſolutionem eſſe reſolutionem (64) dimenſionum æquationis.

Datâ igitur in genere (n) dimenſionum æquatione

$$x^n-px^{n-1}+qx^{n-2}-rx^{n-3}+\&c.=0$$

ejus reſolutio erit reſolutio æquationis, cujus dimenſiones ſunt multò altiores.

Subſequens, ut mihi videtur, eſt unica methodus generalem æquationum reſolutionem inveniendi.

Aſſu-

Aſſumatur reſolutio, plures quam datæ æquationis (n) dimenſiones incognitas quantitates habens, transformetur hæc æquatio ita ut exterminentur irrationales quantitates, & reſultat æquatio, cujus data æquatio eſt diviſor; unde erui poſſunt (n) æquationes, plures quam (n) incognitas quantitates habentes, quibus reductis, reſultat æquatio duas vel plures incognitas quantitates habens.

Incognitarum reſultantis æquationis quantitatum inveniantur rationales radices, ſibi ipſis correſpondentes; quibus pro ſuis valoribus ſubſtitutis, reſultant incognitæ quantitates quæſitæ.

Facile patebit reſolutio, quam exigat quæcunque data æquatio. Sed in multarum dimenſionum æquationibus calculus requiritur fere infinite laborioſus.

COROLLARIUM II.

Datâ æquatione, invenire, annon reſolvi poteſt extractionis radicis quadraticæ, cubicæ &c, ope.

Datâ æquatione, dantur etiam ejus dimenſiones, aſſumatur generaliter reſolutio, quæ ſolummodo habet radicem quadraticam, cubicam &c, & ita ut, ſi modo exterminentur ſurdæ quantitates, reſultat æquatio datarum dimenſionum.

Reſultans & data æquatio æquales inter ſe fiant, & fit problema.

EXEMPLUM.

Invenire, annon data quatuor vel octo dimenſionum æquatio reſolvi poteſt extractionis radicis quadraticæ ope.

Aſſumatur generaliter reſolutio, quæ radices habet quadraticas, ita ut reſultans æquatio quatuor vel octo &c. habeat dimenſiones, & erit reſolutio

$$x = \sqrt{a + m\sqrt{b}} + n\sqrt{b} + c \text{ vel } x = \sqrt{a + m\sqrt{b + n\sqrt{c}} + o\sqrt{c}}$$
$$+ r\sqrt{b + n\sqrt{c}} + s\sqrt{c} + d$$

reducantur hæ æquationes, ita ut exterminentur irrationales quantitates & reſultat æquatio

$$x^4 - 4cx^3 - \overline{2n^2b - 6c^2 + 2a}\, x^2 - \overline{4c^3 - 4ac}$$
$$+$$

$$\overline{+4mnb-4n^2cb}\;x+\left.\overline{n^2b+c^2 \atop -a}\right)^2 - \overline{m-2nc}^2\times b=o.$$

vel x^8-8cx^7 &c. $=o.$

Fiant reſultans & data æquatio inter ſe æquales & fit.

COROLLARIUM.

Hinc reduci poteſt æquatio $2n$, $4n$, $8n$, &c. dimenſionum in æquationem (n) dimenſionum extractionis quadraticæ radicis ope:

EXEMPLUM I.

Sit data æquatio

$$x^{2n}-px^{2n-1}+qx^{2n-2}-rx^{2n-3}+\text{\&c.}=o.$$

aſſumatur

$$x^n-Px^{n-1}+Qx^{n-2}\text{\&c.}=\sqrt{a}\times\overline{\overset{1}{p}x^{n-1}-\overset{1}{q}x^{n-2}}$$

transformetur hæc æquatio, ita ut exterminetur irrationalis quantitas ($\sqrt{a}$) & fit

$$x^{2n}-2Px^{2n-1}+\overline{2Q+P^2-a\overset{1}{p}{}^2}\times x^{2n-2}\text{\&c.}=o.$$

Fiat hæc æquatio eadem ac data, & conſequenter earum coefficientes inter ſe æquales

i. e. $p=2P$, $q=2Q+P^2-a\overset{1}{p}{}^2$ & ſic deinceps. Ex æquationibus reſultantibus inveniri poſſunt rationales incognitarum coefficientium ſi quos forte habent, valores.

EXEMPLUM II.

Sit data æquatio

$$x^{4n}-px^{4n-1}+qx^{4n-2}-rx^{4n-3}+\text{\&c.}=o.$$

aſſumatur

$$x^n-Px^{n-1}+\underline{Qx^{n-2}+\sqrt{a}\times\overline{\overset{1}{p}x^{n-1}+\overset{1}{q}x^{n-2}}}\text{\&c.}=$$
$$\sqrt{\overline{b+m\sqrt{a}}}\times\overset{11}{p}x^{n-1}+\overset{11}{q}x^{n-2}+\text{\&c.}$$

transformetur hæc æquatio, ita ut exterminentur irrationales quantitates, & reſultantis æquationis coefficientes correſpondentibus datæ æquationis coefficientibus fiant æquales; & reſultantium æquationum incognitarum quantitatum quicunque rationales valores erunt coefficientes quæſitæ.

CAP. V.

PROB. IX.

DUAS æquationes in unam tranſformare, ut incognita quantitas exterminetur.

1. Collocentur ſingulæ æquationis termini juxta incognitæ quantitatis exterminandæ dimenſiones, ita ſubducatur alterutra æquatio aut ejus multiplex de alterâ, & reliquum aut ejus multiplex de ablatâ æquatione & ſic deinceps; ut tandem exterminetur incognita quantitas.

EXEMPLUM.

Sint duæ æquationes

$$y^2+\overline{3+2x}\times y+4x^2+5x+6=0$$
$$y^2+\overline{5+2x}\times y+10x^2+7x+12=0.$$

et operatio erit hujuſmodi

$$y^2+\overline{3+2x}\times y+4x^2+5x+6\;(y^2+\overline{5+2x}\times y+10x^2+7x+12)\,1$$
$$y^2+\overline{3+2x}\times y+4x^2+5x+6$$
$$2y\quad+6x^2+2x+6=0$$

$$2y+6x^2+2x+6\;(y^2+\overline{3+2x}\times y+4x^2+5x+6)\;\tfrac{1}{2}y+\frac{x-3x^2}{2}$$
$$y^2+\overline{3x^2+x+3}\times y$$
$$\overline{x-3x^2}\times y+4x^2+5x+6$$
$$\overline{x-3x^2}\times y+\frac{x-3x^2}{2}\times\overline{6x^2+2x+6}$$
$$4x^2+5x+6-\overline{x-3x^2}\times\overline{3x^2+x+3}=0$$

& transformantur duæ datæ æquationes in unam ($4x^2+5x+6-\overline{x-3x^2}\times\overline{3x^2+x+3}=0$) ut exterminetur incognita quantitas (y).

Et ſic de (m) æquationibus in unam transformandis, ut ($m-1$) incognitæ quantitates exterminentur.

COROLLARIUM.

Æquationes ſic derivatæ plerumque plures habent dimenſiones, quam neceſſario exigat problema.

2. Eſt

2. Eſt alia methodus transformandi duas æquationes in unam, ut incognita quantitas exterminetur

Sint duæ æquationes

$$y^n - py^{n-1} + qy^{n-2} - ry^{n-3} + \&c. = 0$$
$$V - Ty + Sy^2 - Ry^3 + \&c. = 0$$

Sint etiam $\alpha, \beta, \gamma, \delta$, &c. prioris æquationis radices (y,) quarum numerus eſt (n;) quibus in poſteriori pro ſuo valore (y) ſubſtitutis, reſultant quantitates

$$V - T\alpha + S\alpha^2 - R\alpha^3 + \&c.$$
$$V - T\beta + S\beta^2 - R\beta^3 + \&c.$$
$$V - T\gamma + S\gamma^2 - R\gamma^3 + \&c.$$
$$V - T\delta + S\delta^2 - R\delta^3 + \&c.$$

multiplicentur hæ quantitates reſultantes continuo in ſeſe & contentum erit

$$V^n - V^{n-1}T \times \begin{array}{l}\alpha\\ \beta\\ \gamma\\ \delta\end{array} + V^{n-1}S \times \begin{array}{l}\alpha^2\\ \beta^2\\ \gamma^2\\ \delta^2\\ \&c.\end{array} \quad \begin{array}{l}\&c.\\ \&c.\\ \&c.\end{array}$$
$$+ V^{n-2}T^2 \times \begin{array}{l}\alpha\beta\\ \alpha\gamma\\ \alpha\delta\\ \beta\gamma\\ \&c.\end{array} \quad \begin{array}{l}\&c.\\ \\ \\ \&c.\end{array}$$

Ope lemmatum in primo capite inveniri poteſt hoc contentum, quod nihilo fiat æquale; & transformantur duæ æquationes in unam, ut exterminetur incognita quantitas (y).

EXEMPLUM.

Sint duæ æquationes

$$y^2 + \overline{3+2x} \times y + 4x^2 + 5x + 6 = 0.$$
$$y^2 + \overline{5+2x} \times y + 10x^2 + 7x + 12 = 0.$$

Sint α & β prioris æquationis radices (y); quibus pro ſuo valore (y) in poſteriori ſubſtitutis, reſultant quantitates

$$\alpha^2 + \overline{5+2x} \times \alpha + 10x^2 + 7x + 12$$
$$\beta^2 + \overline{5+2x} \times \beta + 10x^2 + 7x + 12$$

 quibus

quibus in ſeſe ductis, ſit rectangulum

$$\alpha^2\beta^2+\overline{5+2x}\times\alpha\beta\times\overline{\alpha+\beta}+\overline{10x^2+7x+12}\times\overline{\alpha^2+\beta^2}+\overline{5+2x}^2\times\alpha\beta+\overline{10x^2+7x+12}\times\overline{5+2x}\times\overline{\alpha+\beta}+\overline{10x^2+7x+12}^2.$$

Ope lemmatis primi invenitur hoc rectangulum

$$\overline{4x^2+5x+6}^2-\overline{5+2x}\times\overline{4x^2+5x+6}\times\overline{3+2x}+\overline{10x^2+7x+12}\times\overline{3+2x}^2-2\times\overline{4x^2+5x+6}+\overline{5+2x}^2\times\overline{4x^2+5x+6}-\overline{10x^2\times 7x+12}\times\overline{5+2x}\times\overline{3+2x}+\overline{10x^2+7x\times 12}^2$$

Fiat hoc rectangulum nihilo æquale & erit quæſita æquatio.

COR. I.

Hac methodo transformentur duæ æquationes, quarum dimenſiones ſunt n & m reſpective, duas incognitas quantitates (x & y) habentes, in unam, ut exterminetur una incognita quantitas (y,) & plures quam ($n\times m$) dimenſiones non habere poteſt reſultans æquatio.

COR. II.

Sit data æquatio

$$My^n+\overline{a+bx}\times y^{n-1}+\overline{c+dx+ex^2}\times y^{n-2}+\&c.=0$$
$$Ny^m+\overline{A+Bx}\times y^{m-1}+\overline{C+Dx+Ex^2}\times y^{m-2}+\&c.=0$$

quarum dimenſiones ſunt n & m, duas incognitas quantitates (x & y) habentes, rejiciantur harum æquationum omnes termini, in quibus incognitarum quantitatum (x & y) dimenſio non invenitur maxima & quantitates reſultantes erunt

$$My^n+bxy^{n-1}+ex^2y^{n-2}+\&c$$
$$Ny^m+Bxy^{m-1}+Ex^2y^{m-2}+\&c$$

ſi communem habent diviſorem hæ quantitates reſultantes; ($n\times m$) dimenſiones non habere poteſt reſultans æquatio, cujus radix eſt incognita quantitas (x vel y.)

COR. III.

Si æquatio reſultans, cujus radix eſt incognita quantitas (x,) ($n\times m$) dimenſiones habeat, ſumma radicum ſolummodo pendet e terminis priorum æquationum, in quibus incognitarum quantitatum (x & y) dimenſiones ſunt n & $n-1$, m & $m-1$; & ſic de ſummâ contentorum ſub ſingulis binis, ternis &c. radicibus.

COR. IV.

Si duæ vel tres vel plures incognitæ quantitatis (x) radices, ſint inter ſe æquales; æquatio cujus radix eſt (x,) deprimi poteſt in alteram pauciores dimenſiones habentem.

EX-

EXEMPLUM.

Sint duæ æquationes duas incognitas quantitates habentes

$$y^3+\overline{6+2x}\times y+12x^3+14x^2-7x+6=0$$
$$y^3+\overline{6+2x}\times y+9x^3+14x^2+14x-12=0$$

auferatur poſterior æquatio de priori, & reſiduum erit $3x^3-21x+18=0$. quæ eſt æquatio, cujus radix eſt incognita quantitas (x.)

Sed æquatio, per methodum ſupra traditam derivata, erit $\overline{3x^3-21x+18}^3=0$. in quâ tres radices incognitæ quantitatis (x) ter ſunt æquales; tres enim diverſi incognitæ quantitatis (y) valores, æquationis ($3x^3-21x+18=0$) radici ſingulæ reſpondent.

Æquationis ($3x^3-21x+18=0$) radices ſunt 1, 2 & -3; quibus pro ſuo valore (x) in datis æquationibus ſubſtitutis, reſultant tres æquationes

$$y^3+8y+25=0$$
$$y^3+10y+144=0$$
$$y^3 \qquad -171=0.$$

unde ſequitur tres radices æquationis] ($y^3+8y+25=0$,) reſpondere radici (1) incognitæ quantitatis (x,); & conſequenter tres radices incognitæ quantitatis (x) eſſe (1); & ſic de reliquis.

COR. V.

Transformentur duæ æquationes, duas incognitas quantitates (x & y) habentes & ſimiliter involventes, in unam; ut exterminetur altera incognita quantitas (y,) & æquatio reſultans, cujus radix eſt (x) duplas habet dimenſiones æquationis, cujus radix eſt ($x+y$) aut (x^2+y^2) &c.

DEMONSTRATIO.

Sint radices incognitæ quantitatis (x) reſpective

$\alpha, \beta, \gamma, \delta \ldots \pi, \rho, \sigma, \tau$; correſpondentes vero radices incognitæ quantitatis (y) reſpective

$\tau, \sigma, \rho, \pi \ldots \delta, \gamma, \beta, \alpha$. & incognitæ quantitatis ($z=x+y$) reſpectivi valores ſunt

$\alpha+\tau, \beta+\sigma, \gamma+\rho, \delta+\pi \ldots \delta+\pi, \gamma+\rho, \beta+\sigma, \alpha+\tau$, unde omnis valor incognitæ quantitatis (z) alterum habet ſibi ipſi æqualem

lem ($\alpha+\tau$, $\alpha+\tau$, $\beta+\sigma$, $\beta+\sigma$, &c.) & conſequenter incognita quantitas (x aut y) duplum habet valorum numerum, quem incognita quantitas ($z=x+y$,) & exinde æquatio, cujus radix eſt (x aut y) duplas habet dimenſiones æquationis, cujus radix eſt ($z=x+y$.)

Hujuſce corollarii exempla præbent Moivræi duo problemata.

1. Sint duæ æquationes $\overline{x^2+y^2} \times \overline{x^4+y^4} = b$

$$\overline{x^2+y^2} \times \overline{x+y} = a$$

utriuſque æquationis terminorum, in quibus incognitarum quantitatum (x aut y) dimenſiones ſunt maximæ, communis eſt diviſor (x^2+y^2); unde ſequitur, æquationem, cujus radix eſt (x vel y) ($6\times3=18$) dimenſiones non habere: ejus dimenſiones ſunt (12.); ſed hæ duæ æquationes incognitas quantitates (x&y) ſimiliter involvunt, & conſequenter æquatio, cujus radix eſt ($z=x+y$) ſolummodo ſex habet dimenſiones, & ex calculo ſequitur formulam ejus eſſe quadraticam.

2. Sint æquationes

$$x^4+x^3y+x^2y^2+xy^3+y^4 = a$$

$$\overline{x^4+x^3y+x^2y^2+xy^3+y^4} \times \overline{x^4-x^3y+x^2y^2-xy^3+y^4} = b$$

harum æquationum termini, in quibus incognitarum quantitatum (x aut y) dimenſiones ſunt maximæ, communem habent diviſorem ($x^4+x^3y+x^2y^2+xy^3+y^4$), & conſequenter æquatio, cujus radix eſt (x aut y) non habet ($4\times8=32$) dimenſiones; habet vero ($4\times4=16$) dimenſiones; & eadem eſt ejus formula, ac octo dimenſionum æquatio, unde ejus negativæ & affirmativæ radices ſunt inter ſe æquales; ſed hæ duæ æquationes incognitas quantitates (x&y) ſimiliter involvunt, & conſequenter æquatio, cujus radix eſt ($z=x^2+y^2$) ſemper quatuor habet radices ſibi ipſis æquales, duas, quoniam ſimiliter involvunt incognitas quantitates (x&y) datæ æquationes, & duas reliquas, quoniam affirmativæ & negativæ radices (x) ſunt inter ſe æquales, unde æquatio, cujus radix eſt ($z=x^2+y^2$) non habere poteſt plures quam $\left(\frac{16}{4}=4\right)$ dimenſiones; & ex calculo ſequitur ejus formulam eſſe quadraticam.

3. Eſt etiam alia methodus transformandi duas æquationes in unam, ut incognita quantitas exterminetur.

Per

Per methodum ſerierum convergentium, inveniantur ſingulæ radices incognitæ quantitatis exterminandæ in unâ datâ æquatione, quibus pro ſuo valore in altera ſubſtitutis, & multiplicentur quantitates reſultantes continuo in ſeſe, contentum nihilo æquale fiat, & erit quæſita æquatio.

EXEMPLUM.

Sint duæ æquationes

$$y^2 - \overline{4x+2} \times y + 3x^2 + 4x + 4 = 0$$
$$y^2 - 3xy + 4x^2 + 6 = 0.$$

eas in unam transformare, ut exterminetur incognita quantitas (y).

Incognitæ quantitatis (y) duæ radices in priori æquatione per convergentes ſeries inveniuntur

$$y = x + 1 + \frac{3}{2x} + \frac{9}{8x^3} + \&c.$$

$$y = 3x + 1 - \frac{3}{2x} - \frac{9}{8x^3} - \&c.$$

quibus radicibus pro (y) in poſteriori æquatione ſubſtitutis, reſultant quantitates

$$2x^2 - x + \frac{11}{2} + \frac{3}{x} + \frac{9}{8x^2} \quad \&c.$$

$$4x^2 + 3x + \frac{5}{2} - \frac{3}{x} - \frac{9}{8x^2} \quad \&c.$$

ducantur hæ duæ quantitates reſultantes in ſeſe & rectangulum ſub iis contentum, erit

$8x^4 + 2x^3 + 24x^2 + 20x + 28$ quod nihilo fiat æquale, & erit æquatio quæſita

$$8x^4 + 2x^3 + 24x^2 + 20x + 28 = 0.$$

PROB. X.

Invenire coefficientium conſtitutionem, in datâ æquatione, unam vel plures incognitas quantitates habente.

1. Datâ æquatione $(x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} \&c. = 0)$ unam incognitam quantitatem habente, invenire ejus coefficientium $(p, q, r, s$ &c.) conſtitutionem.

Sint

Sint $\alpha, \beta, \gamma, \delta$ &c radices datæ æquationis, quarum numerus ſit $(n,)$ quibus pro ſuo valore (x) ſubſtitutis, reſultant æquationes

$$\alpha^n - p\alpha^{n-1} + q\alpha^{n-2} - r\alpha^{n-3} + s\alpha^{n-4} \text{ \&c.} = 0$$
$$\beta^n - p\beta^{n-1} + q\beta^{n-2} - r\beta^{n-3} + s\beta^{n-4} \text{ \&c.} = 0.$$
$$\gamma^n - p\gamma^{n-1} + q\gamma^{n-2} - r\gamma^{n-3} + s\gamma^{n-4} \text{ \&c.} = 0.$$
$$\delta^n - p\delta^{n-1} + q\delta^{n-2} - r\delta^{n-3} + s\delta^{n-4} \text{ \&c.} = 0.$$

reducantur hæ æquationes in unam, ita ut exterminentur omnes præter unam (p) coefficientes, & ſequitur $p=\alpha+\beta+\gamma+\delta+$&c, eodem modo invenitur $q=\alpha\beta+\alpha\gamma+\beta\gamma+$&c, $r=\alpha\beta\gamma+$&c & ſic deinceps.

2. Datâ æquatione

$$y^n + \overline{a+bx}\times y^{n-1} + \overline{c+dx+ex^2}\times y^{n-2} + \overline{f+gx+hx^2+kx^3}\times y^{n-3} \text{ \&c.} = 0.$$

duas incognitas quantitates $(x$ & $y)$ habente, invenire coefficientium conſtitutionem.

Sint incognitæ quantitatis (y) radices $\alpha, \beta, \gamma, \delta$ &c, & incognitæ quantitatis (x) correſpondentes radices π, ρ, σ, τ &c. & ſingulæ incognitæ quantitatis $(x$ & $y)$ ſit numerus correſpondentium radicum $n\times\frac{n+3}{2}$; ſubſtituantur hæ correſpondentes radices pro ſuis reſpectivis valoribus $(x$ & $y,)$ & reſultant æquationes

$$\alpha^n + \overline{a+b\pi}\times\alpha^{n-1} + \overline{c+d\pi+e\pi^2}\times\alpha^{n-2} + \overline{f+g\pi+h\pi^2+k\pi^3}\times\alpha^{n-3} \text{ \&c.} = 0$$
$$\beta^n + \overline{a+b\rho}\times\beta^{n-1} + \overline{c+d\rho+e\rho^2}\times\beta^{n-2} + \overline{f+g\rho+h\rho^2+k\rho^3}\times\beta^{n-3} \text{ \&c.} = 0$$
$$\gamma^n + \overline{a+b\sigma}\times\gamma^{n-1} + \overline{c+d\sigma+e\sigma^2}\times\gamma^{n-2} + \overline{f+g\sigma+h\sigma^2+k\sigma^3}\times\gamma^{n-3} \text{ \&c.} = 0$$
$$\delta^n + \overline{a+b\tau}\times\delta^{n-1} + \overline{c+d\tau+e\tau^2}\times\delta^{n-2} + \overline{f+g\tau+h\tau^2+k\tau^3}\times\delta^{n-3} \text{ \&c.} = 0.$$

harum æquationum numerus per hypotheſin æqualis eſt fractioni $\frac{n\times\overline{n+3}}{2}$; et numerus incognitarum coefficientium $(a, b, c, d, e$ &c,) eidem quantitati æqualis eſt; reducantur hæ $\left(\frac{n\times\overline{n+3}}{2}\right)$ æquationes in unam, ita ut exterminentur omnes præter unam (a) incognitæ coefficientes, & conſtat ex quot & qualibus membris *i. e.* ex quâ radicum functione iſta una (a) coefficiens conſtat; & ſic de reliquis.

COROLLARIUM. II.

1. Sit numerus (m) correſpondentium radicum incognitarum quantitatum $(x$ & $y)$ æquationis, cujus dimenſiones ſunt $(n,)$ major quam

$n\times$

$n\times\overline{\frac{n+3}{2}}$; tum ex quibuſque $\left(n\times\overline{\frac{n+3}{2}}\right)$ correſpondentibus radicibus incognitarum quantitatum (x & y) oritur æquationis conſtitutio, *i, e.* coefficientes quæſitæ æquationis.

2. Sit numerus (m) correſpondentium radicum minor quam $\left(n\times\overline{\frac{n+3}{2}},\right)$ & infinitis modis dari poſſunt diverſæ conſtitutiones æquationis, vel pro $\left(n\times\overline{\frac{n+3}{2}}-m\right)$ coefficientibus ſcribi poſſunt quilibet numeri.

COR. III.

Sint duæ æquationes, duas incognitas quantitates (x & y) habentes, quarum dimenſiones ſunt n & m reſpective, et ſit (n) minor quam m; tum æquationis n dimenſionum ita aſſumi poſſunt coefficientes, ut quicunque $\left(n\times\overline{\frac{n+3}{2}}\right)$ numeri, ſint radices incognitæ quantitatis (x), & ſimiliter quicunque $\left(n\times\overline{\frac{n+3}{2}}\right)$ numeri, ſint correſpondentes radices incognitæ quantitatis (y); æquationis vero m dimenſionum coefficientes infinitis modis variari poſſint:

Sint n & m inter ſe æquales, tum haud aſſumi poſſunt coefficientes utriuſque æquationis, ita ut quicunque $\left(n\times\overline{\frac{n+3}{2}}\right)$ numeri ſint radices incognitæ quantitatis (x), & quicunque etiam $\left(n\times\overline{\frac{n+3}{2}}\right)$ numeri ſint correſpondentes radices incognitæ quantitatis (y): ſi enim hoc ſit, utraque æquatio eadem fiet: ſed aſſumi poſſunt coefficientes infinitis modis, ita ut quicunque $\left(n\times\overline{\frac{n+3}{2}}-1\right)$ numeri ſint reſpective correſpondentes valores incognitarum quantitatum (x & y.)

EXEMPLUM.

Sint duæ quadraticæ æquationes, duas incognitas quantitates habentes

$$y^2+\overline{a+bx}\times y+c+dx+ex^2=0 \text{ \&}$$
$$y^2+\overline{A+Bx}\times y+C+Dx+Ex^2=0$$

in utraque æquatione ſunt quinque $\left(2 \times \overline{\frac{2+3}{2}}\right)$ incognitæ coefficientes, ſubſtituantur pro incognitis quantitatibus (x & y) quicunque quinque numeri reſpective in utrâque æquatione, & oriuntur ex utrâque eædem quinque ſimplices æquationes, unde patet, quod utraque æquatio eadem erit; ergo tantùm ſubſtituti poſſunt in utrâque æquatione quatuor correſpondentes valores incognitarum quantitatum (x & y), quibus ſubſtitutis, reſultant ex utrâque quadraticâ, quatuor ſimplices æquationes quinque incognitas coefficientes habentes, & conſequenter in utrâque pro unâ incognitâ coefficiente aſſumi poteſt quicunque numerus, at cavendum eſt, ne idem numerus pro eâdem coefficiente in utriſque æquationibus aſſumatur.

Sint quatuor valores incognitæ quantitatis (y) 0, 2, 4, 2, & quatuor correſpondentes valores incognitæ quantitatis (x) 0, 1, 1, 2, quibus ſubſtitutis, & in utrâque quadraticâ pro unâ coefficiente (b & B) aſſumatur reſpective $+4$ & -2: & reſultant ex primâ quadraticâ, quatuor ſimplices æquationes

$$c=0$$
$$4+2a+8+e+d=0$$
$$16+4a+16+e+d=0$$
$$4+2a+16+4e+2d=0$$ unde ſequitur

primam æquationem eſſe $y^2+\overline{4x-10}\times y-8x^2+16x=0$; & eodem modo invenitur ſecunda $y^2-\overline{2x+4}\times y-2x^2+10x=0$.

COROLLARIUM.

Numerus correſpondentium radicum incognitarum quantitatum (x & y), quæ generaliter aſſumi poſſunt, æqualis eſt numero incognitarum datæ æquationis coefficientium.

Eadem eſt methodus ratiocinandi de pluribus æquationibus plures incognitas quantitates habentibus: ſed obſervandum eſt æquationem n dimenſionum, m incognitas quantitates involventem,

$\overline{n+1} \times \frac{\overline{n+2}}{2} \times \frac{\overline{n+3}}{3} \ldots \frac{\overline{n+m}}{m} - 1$ incognitas coefficientes habere poſſe.

LEM-

LEMMA VI.

Sint duæ æquationes, duas incognitas quantitates (x & y) *habentes, & ſint* α, β, γ, δ *&c. radices incognitæ quantitatis* (y), *&* π, ρ, σ, τ *&c. correſpondentes radices incognitæ quantitatis* (x)

invenire ſummam

$$\alpha^m \times \pi^n + \beta^m \times \rho^n + \gamma^m \times \sigma^n + \delta^m \times \tau^n + \text{\&c.}$$

Supponantur $y^m \times x^n = z$, & in datis æquationibus pro (y) ſubſtituatur ejus valor $\left(\dfrac{z^{\frac{1}{m}}}{x^{\frac{n}{m}}}\right)$; & transformentur duæ æquationes reſultantes in unam, ut exterminetur incognita quantitas (x); & reſultat æquatio cujus radix eſt z, & cujus radicum ſumma erit ſumma quæſita.

LEMMA VII.

Invenire ſummam omnis hujuſmodi producti

$$\alpha^m \pi^n \beta^n \rho^b \gamma^o \sigma^c \text{ \&c.} + \alpha^m \pi^a \beta^m \rho^b \delta^o \tau^c + \text{\&c.}$$

ſint

$$p^m \times Q^a = \alpha^m \pi^a + \beta^m \rho^a + \gamma^m \sigma^a + \delta^m \tau^a + \text{\&c.}$$
$$p^n \times Q^b = \alpha^n \pi^b + \beta^n \rho^b + \gamma^n \sigma^b + \delta^n \tau^b + \text{\&c.}$$
$$p^o\ Q^c = \alpha^o \pi^c + \beta^o \rho^c + \gamma^o \sigma + \delta^o \tau^c + \text{\&c.}$$

&c. &c. &c.

$$p^{m+n} \times Q^{a+b} = \alpha^{m+n} \pi^{a+b} + \beta^{m+n} \rho^{a+b} + \gamma^{m+n} \sigma^{a+b} + \text{\&c.}$$
$$p^{m+o} \times Q^{a+c} = \alpha^{m+o} \pi^{a+c} + \beta^{m+o} \rho^{a+c} + \gamma^{m+o} \sigma^{a+c} + \text{\&c.}$$
$$p^{n+o} \times \gamma^{b+c} = \alpha^{n+o} \pi^{b+c} + \beta^{n+o} \rho^{b+c} + \gamma^{n+o} \sigma^{b+c} + \text{\&c.}$$

&c. &c. &c.

$$p^{m+n+o} \times Q^{a+b+c} = \alpha^{m+n+o} \pi^{a+b+c} + \beta^{m+n+o} \rho^{a+b+c} + \text{\&c.}$$

&c. &c, &c.

& ſic obſervatâ lemmatis ſecundi ſubſtitutionis lege; ſcribatur

$$A = p^m\ Q^a \times p^n Q^b \times p^o Q^c \text{ \&c.}$$

$$B = \frac{p^{m+n}\ Q^{a+b} \times A}{p^m Q^a \times p^n Q^b} + \frac{p^{m+o}\ Q^{a+c} \times A}{p^m Q^a \times p^o Q^c} + \text{\&c.}$$

$$C = \frac{p^{m+n+o} Q^{a+b+c} \times A}{p^m Q^a \times p^n Q \times p^o Q^c} + \frac{p^{m+n+r} Q^{a+b+d} \times A}{p^m Q^a \times p^n Q^b \times p^r Q^d} + \text{\&c.}$$

$$D = \frac{p^{m+n+o+r} \mathfrak{Q}^{a+b+c+d} \times A}{p^m \mathfrak{Q}^a \times p^n \mathfrak{Q}^b \times p^o \mathfrak{Q}^c \times p^r \mathfrak{Q}^d} + \text{\&c.}$$

$$BB = \frac{p^{m+n} \mathfrak{Q}^{a+b} \times p^{o+r} \times \mathfrak{Q}^{c+d} \times A}{p^m \mathfrak{Q}^a \times p^n \times \mathfrak{Q}^b \times p^o \mathfrak{Q}^c \times p^r \mathfrak{Q}^d} + \frac{p^{m+o} \mathfrak{Q}^{a+c} \times p^{n+r} \times \mathfrak{Q}^{b+d} \times A}{p^m \mathfrak{Q}^a \times p^n \mathfrak{Q}^b \times p^o \mathfrak{Q}^c \times p^r \mathfrak{Q}^d} + \text{\&c.}$$

&c. &c.

ſumma quæſita eſt

$$A - B + 2C - 1 \times 2 \times 3 D + 1 \times 2 \times 3 \times 4 E - \text{\&c.}$$
$$+ \; 1 \times 1 BB - 1 \times 1 \times 2 \; BC + \text{\&c.}$$

lex, quam obſervat hæc ſeries, e lemmate ſecundo petenda eſt.

PROB. XI.

Transformare duas æquationes, duas incognitas quantitates (x & y) habentes in unam, cujus incognita quantitas aſſignabilem habet relationem ad incognitas datarum æquationum quantitates, (x & y).

Aſſumantur radices incognitarum quantitatum tanquam cognitæ, *i. e.* ſint radices incognitæ quantitates (x); $\alpha, \beta, \gamma, \delta$, &c. & correſpondentes radices incognitæ quantitatis (y); π, ρ, σ, τ &c. tum ex aſſignabili relatione inveniantur radices quæſitæ quantitatis; & conſequenter per lemma præcedens, ſumma ejus radicum, quæ eſt coefficiens ſecundi æquationis termini.

Ducantur quæque duæ, tres &c. radices quæſitæ quantitatis in ſeſe & lemmatis præcedentis ope, reſultantium contentorum inveniri poſſunt aggregata, quæ ſunt reſpective coefficientes, tertii, quarti &c. quæſitæ æquationis terminorum: unde ſequuntur quæſitæ æquationis coefficientes & conſequenter æquatio ipſa.

COROLLARIUM.

Eodem fere modo transformari poſſunt duæ vel plures æquationes in duas vel plures alias, quarum incognitæ quantitates aſſignabiles habent relationes ad incognitas datarum æquationum quantitates.

PROB. XII.

Invenire datarum æquationum radices, quæ datam inter ſe habent relationem.

1. Sint radices incognitarum quantitatum (x & y) ſigillatim datæ, vel algebraicam relationem inter ſe habeant correſpondentes incognitarum

nitarum quantitatum (x & y) radices, pro incognitis quantitatibus (x & y) ſubſtituantur earum valores & quantitatum reſultantium communes diviſores erunt radices quæſitæ.

2. Si æquationes datam relationem exprimentes diverſas ejuſdem incognitæ quantitatis radices involvunt, a ſecundo problemate & capite tertio ſequitur ejus ſolutio.

COROLLARIUM.

Datis duabus æquationibus, duas vel plures incognitas quantitates habentibus, ex hoc problemate inveniri poteſt, utrum duæ vel plures radices ſunt inter ſe æquales; facilius vero per ſubſequentem methodum.

Sint duæ æquationes duas incognitas quantitates habentes

$$y^n + \overline{a+bx} \times y^{n-1} + \overline{c+dx+ex^2} \times y^{n-2} \text{ \&c.} = 0.$$

$$y^m + \overline{A+Bx} \times y^{m-1} + \overline{C+Dx+Ex^2} \times y^{m-2} + \text{ \&c.} = 0.$$

ſint duæ radices incognitæ quantitatis (y) inter ſe æquales & erit

$$ny^{n-1} + \overline{n-1} \times \overline{a+bx} \times y^{n-2} + \text{ \&c.} = 0$$

$$\text{\&} \quad my^{m-1} + \overline{m-1} \times \overline{A+Bx} \times y^{m-2} + \text{ \&c.} = 0.$$

Inveniantur harum æquationum communes diviſores, & inveniuntur duæ vel plures æquales radices.

PROB. XIII.

Transformare (m) *æquationes, vel* (m+1) *vel plures incognitas quantitates habentes in unam, ita ut exterminentur* (m—1) *incognitæ quantitates, etiamque ſi æquationis reſultantis incognitæ quantitates vel ſint rationales quantitates vel integri numeri, exterminatæ etiam erunt vel rationales quantitates vel integri numeri.*

1. Per noni problematis methodum primam transformentur (m) æquationes in (m—1), ita ut exterminetur incognita quantitas (x), et ſi in operatione reſultat æquatio ſimplicem ſolummodo poteſtatem incognitæ quantitatis (x) habens, tum ita reducantur (m) æquationes in (m—1), ut exterminetur incognita quantitas (x), etiamque ſi (m—1) æquationum reſultantium incognitæ quantitates ſint rationales quantitates, exterminata (x) erit rationalis quantitas; et ſic iteratâ operatione exterminentur tandem (m—1) incognitæ quantitates,

2. Si prior methodus haud ita transformat æquationes, ut exigat problema, recurrendum eſt ad ſubſtitutionem *i. e.* pro incognitâ quan-

titate

titate (x) ſubſtituatur rationalis functio reliquarum incognitarum quantitatum & novæ quantitatis (v); & deinde inveniendum eſt, annon per methodum præcedentem transformationem deſideratam admittit.

3. In genere, pro ſingulâ incognitâ quantitate exterminandâ, ſubſtituatur rationalis functio æquationis reſultantis incognitarum quantitatum, ſcribantur hæ rationales functiones pro ſuis valoribus in datis æquationibus, æquationum reſultantium ſinguli termini nihilo vel inter ſe æquales fiant, & ex iis colligi poſſunt quæſitæ coefficientes.

PROB. XIV.

Datis (m) *æquationibus*, (m+1) *incognitas quantitates habentibus, invenire rationales vel integros earum valores, ſibi ipſis correſpondentes.*

Sit (z) quæcunque aſſumpta quantitas, pro ſingulâ $(m-1)$ incognitâ quantitate ſubſtituatur in genere rationalis functio aſſumptæ quantitatis $(z,)$ & æquationum reſultantium fiant ſinguli termini nihilo reſpective æquales; reſultantium æquationum ope inveniri poſſunt rationalium functionum aſſumptæ quantitatis (z) coefficientes quæſitæ.

COROLLARIUM.

Methodo in Capite tertio datâ, inveniri poſſunt innumeræ (m) æquationes, $(m+1)$ incognitas quantitates habentes, ita ut correſpondentes radices incognitarum quantitatum ſint rationales functiones aſſumptæ quantitatis (z).

Si hæc methodus haud invenit incognitarum quantitatum rationales valores, ſibi ipſis correſpondentes, tum haud admittunt generalem regulam æquationes, ſed in multis caſibus approximatio innumeros invenit rationales valores.

Sint etiam (m) æquationes, eundum (m) numerum incognitarum quantitatum habentes, & earum rationales valores per approximationem inveniri poſſunt.

EXEMPLUM.

Sit irrationalis quantitas

$$a+\sqrt{b}$$

invenire ejus radicem (n) poteſtatis, quæ ſit hujuſce formulæ $(x+\sqrt{y}.)$

Inveniatur quam proxime valores quantitatis

$\overline{a+\sqrt{b}}^{\frac{1}{n}}=x+\sqrt{y}$ & $\overline{a^2-b}^{\frac{1}{n}}=x^2-y$ & conſequenter

a^2-

$$\frac{\overline{a^2-b}^{\frac{1}{n}}}{\overline{a+\sqrt{b}}^{\frac{1}{n}}} = \frac{x^2-y}{x+\sqrt{y}} = x-\sqrt{y} \text{ unde ſequitur}$$

$$\frac{\overline{a+\sqrt{b}}^{\frac{1}{n}} + \overline{a-\sqrt{b}}^{\frac{1}{n}}}{2} = x \ \& \ \frac{\overline{a+\sqrt{b}}^{\frac{1}{n}} - \overline{a-\sqrt{b}}^{\frac{1}{n}}}{2} = \sqrt{y},$$

quibus valoribus (x & y) quam proxime inventis, pro iis vel earum poteſtatibus aſſumantur proximi integri numeri & fit exemplum.

Et ſic de extrahendis radicibus diverſis poteſtatibus affectis, ſed ſubſequentis problematis ope omnia hujuſmodi problemata reſolvi poſſunt.

Si vero ſit incognita quantitas, vel incognitæ quantitates, dati numeri, aut datorum numerorum diviſores; approximationis ope earum integri valores minime inveniri poſſunt, nulla enim regula pro inveniendis numerorum diviſoribus dari poteſt.

Et ſic de multis aliis caſibus & ad integros & rationales pertinentibus.

Æquationum, quarum incognitæ vel incognitarum quantitatum integri valores, ſunt datorum numerorum diviſores, facile eſt dare exempla.

Ex. gr. Sint (n) æquationes ($n+m$) incognitas quantitates habentes, & ſint omnes poteſtates (m) incognitarum quantitatum in datis æquationibus, multiplicatæ in incognitam quantitatem (x,) ita transformentur (n) æquationes in unam, ut exterminentur omnes præter (m) incognitas quantitates prædictas, & (x) in quam ductæ fuerunt, & invenitur numerus, cujus diviſor eſt incognita quantitas (x.) &c.

PROB. XV.

Datis duabus æquationibus, duas incognitas quantitates (x & y) *habentibus, invenire rationales incognitæ, quantitatis* (y), *ſi quas forte habet, radices.*

1. In datis æquationibus pro incognitâ quantitate (y) ſubſtitue tres vel plures terminos hujus progreſſionis arithmeticæ — 1, 0, 1, 2, &c; duas quantitates reſultantes ex ſingulâ ſubſtitutione colloca juxta dimenſiones literæ (x,) ita ut illi termini primum locum occupent, in quibus litera iſta (x) eſt plurimarum dimenſionum &c. & perpetuâ ablatione minoris a majori & reliqui de ablato, exterminetur incognita

quan-

quantitas (x.) & ſit arithmetica ſeries $z+1$, z, $z-1$, $z-2$ diviſores numerorum reſultantium reſpective, tum diviſor (z) erit rationalis incognitæ quantitatis (y) quæſita radix.

EXEMPLUM.

Sint duæ æquationes

$$y^2+\overline{1+3x}\times y+2x^2+4x-18=0.$$
$$y^2+\quad 3x\times y+3x^2+4x-17=0.$$

pro y ſubſtituatur (o) & reſultant quantitates,

$$2x^2+4x-18$$
$$3x^2+4x-17$$

transformentur hi termini, ita ut exterminetur (x), & operatio eſt.

$\overline{2x^2+4x-18}\times 3 = 6x^2+12x-54$

$\overline{3x^2+4x-17}\times 2 = 6x^2+8x-34$ auferatur poſterior de priori & reſiduum erit $4x-20$ tum

$\overline{2x^2+4x-18}\times 2 = 4x^2+8x-36$

$\overline{4x-20}\times x = 4x^2-20x$

reſiduum $28x-36$

$\overline{4x-20}\times 7 = 28x-140$

reſiduum $=$ 104.

Et ſic ſubſtituantur 1 & —1 & numeri reſultantes erunt 512, 972 reſpective, quos cum omnibus eorum diviſoribus colloco

1	512	1, 2, 4, 8, 16, 32, 64, 128, 256, 512,	1
0	104	1, 2, 4, 8, 13, 26, 52, 104	2
—1	972	1, 2, 3, 4, &c.	3

unica tantum eſt arithmetica progreſſio $z-1$, z, $z+1$, (1, 2, 3), cujus communis differentia eſt (1,) ergo numerus qui ſtat e regione termini (o,) eſt rationalis quantitatis (y) radix.

COR. I.

Eodem ferè modo quadratici, cubici &c. diviſores duarum vel plurium æquationum duas vel plures incognitas quantitates habentium inveniri poſſunt.

COR. II.

Sint duæ æquationes, duas incognitas quantitates habentes, & ſit (y) integer numerus, vel fractio; (x) etiam erit integer numerus vel fractio, niſi duæ, tres vel plures incognitæ quantitatis (x) radices ſint inter ſe æquales.

2. Eſt

2. Eſt etiam alia methodus hoc problema ſolvendi:

Sint duæ æquationes duas incognitas quantitates (x & y) habentes

$$T - Sy + Ry^2 - Qy^3 + \ldots y^n = 0$$

$$t - sy + ry^2 - qy^3 + \ldots y^m = 0$$

in quibus T, S, R, Q, &c. t, s, r, &c. denotant reſpective rationales integras functiones literæ (x), & ſit z rationalis integra radix quantitatis (y) quæſita, tum

$$\frac{T}{z},\ \frac{t}{z},\ \frac{Sz - T}{z^2},\ \frac{sz - t}{z^2},\ \frac{Rz^2 - Sz + T}{z^3},\ \frac{rz^2 - sz + t}{z^3} \text{ \&c.}$$

integri erunt numeri.

LIBER SECUNDUS.

CAP. I.

PROB. I.

Fig. 1. *DATA æquatione relationem inter abſciſſam* (AP $=$ x) *& ejus ordinatas* (PM $=$ y) *definiente, invenire æquationem, relationem definientem, inter abſciſſam* (ap), *quæ ſecat abſciſſam* (AP) *ad datum angulum* (p*NP*), *& in dato puncto* (N), *& ordinatas* (pM &c.) *eam* (ap) *ſecantes in dato angulo* (MpN).

Sit radius (1) & ſinus angulorum *APM*, *pNP* & *MpN* reſpective s, S & σ, tum ſinus angulorum *NrP* & *rMp* erunt,

$s\sqrt{1-S^2}+S\sqrt{1-s^2}=c$, & $\sigma\sqrt{1-\overline{s\sqrt{1-S^2}+S\sqrt{1-s^2}}^{\,2}}+\overline{s\sqrt{1-S^2}+S\sqrt{1-s^2}}\times\sqrt{1-\sigma^2}=C$: ſubſtituantur pro lineis *NA*, *Na*, *ap*, *pM* reſpective a, b, z, v, & erit

$$z+b-\frac{C}{c}v=Nr, \text{ \& conſequenter } \frac{cz+cb-Cv}{s}+a=x$$

$$\& \; y=rM+rP=\frac{\sigma v}{c}+\frac{Sz+Sb}{s}-\frac{CS}{cs}v=\frac{\overline{\sigma s-CS}}{cs}\times v+\frac{Sz}{s}$$

$+\frac{Sb}{s}$ quibus quantitatibus pro ſuis valoribus (x & y) in datâ æquatione ſubſtitutis, reſultat æquatio relationem inter abſciſſam (ap) & ejus ordinatas (pM) definiens.

COR. I.

Fig. 2. Datâ æquatione relationem inter abſciſſam (*AP*, x) & ejus ordinatas (*PM*, v) definiente, invenire æquationem definientem relationem inter abſciſſam ($Ap=z$) & ejus ordinatas ($pM=v$).

Sint

Sint ſinus angulorum APM & ApM reſpective s, S; & ſinus anguli (pMP) erit $s\sqrt{1-S^2} + S\sqrt{1-s^2}$; pro ordinata PM, y, in datâ æquatione ſubſtituatur ejus valor $\frac{S \times v}{s}$, & pro abſciſſâ AP, (x) ſubſtituatur $z \pm \frac{s\sqrt{1-S^2} + S\sqrt{1-s^2}}{s}v$, & fit.

COR. II.

Datis ut in ſuperioribus, invenire æquationem, relationem inter abſciſſam Np, z, & ordinatam Mp, v, exprimentem: Fig. 3.

Sint ſinus datorum angulorum APM & PNp reſpective s & S; tum ſinus anguli (MpN) erit $s\sqrt{1-S^2} + S\sqrt{1-s^2}$; pro abſciſſâ AP (x) in datâ æquatione ſcribatur ejus valor $\frac{s\sqrt{1-S^2} + S\sqrt{1-s^2}}{s}z + AN$ (a) & pro ordinatâ (y); $v + \frac{Sz}{s}$ & reſultat æquatio quæſita.

COR. III.

Sit æquatio relationem definiens inter abſciſſam (x) & ejus ordinatas (y) hujuſce formulæ

$$y^n + ay^{n-1} + \overline{c+dx} \times y^{n-2} \text{ \&c.} = 0.$$

vel $$y^n + bxy^{n-1} + \overline{c+ex^2} \times y^{n-2} \text{ \&c.} = 0.$$

vel etiam ſi modo ſit æquatio

$$y^n + \overline{a+bx}\,y^{n-1} + \overline{c+dx+ex^2}\,y^{n-2} + \overline{f+gx+hx^2+kx^3}\,y^{n-3} + \text{ \&c.} = 0.$$

& duæ æquationes $bq^{n-1} + 2eq^{n-2} + 3kq^{n-3} + \text{\&c.} = 0.$

& $aq^{n-1} + dq^{n-2} + hq^{n-3} + \text{\&c.} = 0.$

communem habeant diviſorem, tum abſciſſa erit diameter, ſin aliter, vero non.

Hoc corollarium conſtat ſubſtitutione in corollario primo datâ.

COR. IV.

Sit æquatio relationem inter abſciſſam (NP) & ejus ordinatas (PM) exprimens, (n) dimenſionum Fig. 4.

$$Ay^n + \overline{a+bx}\,y^{n-1} + \overline{c+dx+ex^2}\,y^{n-2} + \text{ \&c.} = 0.$$

Per hoc problema transforn ur hæc æquatio in æquationem relationem inter abſciſſam (Np,z) & rdinatas (Mp=v) exprimentem: Sint anguli (MpN & NPM) enter eorum ſinus s & σ, dati. Æqua-

Æquationis, ex hâc transformatione reſultantis, termini $(z v^{n-1})$ coefficiens erit

$$n \times \left.\overline{\frac{\sigma s - CS}{cs}}\right|^{n-1} \times \frac{S}{s} A + \overline{\left.\overline{\frac{\sigma s - CS}{cs}}\right|^{n-1} \times \frac{c}{s} - \overline{n-1} \times \left.\overline{\frac{\sigma s - CS}{cs}}\right|^{n-2} \times \frac{C}{s}} \times \frac{S}{s} \times b + \overline{\left.\overline{\frac{\sigma s - CS}{cs}}\right|^{n-2} \times \frac{-2Cc}{s^2} + \overline{n-2} \times \left.\overline{\frac{\sigma s - CS}{cs}}\right|^{n-3} \times \frac{\overline{C^2}}{s^2}} \times e \ \&c.$$

Pro C & c ſubſtituantur in hac coefficiente earum valores, & fiat coefficiens nihilo æqualis, in reſultanti æquatione, a ſurdis liberatâ, incognita quantitas (S), ſurget ad $(8n)$ dimenſiones, & tot poſſunt eſſe diverſi valores incognitæ quantitatis S, (*i.e.* ſinus anguli PNp,) quot ſunt ejus dimenſiones; quatuor vero valores incognitæ quantitatis in eundem angulum referuntur, & conſequenter $(2n)$ poſſunt eſſe diametri, quæ ſecant ſuas ordinatas ad datum angulum (NpM).

Hinc ſequitur curvam (n) dimenſionum non poſſe plures quam (n) axes (*i.e.* diametros ſuas ordinatas ad rectum angulum ſecantes) habere.

COROLLARIUM V.

Æquationis exprimentis relationem inter abſciſſam (x) & ejus correſpondentes ordinatas (y), ſint termini, in quibus quantitatum $(x$ & $y)$ dimenſiones inveniuntur maximæ, $(\overline{Ay+Bx}^{n})$; & omnes curvæ diametri erunt inter ſe parallelæ, & vice verſâ omnis linea cuicunque diametro parallela, erit diameter.

Pro $y + \frac{B}{A}x$ ſubſtituatur (v), & pro $(x)\, Sz$, & reſultat æquatio, in quâ terminus, in quo dimenſiones abſciſſæ & ordinatæ inveniuntur maximæ, erit v^n, *i. e.* æquatio reſultans eſt hujuſce formulæ

$$v^n + \alpha v^{n-1} + \overline{\gamma + \delta z}\, v^{n-2} + \&c. = 0,$$

& ſimiliter æquatio, relationem inter quamcunque abſciſſam huic parallelam & ejus correſpondentes ordinatas exprimens, erit ejuſdem formulæ.

Sin æquatio relationem inter abſciſſam & correſpondentes ordinatas exprimens, ſit hujus formulæ, tum abſciſſa erit diameter.

Linea,

Linea, huic diametro haud parallela, non erit diameter: æquatio enim relationem inter abſciſſam & ejus correſpondentes ordinatas exprimens, non erit prædictæ formulæ, & conſequenter abſciſſa non erit diameter.

In hoc caſu duæ & haud plures poſſunt eſſe diametri, quæ datum angulum cum ſuis ordinatis faciunt, ni rectus eſt angulus, & tum unica ſolummodo erit diameter.

Non conſtare poteſt curva, cujus omnes diametri ſunt parallelæ, ex hyperbolicis cruribus, ni omnes eorum aſymptotes ſint inter ſe parallelæ.

Non conſtat ex parabolicis cruribus, ni omnia crura ſunt concava vel convexa eâdem directione.

Conſtant hæc e ſubſequentibus.

COROLLARIUM VI.

Æquationis relationem inter abſciſſam & ejus correſpondentes ordinatas exprimentis, ſint nulli termini, in quibus inveniuntur $(n-1)$ dimenſiones, & omnes curvæ diametri ſeſe interſecant in puncto, quod eſt abſciſſæ vertex. Fig. 4.

Per corollarium ſecundum transformetur hæc æquatio in alteram relationem inter abſciſſam Ap (z), utcunque inclinatam ad datam abſciſſam (AP), & ejus correſpondentes ordinatas pM (v) exprimentem; & ſemper reſultat æquatio hujus formulæ

$$v^n + bzv^{n-1} + \overline{c+ez^2}\, v^{n-2} \text{ \&c.} = o$$

& per corollarium tertium abſciſſam hujus æquationis eſſe diametrum conſtat, & conſequenter omnes lineæ per hoc centrum ductæ, erunt diametri.

Sit curva (n) dimenſionum, (n) habens aſymptotes, ſeſe ſecantes in uno eodemque puncto; tum iſtud punctum erit curvæ centrum.

Si centrum non eſt vertex abſciſſæ, ſubſtituatur pro $(AP=x)$ abſciſſâ, $(ap+\alpha=z+\alpha)$, & pro ordinatâ (y) $(pM+\beta=v+\beta)$, & ita aſſumantur quantitates α & β, ut nihilo fiant æquales omnes termini, in quibus dimenſiones ſunt $(n-1)$, & vertex novæ abſciſſæ erit centrum. Fig. 5.

Sit

Si curva habet parabolicum crus hujuſce formulæ ($y^{m+r}=ax^{m}$); in quâ (y & x) ſunt abſciſſæ & ordinatæ reſpective, & (r) non eſt minor quam (2), tum neceſſario habet generale centrum.

Curva (n) dimenſionum non habere poteſt plures quam ($n-1$) parallelas diametros vel diametros ſeſe interſecantes in uno eodemque puncto, ni omnes ejus diametri ſunt parallelæ vel centrum habent generale.

Fig. 2. Sit enim angulus APM rectus, & pro ordinata (PM) ſubſtituatur Sv, & pro ejus abſciſſâ $z-\sqrt{1-S^2}\,v$ & reſultat coefficiens termini (zv^{n-1})

$$bS^{n-1}-2eS^{n-2}\sqrt{1-S^2}+3kS^{n-3}\times\overline{1-S^2}\ \&c.$$

fiat hæc coefficiens nihilo æqualis, & ita transformetur reſultans æquatio, ut exterminentur irrationales quantitates; & reſultat æquatio ($2\times\overline{n-1}$) dimenſionum, duos vero valores habens, inter ſe æquales, alterum affirmativum, alterum vero negativum, & conſequenter ad eandem ordinatarum inclinationem referuntur hi duo valores, & numerus angulorum & parallelarum diametrorum non major eſt quam ($n-1$,) ni omnes ſunt parallelæ.

Eodem modo ratiocinari licet de diametris in uno eodemque puncto ſeſe ſecantibus.

COROLLARIUM VII.

Curva poteſt habere unicam ſolummodo diametrum, quam dico generalem diametrum, cui etiam lineæ utcunque inclinatæ erunt ordinatæ.

Hujuſmodi vero diametrum habet curva, ſi æquationis relationem inter abſciſſam & ejus correſpondentes ordinatas exprimentis, termini, in quibus dimenſiones inveniuntur maximæ, ſint $\left(\overline{Ay+Bx}^{\,n}\right)$ & nulli ($n-1$) dimenſionum inveniuntur termini.

Hoc in caſu, e prædictis conſtat, curvam & habere omnes ſuas diametros inter ſe parallelas, & omnes ſeſe ſecare in uno eodemque puncto, & conſequenter curva unicam ſolummodo habet diametrum.

Transformetur enim æquatio relationem inter abſciſſam & ejus correſpondentes ordinatas exprimens, in alteram, in quâ, terminus, in quo inveniuntur maximæ dimenſiones, ſit ſolummodo (v^{n}), & ($n-1$) dimenſionum ſint nulli termini tum abſciſſa erit generalis diameter.

Si curva habet parabolicum crus hujus generis ($y^{m+r}=ax^{m}$), in quâ r major eſt quam (3), tum generalem habet diametrum.

Hæ

Hæ ſunt diametri primi generis, quarum ordinatæ curvam ſecant in (n) punctis, & eorum numerus, uno quidem caſu excepto, eſt infinitus.

Si vero ordinatæ ſecant curvam in ($n-1$, $n-2$, $n-3$, &c.) punctis reſpective, & hæ ordinatæ ita ſecantur per rectas, ut ſumma ſegmentorum in omnibus utrinque ſint inter ſe æqualis, tum hæ lineæ reſpective dici poſſunt diametri ſecundi, tertii, &c. generis; eodem modo diſtingui poſſunt abſciſſæ diverſorum generum.

COROLLARIUM VIII.

Non poteſt ſumma diametrorum, ſecundi, tertii, quarti &c. generum in curva (n) dimenſionum, major eſſe quam $\left(\frac{n}{3}\right)$.

Sit enim æquatio relationem inter abſciſſam & ejus correſpondentes ordinatas cum eâ rectum angulum facientes

$$y^n + \overline{a+bx}\,y^{n-1} + \overline{c+dx+ex^2}\,y^{n-2} + \overline{f+gx+hx^2+kx^3}\,y^{n-3} + \&c. = 0.$$

Transformetur hæc æquatio, ita ut abſciſſæ eædem maneant, ordinatarum vero mutetur inclinatio: *i. e.* per corollarium primum pro ordinata (y), ſubſtituatur Sv; & pro abſciſſa (x), $z + \sqrt{1-S^2}\,v$, & reſultat æquatio, Fig. 2.

$$S^n + b\sqrt{1-S^2} \times S^{n-1} + e\,\overline{1-S^2} \times S^{n-2} + k\,\overline{1-S^2}^{1\frac{1}{2}} \times S^{n-3} + \&c.\ v^n$$

Ut fiat diameter vel abſciſſa ſecundi, generis, neceſſe eſt, ut evaneſcat terminus (v^n), & conſequenter ejus coefficiens nihilo fiat æqualis, *i. e.*

$$S^n + bS^{n-1} \times \sqrt{1-S^2} + eS^{n-2} \times \overline{1-S^2} + \&c. = 0.$$

quâ æquatione reductâ, aſcendit æquatio reſultans ad $2n$ dimenſiones, cujus radix $2n$ habet valores, duo vero ſinus in eandem inclinationem referuntur, & conſequenter (n) ſolummodo poſſunt eſſe inclinationes, in quibus ordinatæ curvam (n) dimenſionum haud ſecant in (n) punctis vel poſſibilibus vel impoſſibilibus.

Ut fiat diameter vel abſciſſa tertii generis, neceſſe eſt, ut evaneſcat prædictus terminus (v^n), & termini (zv^{n-1} & v^{n-1}); coefficientes vero horum terminorum reſpective ſunt

$$bS^{n-1} + 2eS^{n-2}\sqrt{1-S^2} + 3kS^{n-3} \times \overline{1-S^2} + \&c.$$

$$\&\quad aS^{n-1} + dS^{n-2}\sqrt{1-S} + hS^{n-3} \times \overline{1-S^2} + \&c.$$

Fiant hæ coefficientes nihilo reſpective æquales, & reſultant duæ æqua-

æquationes $2 \times \overline{n-1}$ dimenſionum, & conſequenter numerus valorum incognitæ quantitatis (S) eſt $2\overline{n-1}$; & exinde numerus diverſarum inclinationum ad abſciſſam (AP) erit $n-1$; ſed ſecundæ æquationis radices ſunt limites inter primæ æquationis radices, & conſequenter non plures poſſunt eſſe diverſæ radices harum duarum æquationum inter ſe æquales, quam $\frac{n}{2}$.

Terminorum (z^2v^{n-2}, zv^{n-2}, v^{n-2}) coefficientes ſunt reſpective

$$eS^{n-2} + 3kS^{n-3}\sqrt{1-S^2} + \&c.$$
$$dS^{n-2} + 2bS^{n-3}\sqrt{1-S^2} + \&c.$$
$$cS^{n-2} + gS^{n-3}\sqrt{1-S^2} + \&c.$$

Fiant hæ coefficientes & prima

$$S^n + bS^{n-1}\sqrt{1-S^2} + eS^{n-2}\,\overline{1-S^2} + kS^{n-3}\,\overline{1-S^2}|^{1\frac{1}{2}} + \&c.$$

nihilo reſpective æquales, & ex hâc & primâ æquatione ſequitur haud plures diverſas radices eſſe harum duarum æquationum quam $\frac{n}{3}$, & ſic deinceps.

Non poſſunt igitur eſſe plures quam $\frac{n}{m}$ inclinationes parallelarum ordinatarum, quæ ſecant curvam in ($n-m$) punctis.

Non poſſunt eſſe plures quam $\frac{n}{m+1}$ diametri, cujus ordinatæ ſecant curvam in ($n-m$) punctis.

Non poſſunt eſſe plures quam (n) diverſæ ordinatarum inclinationes curvam in (n) punctis haud ſecantium vel poſſibilibus vel impoſſibilibus.

Non poteſt igitur ſumma diametrorum ſecundi, tertii, &c. ordinum eſſe major quam $\frac{n}{3}$.

Diametri vel potius axes nonnunquam definiuntur eſſe lineæ, quæ ita ſecant ordinatas ſibi ipſis ad rectum angulum inſiſtentes, ut ordinatæ utrinque ſint inter ſe æquales.

COROLLARIUM IX.

Sit curva (MNS), cujus linea (AP) eſt hujuſmodi diameter, tum ordinatæ utrinque ſunt inter ſe æquales *i.e.* negativi & affirmativi ordinatæ valores ſunt æquales, & æquatio erit hujus formulæ Fig. 6.

$$y^{2n}+\overline{c+dx+ex^2}\,y^{2n-2}+\overline{f+gx+hx^3+kx^3+lx^4}\times y^{2n-4}+\text{\&c.}=0$$

Ita ſcribatur hæc æquatio, ut termini progrediantur ſecundum dimenſiones incognitarum quantitatum (x & y) iis contentarum,

$$\alpha+\beta x+\gamma y^2+\delta x^2+\epsilon xy^2+\zeta x^3+\eta y^4+\theta x^2y^2+ix^4+\text{\&c.}=0$$

transformetur hæc æquatio, ut relationem inter abſciſſam ($Ap=z$) & ejus ordinatas ($pM=v$) rectum angulum cum abſciſſa (Ap) facientes, exprimat, *i.e.* ſi modo (q) ſit ſinus anguli PAp, pro

PM (y) ſubſtituatur $qz+\sqrt{1-q^2}\times v$ &

pro AP (x) ſcribatur $\sqrt{1-q^2}\times z-qv$, quibus quantitatibus pro ſuis valoribus ſubſtitutis, & reſultantis æquationis terminis, in quibus inveniuntur impares ordinatæ (v) dimenſiones, nihilo æquales eſſe ſuppoſitis, reſultant æquationes, e quibus cognoſci poſſunt diametri.

Ex. gr. Terminus unius dimenſionis eſt (βx), in eo pro (x) ſcribatur $\sqrt{1-q^2}\,z-qv$, & reſultat $\beta\sqrt{1-q^2}\,z-\beta qv$; ſed v eſt impar ordinatæ dimenſio, & conſequenter βqv nihilo debet eſſe æqualis, & exinde vel,

1. Sit β data quantitas & conſequenter $q=0$, tum curva non habet aliam diametrum præter datam, vel,

2. Sit $\beta=0$, & nulli ſint termini, in quibus una ſolummodo continetur dimenſio.

In terminis duarum dimenſionum ($\gamma y^2+\delta x^2$) pro (x & y) ſcribantur prædicti valores, & reſultant

$$\begin{array}{llll}\gamma\times\overline{1-q^2}\times v^2 & +2\gamma q\sqrt{1-q^2}\,zv & +\gamma q^2 & z^2\\ \delta\times\quad q^2 & -2\delta q\sqrt{1-q^2} & +\delta\,\overline{1-q^2} & \end{array}$$

ut fiat (Ap) diameter vel axis, terminus $2\times\overline{\gamma-\delta}\times q\sqrt{1-q^2}\,zv$, in quo impar eſt dimenſio ordinatæ, debet eſſe nihilo æqualis.

Si vero $\overline{\gamma-\delta}\times q\sqrt{1-q^2}$ nihilo eſt æqualis; tum, ſi $\overline{\gamma-\delta}$ ſit data quantitas, vel q vel $\sqrt{1-q^2}$ nihilo eſt æquale, & q vel $=0$ vel $=1$. *i.e.* quoad hoc curva poteſt habere duas diametros ſecum angulum rectum facientes.

Si $\gamma-\delta=0$, tum quoad hoc curva poteſt habere infinitas hujuſmodi diametros; & hic caſus verus eſt in circulis, in quibus infinitæ dantur diametri.

In terminis ($\epsilon xy^2+\zeta x^3$) pro (x & y) ſcribantur prædicti valores, & reſultant

$$-\epsilon q\times\overline{1-q^2}\times v^3+\epsilon\,\overline{1-3q^2}\times\sqrt{\overline{1-q^2}}\,zv^2+\epsilon\times\overline{2q-3q^3}\,z^2v+\epsilon\,\overline{1-q^2}^{\,1\frac{1}{2}}z^3$$
$$-\zeta q^3 \qquad + \qquad 3\zeta q^2\sqrt{\overline{1-q^2}} \qquad -3\zeta q\times\overline{1-q^2}+\zeta\times\overline{1-q^2}^{\,1\frac{1}{2}}$$

ſed ut fiat diameter, neceſſe eſt, ut evaneſcant termini, ($\overline{\epsilon q\times\overline{1-q^2}+\zeta q^3}\,v^3$, $\overline{2\epsilon q-3\epsilon q^3-3\zeta q\times\overline{1-q^2}}\,z^2v$), in quibus inveniuntur impares ordinatæ (v) dimenſiones; e priori vero termino ſequitur $\epsilon=\frac{-\zeta q^2}{1-q^2}$, quâ quantitate pro ſuo valore in ſecundâ æquatione ſubſtitutâ, reſultat æquatio $\frac{-\zeta q^2}{1-q^2}\times\overline{2q-3q^3}-3\zeta q\times\overline{1-q^2}=0$, quâ reductâ, ſequitur vel $q=0$ vel $q^2=\frac{3}{4}$, & $q=\sqrt{\frac{3}{4}}=$ ſin. 60°; & conſequenter, ſi in datâ æquatione ſint quicunque termini, tres habentes dimenſiones ordinatæ & abſciſſæ, tum non poteſt habere curva plures quam tres diametros; & ſi tres ſint diametri, ſeſe interſecant in angulo 60°; & ϵ vel $=-3\zeta$ vel $=\zeta$; poſterior pars hoc modo deduci poteſt; ab æquatione $\epsilon q\times\overline{1-q^2}+\zeta q^3=0$ conſtat $\frac{\epsilon}{\epsilon-\zeta}=q^2$; quo valore in æquatione $\epsilon\times\overline{2-3q^2}-3\zeta\,\overline{1-q^2}=0$ ſubſtituto, reſultat æquatio $\epsilon\times\overline{2-\frac{3\epsilon}{\epsilon-\zeta}}-3\zeta\times\overline{1-\frac{\epsilon}{\epsilon-\zeta}}=0$, quâre ductâ, fit $\epsilon^2+2\zeta\epsilon-3\zeta^2=0$, cujus radices ($\epsilon$) ſunt vel $=-3\zeta$ vel $=\zeta$.

In terminis ($ny^4+\theta y^2x^2+ix^4$) quatuor dimenſionum ſubſtituantur pro (x & y) eædem quantitates, quæ prius ſubſtitutæ fuerunt, & quantitatum reſultantium fiant illæ, in quibus inveniuntur impares ordinatæ dimenſiones nihilo æquales, & exinde ſequitur, ſi $2n=2i=\theta$; tum quoad hos terminos infinitæ dari poſſunt diametri, ſin illæ quantitates haud ſunt inter ſe æquales, tum non plures quam quatuor

quatuor poſſunt eſſe diametri, & ſi quatuor ſunt diametri, angulum inter ſe faciunt 45° graduum.

Duæ poſſunt eſſe etiam ſolummodo diametri; ſed hæc reſtrictio oritur e terminis duarum dimenſionum abſciſſæ & ordinatæ.

In genere, ſint curvæ (n) dimenſionum, quæ (n) habent diametros, & ſi modò ſint (n) diametri, anguli, quos hæ diametri inter ſe faciunt, erunt $\frac{180^\circ}{n}$.

Si modo (n) eſt primus numerus, tum habet unam vel (n) vel nullam diametrum omnis hujuſmodi curva: & curva imparium dimenſinum non poteſt habere parem diametrorum numerum.

Conſtat methodus omnia hujus generis inveniendi.

Hæc facile demonſtrari poſſunt e geometricis principiis.

Hinc conſtàt, ſi curva duas vel plures habet diametros, tum generale habet centrum.

Hinc etiam conſtat, ſi ducantur per centrum lineæ æquales angulos primo, tertio, quinto &c. axe facientes, tum hæ diametri æquales angulos cum ſuis ordinatis faciunt.

Primus, tertius, quintus &c. axes erunt inter ſe æquales.

Eodem modo, quo inveniuntur axes, deduci poſſunt diametri earum ordinatas ad datum angulum ſecantes, & ſic deinceps.

COROLLARIUM X.

Sunt curvæ (n) dimenſionum, quæ ſecare poſſunt rectam lineam in (n) punctis; ſunt etiam curvæ $(2n)$ parium dimenſionum, quæ nullam interſecant rectam lineam in pluribus quam (2) punctis, & imparium dimenſionum, quæ nullam interſecant rectam lineam in pluribus quam (3) punctis; ſed obſervandum eſt, ut par erit maximus numerus punctorum, in quibus recta linea ſecare poteſt curvam parium dimenſionum, & impar, imparium dimenſionum.

COROLLARIUM XI.

Sit curva MmM, cujus æquatio relationem inter abſciſſam (AP) Fig. 7. & ejus correſpondentes ordinatas (PM) exprimens eſt,

$$Ay^n + \overline{a+bx}\,y^{n-1} + \overline{c+dx+ex^2}\,y^{n-2} + \&c. = 0.$$

ſint termini, in quibus maximæ inveniuntur dimenſiones incognitarum quantitatum $(y \,\&\, x)$

$$Ay^n + bxy^{n-1} + ex^2y^{n-}$$

ducantur duæ aliæ lineæ (PC, Pc) curvam in (n) punctis, & ſeſe interſecantes in puncto abſciſſæ (P), & cum eâ angulos facientes, quorum ſinus ſunt q & r reſpective, ſit (s) ſinus anguli (MPA), quem ordinatæ cum ſuâ diametro faciunt; & C, c ſinus angulorum, quos lineæ ſeſe interſecantes, in puncto (P) cum ordinatis (PM) faciunt, tum contentum ſub (n) lineis inter punctum (P) & curvam in directione (Pc) interceptis, erit ad contentum ſub (n) lineis inter punctum (P) & curvam in directione (PC) interceptis, in eadem ratione, quam habet

$$Aq^n + bq^{n-1}C + eq^{n-2}C^2 + \&c. \text{ ad } Ar^n + br^{n-1}c + er^{n-2}c^2 + \&c.$$

COROLLARIUM XII.

Datis duabus æquationibus eaſdem dimenſiones habentibus, relationem inter abſciſſas & earum correſpondentes ordinatas exprimentibus, invenire annon hæ duæ curvæ ſunt eædem.

Pro x & y in alterâ æquatione ſubſtituantur $nx+qv+r$ & $mz+ov+p$, & fiant reſultantis & datæ æquationis coefficientes eædem; ſi hoc poſſibile ſit, per problema inveniatur, utrùm ordinatarum & abſciſſæ mutatio admittit hujuſmodi ſubſtitutionem, & ſit.

Hic innumeras hujuſmodi propoſitiones adjicere licet.

PROB. II.

Datâ æquatione relationem inter curvæ abſciſſam & ejus ordinatas exprimente, invenire ejus crura in infinitum abeuntia.

Per infinitas ſeries inveniantur ſinguli ordinatæ (y) valores; terminis vero ſecundum abſciſſæ (x) dimenſiones progredientibus, ita ut abſciſſæ (x) dimenſiones fiant continuo minores i. e. ſit

$$y = Ax^m + \&c. + P + \frac{Q}{x^r} + \&c.$$

crus, quod deſignat hæc æquatio, erit parabolicum vel hyperbolicum, prout (m) eſt unitas vel (o), necne.

COROLLARIUM I.

Sit æquatio relationem inter abſciſſam & ejus correſpondentes ordinatas exprimens,

$$y^n + \overline{a+bx}\, y^{n-1} + \overline{c+dx+ex^2}\, y^{n-2} + \&c. = 0.$$

Sint

Sint termini in quibus maximæ ordinatæ & abſciſſæ (x & y) inveniuntur dimenſiones

$$y^n + bxy^{n-1} + ex^2y^{n-2} + \&c.$$ fiant

hi termini nihilo æquales, & reſultat

$$y^n + bxy^{n-1} + ex^2y^{n-2} + \&c. = o.$$

Sint omnes radices hujus æquationis impoſſibiles, & curva nullum habet crus in infinitum progrediens;

Sint ($2m$) radices impoſſibiles, & ($n-2m$) poſſibiles & inæquales, ſint $(\overline{n-2m})$ poſſibiles radices ordinatæ (y) αx, βx, γx, δx, &c. reſpective, *i. e.*

$y = \alpha x + \&c.$

$y = \beta x + \&c.$

&c. &c. & conſequenter curva habet ($n-2m$) aſymptotes ($2n-4m$) hyperbolica crura habentes.

Sint poſſibiles radices inter ſe æquales & eodem modo e ſubſequentibus terminis datæ æquationis inveniri poteſt formula ſeriei, a quâ deſignatur ordinata (y).

COROLLARIUM II.

Sit (n) æqualis $m + 2o + 3r + 4s + 5t + \&c.$ & facile deduci poteſt curva (n) dimenſionum, quaſcunque (m) lineas aſymptotes, quæcunque (o) parabolica conici generis crura, (r) ſecundi generis parabolica, & ſic deinceps habens.

Sit enim ($AP = x$) abſciſſa, & PM (y) ejus correſpondens ordinata, & ſint æquationes ad lineas aſymptotes

$$y + px + q = o$$

$$y + \overset{\text{I}}{p}x + \overset{\text{I}}{q} = o$$

$$y + \overset{\text{II}}{p}x + \overset{\text{II}}{q} = o$$

& æquatio ad parabolam conici generis $\overline{y + \alpha x}^2 + \beta y + \gamma x + \delta = o$ & ſic deinceps, e multiplicatione horum terminorum in ſeſe conſtant dimenſiones, ad quas aſſurgit data æquatio, & termini ipſi, in quibus inveniuntur (n) & ($n-1$) dimenſiones.

PROB. III.

Datâ æquatione relationem inter abſciſſam & ejus correſpondentes ordinatas exprimente

$$Ay^n + \overline{a + bx}\,y^{n-1} + \overline{c + dx + ex^2}\,y^{n-2} + \&c. = o.$$

inve-

invenire, qui ſit valor abſciſſæ, quando duo ordinatæ (y) *valores ſiunt inter ſe æquales.*

Si duo ordinatæ valores ſunt inter ſe æquales, tum

$$nAy^{n-1}+\overline{n-1}\times\overline{a+bx}\,y^{n-2}+\overline{n-2}\times\overline{c+dx+ex^2}\,y^{n-3}+\&c.=0.$$

transformentur hæ duæ æquationes in unam, ita ut exterminetur incognita quantitas (y), & reſultat æquatio, cujus radix eſt abſciſſa, quando duo ordinatæ valores fiunt inter ſe æquales.

Subſtituantur hæ radices pro ſuo valore (x) in duabus datis æquationibus, & æquationum reſultantium communes diviſores erunt valores ordinatarum, quando duo ſunt inter ſe æquales.

COROLLARIUM I.

Curvæ crura, cujus æquatio deſignans relationem inter ejus abſciſſam & ordinatas eſt

$$nAv^{n-1}+\overline{n-1}\times\overline{a+bx}\,v^{n-2}+\overline{n-2}\times\overline{c+dx+ex^2}\,v^{n-3}+\&c.=0$$

ſemper interponentur inter crura curvæ, cujus æquatio relationem inter abſciſſam & ejus ordinatas exprimens, eſt

$$y^n+\overline{a+bx}\,y^{n-2}+\overline{c+dx+ex^2}\,y^{n-3}+\&c.=0.$$

COROLLARIUM II.

Hinc inveniri poteſt, quando duo quatuor &c. ordinatæ valores fiunt impoſſibiles.

Inveniatur enim, quando duo valores fiunt æquales, & quando duo bis fiunt æquales, & ſic deinceps, & invenitur, quando duo, quatuor &c. ordinatæ valores fiunt impoſſibiles.

COROLLARIUM III.

Hinc etiam inveniri poſſunt puncta duplicia, triplicia, &c. & puncta conjugata, & ovales.

Inveniatur, quando duo, tres, quatuor & ordinatæ (y) valores fiunt inter ſe æquales.

Si duo, quatuor, ſex &c. ordinatæ valores fiunt inter ſe æquales, & utrinque ſint correſpondentes ordinatæ impoſſibiles, tum erit purum punctum conjugatum, ſimplex, duplex &c.

Sint præcedentes ordinatæ utrinque poſſibiles, & curva habet punctum duplex, quadruplex &c.

Si

Si ordinatæ partim ſunt poſſibiles & partim impoſſibiles utrinque, tum ſingula utrinque numerum æqualium radicum unitate auget, ſi vero ordinata tangit curvam, tum auget numerum æqualium ordinatarum per duo, & ſic de punctis contrariæ flexuræ, cuſpidibus.

Si plures ſunt ordinatæ æquales, quam requirit figura, tum datur punctum ad curvam.

Si duo vel plures ordinatæ valores bis fiunt æquales, & quarum Fig. 8. abſciſſæ reſpective ſunt PA & $P\overset{1}{A}$, & inter eas ſint ordinatæ poſſibiles: ordinatæ vero PM & $\overset{1}{P}\overset{1}{M}$ eundem habeant numerum crurum utrinque puncta M & $\overset{1}{M}$ jacentium, tum jacet inter puncta M & $\overset{1}{M}$ ovalis.

COROLLARIUM IV.

Sit punctorum numerus, in quibus duo, tria, quatuor &c. coeant reſpective ($m, r, s,$ &c.) & N vel $= \sqrt{m+r+s+t+\frac{9}{4}} - \frac{3}{2}$, ſi ſit integer numerus, vel integer numerus ei proxime ſuperior, ſi modo ſit ſurda quantitas vel fractio, tum dimenſiones curvæ, in quâ includuntur hæc puncta, non poſſunt eſſe minores quam

$$\frac{N \times \frac{N+3}{2} + m + 2r + 3s + 4t + \&c.}{N}.$$

Et ſic de ovalibus.

PROB. IV.

Datâ æquatione

$$Ay^n + \overline{a+bx}\,y^{n-1} + \overline{c+dx+ex^2}\,y^{n-2} + \&c. = 0.$$

relationem inter abſciſſam (x) *& ejus ordinatas* (y), *exprimente, invenire æquationem relationem inter quaſcunque duas alias lineas quocunque modo ex iis derivatas exprimentem.*

Inveniantur æquationes relationem inter abſciſſam, ordinatas & lineas, inter quas requiritur relatio, exprimentes; transformentur hæ æquationes in unam, ita ut exterminentur omnes præter eas lineas, inter quas requiritur relatio, incognitæ quantitates, & fit problema.

EX-

EXEMPLUM.

Iiſdem poſitis, invenire æquationem relationem exprimentem inter lineas (EM & FM), a quibuſcunque duobus punctis (E & F) ad quodlibet curvæ punctum (M), ductas.

Sit ($AP=x$) abſciſſa, & ($PM=y$) ordinata, per puncta E & F ducantur lineæ EK & FL parallelæ ordinatis (PM), occurrentes abſciſſæ in punctis (K & L).

Scribantur pro KA, EK; LA, FL; EM, FM reſpective E, F, G, H, v, z, & pro ſinu anguli (MPA) & ejus coſinu, s & c reſpective, tum

$$EM^2 = s^2 \times \overline{EK+PM}^2 + \overline{c \times \overline{PM+EK}+KP}^2 \quad \&$$

$$FM^2 = s^2 \times \overline{FL+PM}^2 + \overline{c \times \overline{PM+FL}+LP}^2 \quad \text{pro}$$

quibus lineis ſubſtituantur earum valores, & reſultant duæ æquationes

$$v^2 = y^2 + \overline{2F - 2cE + 2cx} \times y + F^2 - 2cFE + E^2 + \overline{2cF - 2E}x + x^2$$

$$z^2 = y^2 + \overline{2H - 2cG + 2cx} \times y + H^2 - 2cHG + G^2 + \overline{2cH - 2G}x + x^2$$

ex his duabus æquationibus inveniantur valores incognitæ quantitatis (x & y) terminis vero incognitarum quantitatum (z & y,) quibus pro ſuis valoribus (x & y) datâ æquatione ſubſtitutis, reſultat æquatio relationem inter (x & v) EM & FM exprimens.

Sit vero angulus (MPA), quem ordinatæ (PM) cum ſuâ abſciſſâ AP faciunt, rectus, & coſinus (c) nihilo eſt æqualis, tranſeat etiam abſciſſa per puncta E & F, *i.e.* F & H nihilo ſint æquales, tum pro x ſubſtituatur $\dfrac{z^2 - v^2 - G^2 + E^2}{2 \times \overline{E-G}}$ & pro $y^2 = v^2 - \overline{\dfrac{z^2 - v^2 - G^2 + E^2}{2 \times \overline{E-G}} - E}\Big|^2$ & reſultat æquatio quæſita.

COROLLARIUM.

Hinc conſtat dimenſiones æquationis relationem inter lineas EM FM exprimentis non plures eſſe dimenſionibus æquationis relationem inter abſciſſam & ejus correſpondentes ordinatas exprimentis quater ſumptis.

Et vice verſa non poſſunt dimenſiones æquationis relationem inter abſciſſam & ejus correſpondentes ordinatas exprimentis, eſſe plures, dimenſionibus æquationis relationem inter lineas EM & FM exprimentis quater ſumptis.

Et

Et ſic de lineis ad plura puncta ductis &c. & proprietatibus exinde deducendis.

PROB. V.

Datâ æquatione relationem inter abſciſſam & ejus correſpondentes ordinatas exprimente, invenire æquationem relationem inter abſciſſam curvæ & ejus correſpondentes ordinatas exprimentem, in quâ abſciſſa & ejus ordinatæ datam algebraicam relationem habent ad abſciſſam datæ curvæ & ejus ordinatas.

Datis algebraicis æquationibus relationem inter abſciſſas & earum ordinatas datæ & requiſitæ curvæ exprimentibus, e reductione harum æquationum in unam, ita ut exterminentur omnes præter abſciſſam ejus correſpondentes requiſitæ curvæ ordinatas, vel problemate primo libri primi, ſequitur requiſita æquatio.

EXEMPLUM I.

Data æquatione, relationem inter abſciſſam (x) & ejus correſpondentes ordinatas (y) definiens

$$y^n + Ay^{n-1} + By^{n-2} + Cy^{n-3} + \&c. = 0.$$

in qua coefficientes ($A, B, C, D,$ &c.) ſunt reſpective abſciſſæ (x) quæcunque rationales functiones, invenire æquationem relationem definientem inter abſciſſam & correſpondentes ordinatas curvæ, quæ ita ſecat datæ curvæ ordinatas, ut ſummæ cuborum ex ſingulis ordinatis utrinque jacentibus, ſint inter ſe æquales.

Pro (y) ſubſtituatur $v + \mathcal{Q}$ & reſultat æquatio

$$v^n + \overline{n\mathcal{Q}+A}\times v^{n-1} + \overline{n\times\frac{n-1}{2}\mathcal{Q}^2 + n-1\times A\mathcal{Q} + B}\times v^{n-2} + \overline{n\times\frac{n-1}{2}\times\frac{n-2}{3}\mathcal{Q}^3 + \overline{n-1}\times\frac{n-2}{2}A\mathcal{Q}^2 + \overline{n-2}B\mathcal{Q} + C}\times v^{n-3} + \&c. = 0.$$

Summa cuborum ex ſingulis incognitæ quantitatis valoribus nihilo æqualis fiat, & per lemma primum libri primi reſultat æquatio

$$\overline{n\mathcal{Q}+A}\big|^3 - 3\times\overline{n\mathcal{Q}+A}\times\overline{n\times\frac{n-1}{2}\mathcal{Q}^2 + \overline{n-1}A\mathcal{Q} + B} + 3\times\overline{n\times\frac{n-1}{2}\times\frac{n-2}{3}\mathcal{Q}^3 + \overline{n-1}\times\frac{n-2}{2}A\mathcal{Q}^2 + \overline{n-2}B\mathcal{Q} + C} = 0.$$

EXEMPLUM II.

Transformare unam curvam in alteram. Sint $(x, y, z, v;)$ duæ abſciſſæ & earum ordinatæ curvæ transformandæ & requiſitæ.

1mo Dentur tres æquationes relationem inter abſciſſam & ordinatas datæ curvæ, & abſciſſas & ordinatas datæ & requiſitæ curvæ, tres igitur ſunt æquationes, quatuor incognitas quantitates habentes, quibus in unam reductis, ita ut exterminentur incognitæ quantitates, reſultat æquatio relationem inter abſciſſam & ejus ordinatas requiſitæ curvæ exprimens.

Ex. gr. Sit data æquatio relationem inter abſciſſam (x) & ejus correſpondentes ordinatas (y) exprimens, & requiſitæ curvæ ſit abſciſſa (z) & ejus ordinata (y), tum ſit

$$x = \frac{pz+qv+r}{Az+Bv+C}$$

$$y = \frac{Pz+Qv+R}{Az+Bv+c}$$

quibus quantitatibus pro ſuis valoribus in datâ æquatione ſubſtitutis, reſultat æquatio relationem inter novam abſciſſam (z) & ejus ordinatis (v), exprimens, cujus dimenſiones æquales ſunt datæ æquationis dimenſionibus.

Data æquatione curvæ, invenire æquationem omnium curvarum datæ ſimilium.

Pro abſciſſâ & ordinatâ $(x$ & $y)$ ſubſtituantur reſpective nz & nv & fit.

Invenire, quando ordinatæ in una curvâ ſunt in datâ ratione ad ordinatas in alterâ.

Subſtituatur pro abſciſſâ & ordinatâ, mz & nv & fit.

Si geometricè requiritur hoc problema perficere, tum ſolummodo requiritur conſtructio.

2do Datis duabus æquationibus relationes inter duas abſciſſas & earum correſpondentes ordinatas exprimentibus, & tertiâ æquatione relationem exprimente inter abſciſſas & ordinatas duarum curvarum, invenire æquationem relationem inter duas vel tres ex incognitis quantitatibus, duabus abſciſſis & earum correſpondentibus ordinatis:

Hoc fit ex reductione æquationum in unam, ita ut incognitæ quantitates exterminentur.

COR.

COR. I.

E priori analyſi facile conſtant æquationum reſultantium dimenſiones, & conſequenter quot ſunt incognitarum quantitatum diverſi valores.

COR. II.

Hinc etiam deduci poteſt ſumma radicum, ſumma productorum ex ſingulis duabus in ſeſe ductis, & ſic deinceps.

PROB. VI.

Invenire algebraicarum curvarum proprietates, quæ pendent e dimenſionibus, quas in data quantitate habet quæcunque incognita quantitas (x), *vel ex æquationis dimenſionibus, cujus radices ſunt quæcunque incognita quantitas.*

Datis incognitæ quantitatis (x) dimenſionibus in datâ quantitate, exinde ſequitur e quot & qualibus factoribus in ſeſe ductis componitur data quantitas; & ſic de æquationibus.

EXEMPLUM.

Sit æquatio $y^n + \overline{a+bx}\, y^{n-1} + \overline{c+dx+ex^2}\, y^{n-2} + \&c. = o.$ relationem inter abſciſſam (x) & ejus correſpondentes ordinatas exprimens, & ejus ſubtangens (v) erit

$$\frac{ny^n + \overline{n-1} \times \overline{a+bxy^{n-1}} + \overline{n-2} \times \overline{c+dx+ex^2 y^{n-2}} + \&c.}{by^{n-1} + \overline{d+2exy^{n-2}} + \&c.}$$

Sint $\alpha, \beta, \gamma, \delta,$ &c. ordinatæ (y) diverſi valores, quibus in poſteriori quantitate ſubſtitutis, reſultant ſubtangentes reſpective

$$\frac{n\alpha^n + \overline{n-1} \times \overline{a+bx} \times \alpha^{n-1} + \overline{n-2} \times \overline{c+dx+ex^2} \times \alpha^{n-2} + \&c.}{b\alpha^{n-1} + d + 2ex\alpha^{n-2} + \&c.}$$

$$\frac{n\beta^n + \overline{n-1} \times \overline{a+bx} \times \beta^{n-1} + \overline{n-2} \times \overline{c+dx+ex^2} \times \beta^{n-2} + \&c.}{b\beta^{n-1} + \overline{d + 2ex}\beta^{n-2} + \&c.}$$

Ducantur omnes hi (n) diverſi ordinatæ valores in ſeſe, & aſcendit numerator ad ($n \times n$) dimenſiones abſciſſæ (x), denominator vero aſſurgit ad ($n \times \overline{n-1}$) dimenſiones, ſupponatur & numerator & denominator nihilo reſpective æqualis, & ſint radices numeratoris & denominatoris reſpective $\pi, \varrho, \sigma, \tau$ &c. & $\zeta, \eta, \Theta, \iota$ &c. ſint (x) (& z) duo diverſi abſciſſæ valores, & A & B contentum ſub (n) ſubtangentibus ad abſciſſas (x & z) reſpective, tum

$$A : B :: \frac{\overline{x-\pi}\times\overline{x-\rho}\times\overline{x-\sigma}\times\overline{x-\iota}\ \&c.}{\overline{x-\zeta}\times\overline{x-n}\ \overline{x-\Theta}\times\overline{x-i}\ \&c.} : \frac{\overline{z-\pi}\times\overline{z-\rho}\times\overline{z-\sigma}\times\overline{z-}}{\overline{z-\zeta}\times\overline{z-n}\times\overline{z-\Theta}\times\overline{z-i}\ \&c.}$$

COROLLARIUM.

Facile ex analyſeos principiis erui poteſt, quot caſus requiruntur ad probandam propoſitionem eſſe veram.

Et ſic de curvaturæ radiis & innumeris hujuſmodi propoſitionibus.

THEO.

Sint duæ curvæ, quarum quicunque termini ſint iidem in utrâque, tum omnis proprietas unius curvæ, quæ ex iis terminis pendet, i. e. *deduci poteſt, erit etiam alterius curvæ proprietas.*

COR. I.

Sit curva (n) dimenſionum, cujus crura in infinitum pergentia, (n) habent aſymptotes, & æquatio ad curvam & ejus crurum aſymptotes habent terminos (n & $n-1$) dimenſionum eoſdem, & conſequenter omnis proprietas curvæ, quæ ex iis terminis pendet, erit etiam proprietas aſymptotum.

COR. II.

Curvarum diametri pendent e terminis æquationis, in quibus dimenſiones abſciſſæ & ordinatæ inveniuntur maximæ (n), & proxime inferiores ($n-1$), & conſequenter diameter ad aſymptotes erit etiam diameter ad ipſam curvam.

COR. III.

Hinc ſequitur eandem habere inclinationem in datum diametrum aſymptotum ac curvæ ordinatæ.

COR. IV.

Sint ordinatæ ad abſciſſam utcunque inclinatæ, & earum ſummæ & ad aſymptotes & curvam erunt inter ſe æquales, & conſequenter nihilo æqualis eſt differentia inter ſummas ordinatarum ad aſymptotes & curvam, & exinde ſummæ ordinatarum partium inter curvæ crura & correſpondentes aſymptotes interceptarum utrinque diametri, erunt inter ſe æquales.

Centrum eſt interſectio duarum conjugatarum diametrorum.

Locus hujuſmodi punctorum eſt curva, quam dico centralem curvam.

Duæ

Duæ diametri ſe ipſos interſecent, & ad ſe invicem conſtanter tendant & ultimo coeant, tum locus punctorum hujuſmodi ultimarum inter ſectionum eſt curva, quam dico diametrialem curvam.

COR. V.

Sit curva (n) dimenſionum, (n) habens aſymptotes, & eandem & diametrialem & centralem curvam habent & aſymptotes & curva.

Diametri enim & curvæ & aſymptotum ſunt eædem, & exinde eædem ſunt & interſectiones conjugatarum diametrorum, & ultimæ diametrorum interſectiones, hæ vero interſectiones conſtituunt reſpective centralem & diametrialem curvam, & conſequenter hæ curvæ ſunt eædem.

Si curva non habet (n) aſymptotes, tum inveniri poſſunt aſymptotes & ellipſes, quarum æquationibus in ſeſe ductis, termini reſultantis æquationis, in quibus dimenſiones abſciſſæ & ordinatæ inveniuntur maximæ & proxime inferiores &c, iidem erunt ac datæ curvæ, & conſequenter eandem habent diametrum &c. aſymptotes & ellipſes conjunctim, ac data curva.

COR. VI.

Abſciſſa & ejus ordinatæ curvam interſecent in (n) punctis, & ratio, quam habet contentum ſub (n) partibus abſciſſæ inter ordinatam & (n) vertices abſciſſæ interceptis, ad contentum ſub ſingulis (n) ordinatis, pendet e terminis æquationis, in quibus inveniuntur maximæ & abſciſſæ & ordinatæ conjunctim dimenſiones; & conſequenter ſi curva habet (n) aſymptotes, & eadem linea eſt abſciſſa ad aſymptotes & curvam, & ad abſciſſam ordinatæ eandem habeant inclinationem, tum contentum ſub ſingulis (n) ordinatis curvæ, erit ad contentum ſub (n) ordinatis aſymptotum, in ratione contenti ſub (n) partibus abſciſſæ inter ordinatam & (n) vertices abſciſſæ curvæ interceptis, ad contentum ſub (n) partibus abſciſſæ inter ordinatam & (n) vertices abſciſſæ aſymptotum interceptis.

Et ſic de aſymptotibus & ellipſibus conjunctim.

COROLLARIUM VII.

Sint termini æquationis, in quibus maximæ invenientur abſciſſæ & ordinatæ (x & y) dimenſiones, contentum $\overline{y^2+x^2}\times\overline{y^2+x^2}\times\overline{y^2+x^2}$, &c.

tum

tum contentum ſub (n) ordinatis ſemper erit æqualis contento ſub (n) partibus abſciſſæ inter ordinatam & (n) vertices abſciſſæ intercep-tis; & omnes ordinatæ cum ſuis diametris rectum angulum faciunt.

Et ſic de innumeris hujuſmodi proprietatibus.

PROB. VIII.

Fig. 10. *Sint deſcriptæ duæ curvæ, quarum æquationes relationem inter abſciſſas* ($x=AP, z=a\Pi$) *& earum reſpectivas correſpondentes ordinatas* ($y=PM$, $v=\Pi M$) *exprimentes ſunt*

$$Ay^n+By^{n-1}+Cy^{n-2}+Dy^{n-3}+\&c.=0$$
$$av^m+bv^{m-1}+cv^{m-2}+dv^{m-3}+\&c.=0$$

in quibus A, B, C, D, &c. a, b, c, d, &c. *reſpective denotant rationales abſciſſarum* (x & z) *functiones, invenire omnes æquationes, quas harum duarum curvarum ope conſtruere licet.*

Ita transformentur hæ æquationes, ut lineæ ($Ip=X$, $\gamma\pi=Z$) fiant abſciſſæ & ($pM=Y$, $\pi M=V$) fiant ordinatæ.

Scribantur pro lineâ AL (a), IL (b), ſinu anguli MpL (s), ſinu anguli MQp (c), & pro ſinu anguli pMQ quæ ſit $s\sqrt{1-c^2}+c\sqrt{1-s^2}$, ($S$), & ſinu anguli MPL (r), & pro ſinu anguli PLp quæ ſit $r\sqrt{1-c^2}+c\sqrt{1-r^2}$, ($C$) tum in æquatione

$$Ay^n+By^{n-1}+Cy^{n-2}+Dy^{n-3}+\&c.=0$$

pro (x) ſcribatur $\frac{cX-SY}{r}+\frac{cb}{r}-a$, & pro ($y$), $\frac{CcX+\overline{rs-CS}\times Y}{rc}+\frac{Cb}{r}$.

Sic etiam transformari poteſt alia æquatio, ut ejus abſciſſa fiat ($\gamma\pi$) & ordinata (πM), quæ angulum facit cum ejus abſciſsâ ($\gamma\pi$) æqualem angulo (MpL), cujus ſinus eſt (s).

Pro abſciſſis (X, Z,) ſcribantur in utrâque reſultanti æquatione (t) & pro ordinatis (Y, V,) in utraque æquatione (w), reducantur duæ æquationes reſultantes in unam, ita ut exterminetur incognita quantitas (t), & reſultat æquatio, cujus radix eſt (w), & quæ continet in ſe omnes æquationes, quæ harum curvarum ope conſtrui poſſunt.

COROLLARIUM I.

Sint duæ curvæ datæ, & e prædictis conſtat æquationem generalem, quas earum ope conſtruere licet, non plures quam ſex incognitas

tas quantitates habere, & conſequenter duarum datarum curvarum ope haud generaliter conſtrui poſſunt æquationes plures quam ſex dimenſiones habentes.

COROLLARIUM II.

Sit una curva, circulus, cujus haud datur radius, & non plures ſunt quam ſeptem incognitæ quantitates, & conſequenter ope circuli & datæ curvæ haud conſtrui poteſt curva plures quam ſeptem dimenſiones habens, & ſic denique de curvis plures incognitas quantitates habentibus.

COROLLARIUM III.

Datâ & æquatione conſtruendâ & unâ curvâ, cujus ope illam conſtruere licet, innumeræ aliæ inveniri poſſunt, quarum abſciſſæ & ordinatæ eædem ſunt ac abſciſſæ & ordinatæ datæ curvæ, & ordinatæ ad earum interſectionum puncta pertinentes erunt datæ æquationis radices.

Hæ datæ æquationes in quaſlibet quantitates ducantur & a ſeipſis ſubducantur, & æquatio reſultans erit quæſita æquatio.

COROLLARIUM IV.

Æquatio reſultans, cujus radix eſt (x) ſæpe habet poſſibiles radices, cum nulla datur poſſibilis interſectio, *i. e.* nullus poſſibilis valor ordinatæ, & hoc fit ut conſtat e corollario ad problema xiii. libri primi quando duæ vel plures abſciſſæ valores ſunt inter ſe æquales.

COROLLARIUM V.

Æquatio (nm) dimenſionum conſtrui poteſt, duarum curvarum ope, quarum dimenſiones in ſeſe ductæ ad minimum æquant datæ æquationis (nm) dimenſiones.

COROLLARIUM VI.

Curva (n) dimenſionum duci poteſt per $\left(n \times \frac{n+3}{2}\right)$ puncta: ſint duæ curvæ (n) & (m) dimenſionum, & in curva (m) dimenſionum aſſumantur plura quam (nm) puncta, ſed pauciora quam $\left(n \times \frac{n+3}{2}\right)$, per quæ requiritur ducere curvam (n) dimenſionum, tum curva (n) dimenſionum reduci poteſt in duas vel plures, quarum una eſt altera curva (m) dimenſionum.

CO-

COROLLARIUM VII.

Sint $y^n + bxy^{n-1} + ex^2y^{n-2} + \&c.$

& $y^m + Bxy^{m-1} + Ex^2y^{m-2} + \&c.$

termini, in quibus maximæ inveniuntur æquationum incognitarum quantitatum (y & x) dimenſiones & communem habeant diviſorem, tum haud aſcendit æquatio reducta ad (nm) dimenſiones.

COROLLARIUM VIII.

Datis duabus æquationibus

$$py^n + \overline{a + bx}\,y^{n-1} + \overline{c + dx + ex^2}\,y^{n-2} + \&c. = 0$$

& $$Py^m + \overline{A + Bx}\,y^{m-1} + \overline{C + Dx + Ex^2}\,y^{m-2} + \&c. = 0$$

relationem inter abſciſſas & earum correſpondentes ordinatas duarum datarum curvarum exprimentibus, facile reduci poſſunt hæ duæ æquationes in unam, ita ut exterminetur incognita quantitas (x).

Interſectiones enim harum curvarum erunt quæſitæ æquationis radices, & datis æquationis radicibus, facile cognoſcitur æquatio ipſa.

Non opus eſt hic conſtruere æquationes. Facile conſtant infinitæ hujuſmodi propoſitiones.

PROB. VIII.

Fig. 11. *Datâ æquatione relationem inter abſciſſam* (AP) *& ejus correſpondentes ordinatas* (PM) *exprimente, invenire æquationem relationem inter abſciſſam* (A π) *& ejus ordinatas* (π m), *ad locum punctorum a quocunque modo e datâ æquatione deductorum.*

Datâ methodo inveniendi punctum (m) e datâ æquatione, datur etiam methodus inveniendi lineas ($A\pi$ & πm) abſciſſam & ordinatam quæſitæ curvæ, æquationes reſultantes ita reducantur in unam, ut exterminentur omnes præter duas ($A\pi$ & πm) abſciſſam & ordinatam quæſitæ curvæ, & fit.

EXEMPLUM.

Sit æquatio relationem inter abſciſſam ($AP = x$) & ejus correſpondentes ordinatas ($PM = y$) exprimens,

$$Ay^n + \overline{a + bx}\,y^{n-1} + \overline{c + dx + ex^2}\,y^{n-2} \ldots \overline{px^{n-1} + qx^{n-2} + rx^{n-3} + \&c.}\,xy$$
$$+ Px^n + Qx^{n-1} + Rx^{n-2} + \&c. = 0.$$

Circa datum punctum (A) revolvatur recta (AM), quæ huic curvæ occurrat

occurrat in (n) punctis ($M, \overset{\prime}{M}, \overset{\prime\prime}{M}$, &c.) & ſi in eâdem recta ſumatur ſemper Pn, ita ut

$$\frac{1}{Pn} = \frac{1}{PM} + \frac{1}{P\overset{\prime}{M}} + \frac{1}{P\overset{\prime\prime}{M}} + \text{\&c.}$$

tum erit locus puncti (n) recta.

Ita transformetur data æquatio, ut exprimat relationem inter abſciſſam AP, & ordinatam cum eâ angulum APL vel APN, &c. facientem, cujus ſinus eſt (σ), ſit etiam ſinus anguli (APM) rectus; tum in datâ æquatione pro (y) ſcribatur σv (PL, PN &c.), & pro x, z (AP) $+ \sqrt{1-\sigma^2}\,v$; & termini reſultantis æquationis, in quibus invenitur nulla & una dimenſio ordinatæ (v), erunt reſpective

$$Pz^n + Qz^{n-1} + Rz^{n-2} + \text{\&c.}$$

& $\overline{p\sigma + nP\sqrt{1-\sigma^2}}\,z^{n-1} + \overline{q\sigma + \overline{n-1}\,Q\sqrt{1-\sigma^2}}\,z^{n-2} + \overline{r\sigma + \overline{n-2}\,R\sqrt{1-\sigma^2}}\,z^{n-3} + \text{\&c.}$

ergo

$$\frac{1}{Pn} = \frac{1}{PM} + \frac{1}{P\overset{\prime}{M}} + \frac{1}{P\overset{\prime\prime}{M}} + \text{\&c.} =$$

$$\frac{\overline{p\sigma + nP\sqrt{1-\sigma^2}}\,z^{n-1} + \overline{q\sigma + \overline{n-1}\,Q\sqrt{1-\sigma^2}}\,z^{n-2} + \overline{r\sigma + \overline{n-2}\,R\sqrt{1-\sigma^2}}\,z^{n-3} + \text{\&c.}}{Pz^n + Qz^{n-1} + Rz^{n-2} + \text{\&c.}}$$

Sint W (Pp) & V (pn) abſciſſa & correſpondentes ordinatæ lineæ, cujus requiritur æquatio, tum $\frac{V}{\sigma} = Pn$ & $\frac{W}{\sqrt{1-\sigma^2}} = Pn$; quibus reductis, reſultat $Pn = \sqrt{W^2+V^2}$; & $\sigma = \frac{V}{\sqrt{W^2+V^2}}$, & $\sqrt{1-\sigma^2} = \frac{W}{\sqrt{W^2+V^2}}$; quibus quantitatibus pro ſuis valoribus in æquatione relationem inter Pn, z & σ exprimente, ſubſtitutis, reſultat æquatio relationem inter z, V & W exprimens

$$1 = \frac{\overline{pV + nPW}\,z^{n-1} + \overline{qV + \overline{n-1}\,QW}\,z^{n-2} + \overline{rV + \overline{n-2}\,RW}\,z^{n-3}}{Pz^n + Qz^{n-1} + Rz^{n-2} + \text{\&c.}}$$

ſed punctum (P) per hypotheſin datur, & conſequenter $AP = z$, ergo hæc æquatio relationem inter W & V abſciſſam & ordinatam lineæ requiſitæ deſignat, & quoniam in nullis terminis continentur plures quam una dimenſio abſciſſæ vel ordinatæ, linea requiſita erit recta.

COROLLARIUM.

Sit $$\frac{1}{Pn} \text{ vel } \frac{1}{Pn^2} = \frac{1}{PM^2} + \frac{1}{P\dot{M}^2} + \frac{1}{P\ddot{M}^2} + \&c.$$

& locus punctorum (n) vel erit conica ſectio, vel duæ rectæ lineæ.

Sit $$\frac{1}{Pn} \text{ vel } \frac{1}{Pn^3} = \frac{1}{PM^3} + \frac{1}{P\dot{M}^3} + \frac{1}{P\ddot{M}^3} + \&c.$$

& locus punctorum (n) vel erit curva ſecundi generis, vel conica ſectio & recta linea, vel tres rectæ lineæ, & ſic deinceps.

Et ſimiliter de infinitis aliis hujuſmodi propoſitionibus.

E ſubſtitutione præcedenti conſtant hæ propoſitiones.

EXEMPLUM II.

Fig. 12. Circa datum punctum (A) abſciſſæ verticem revolvatur recta linea AM, quæ occurrat algebraicæ lineæ cujuſcunque ordinis, in tot punctis, M, &c. quot ſunt ejus dimenſiones, & ſi in eâdem rectâ ſumatur AN, ita ut

$AN = AM + A\dot{M} + A\ddot{M} + \&c.$ tum locus punctorum (N) poteſt eſſe curva (n) dimenſionum, vel duæ vel plures curvæ, quarum dimenſionum ſumma eſt (n).

Sit æquatio relationem inter abſciſſam ($AP = x$) & ejus correſpondentes ordinatas (PM, y) exprimens,

$$Ay^n + \overline{a+bx}\,y^{n-1} + \overline{c+dx+ex^2}\,y^{n-2} + \overline{f+gx+hx^2+kx^3} \times y^{n-3} \&c. = 0.$$

ita transformetur hæc æquatio; ut relationem inter abſciſſam (Ap) & ejus ordinatas (pM) exprimat; ſit angulus APM rectus, & pro ſinubus angulorum (ApP, PAp) ſcribantur s & $C = \sqrt{1-s^2}$ reſpective, & pro lineis (Ap & pM) z & v, tum reſultant quantitatum (x) & (y) valores sz & $v + C(\sqrt{1-s^2})\,z$, quibus quantitatibus pro ſuis valoribus ſubſtitutis, coefficientes terminorum (z^n & z^{n-1}) reſultantis æquationis, erunt reſpective

$$AC^n + bC^{n-1}s + eC^{n-2}s^2 + kC^{n-3}s^3 + \&c.$$
$$\& \quad aC^{n-1} + dC^{n-2}s + hC^{n-3}s^2 + \&c.$$

& conſequenter ſumma radicum (z) erit

$$\frac{aC^{n-1} + dC^{n-2}s + hC^{n-3}s^2 + \&c.}{AC^n + bC^{n-1}s + eC^{n-2}s^2 + kC^{n-3}s^3 + \&c.} = AM + A\dot{M} + A\ddot{M} + \&c. = An;$$

ſint (V & w) reſpective Am & mn, abſciſſa & ejus ordinata curvæ quæſitæ

quæſitæ, tum eodem modo, quo in priori exemplo, colligi poteſt

$$An = \sqrt{w^2 + V^2} \;\&\; s = \frac{V}{\sqrt{w^2 + V^2}} \;\&\; C = \frac{w}{\sqrt{V^2 + w^2}}$$

quibus quantitatibus pro ſuis valoribus in priori æquatione ſubſtitutis, reſultat æquatio relationem inter abſciſſam Am & ejus ordinatas (mn) quæſitæ curvæ exprimens

$$Aw^n + \overline{bV - a}\, w^{n-1} + \overline{eV^2 - dV}\, w^{n-2} + \overline{kV^3 - bV^2}\, w^{n-3} + \&c. = 0,$$

cujus dimenſiones nonplures quam (n) eſſe poſſunt.

COROLLARIUM.

Ducatur quicunque parallelarum linearum numerus $(mn, \overset{1}{m}\overset{1}{n}, \&c.)$ transeuntium per (n) hujuſmodi puncta, *i.e.* occurrentium curvæ quæſitæ in (n) punctis $(M, K, H, \&c.\ \overset{1}{M}, \overset{1}{K}, \overset{1}{H}, \&c.)$ & ſi in eadem recta ſemper ſumatur $(mo, \overset{1}{m}\overset{1}{o}, \&c.)$ ita ut Fig. 13.

$$mo = mn + mK + mH + \&c.$$

$$\& \quad \overset{1}{m}\overset{1}{o} = \overset{1}{m}\overset{1}{n} + \overset{1}{m}\overset{1}{K} + \overset{1}{m}\overset{1}{H} + \&c.$$ erit locus punctorum $(o, \overset{1}{o}, \&c.)$ recta linea.

Et ſic de infinitis hujuſmodi propoſitionibus.

EXEMPLUM III.

Iiſdem poſitis, invenire curvam datæ curvæ centrorum locum.

Sit diameter, cujus abſciſſa (x) incipit a centro, & æquationis relationem inter abſciſſam (x) & ejus correſpondentes ordinatas (y) exprimentis, nihilo æquales erunt tres termini $(xy^{n-1}, y^{n-1}, x^{n-1}.)$ Fig. 14.

Ita transformetur data æquatio, ut exprimat relationem inter abſciſſam (Ap, z) & ejus ordinatas (pM, v); ſit angulus APM rectus, & pro ſinubus angulorum $(PAp$ & $ApM)$ ſcribantur reſpective s & c; & erunt ſinus angulorum $(prM$ & $rMp)$ reſpective, $\sqrt{1 - s^2}$ & $sc + \sqrt{1 - s^2}\sqrt{1 - c^2} = C$, & conſequenter abſciſſa

$$AP = \sqrt{1 - s^2}\, z - Cv \quad \text{\& ordinata}$$

$$PM = sz + \frac{\overline{c - sC}}{\sqrt{1 - s^2}}\, v \quad \text{quibus}$$

quantitatibus pro ſuis valoribus in datâ æquatione ſubſtitutis; & in æquatione reſultanti

ſubſtituatur pro abſciſſa (z), $X+w$ (AN), & pro ordinatâ (y), $Y+u$ (Nm); & reſultant

$$AP=\sqrt{1-s^2}\,X-CY+\sqrt{1-s^2}\,w-Cu \text{ \& }$$

$$PM=sX+\frac{c-Cs}{\sqrt{1-s^2}}Y+sw+\frac{c-Cs}{\sqrt{1-s^2}}u.$$

Subſtituantur hæ quantitates pro ſuis valoribus in datâ æquatione, & reſultantis æquationis terminorum (XY^{n-1}, Y^{n-1}, X^{n-1}) coefficientes erunt reſpective

$$nA\times\left(\frac{c-Cs}{\sqrt{1-s^2}}\right)^{n-1}s+b\times\overline{\left(\frac{c-Cs}{\sqrt{1-s^2}}\right)^{n-1}\sqrt{1-s^2}-\overline{n-1}\times\left(\frac{c-Cs}{\sqrt{1-s^2}}\right)^{n-2}\times sC}$$

$+$ &c. . . . .

$$a\times\left(\frac{c-Cs}{\sqrt{1-s^2}}\right)^{n-1}-d\times\left(\frac{c-Cs}{\sqrt{1-s^2}}\right)^{n-2}C+b\times\left(\frac{c-Cs}{\sqrt{-1s^2}}\right)^{n-3}C^2+\text{ \&c.}$$

$$n\times A\left(\frac{c-Cs}{\sqrt{1-s^2}}\right)^{n-1}\times\overline{sw+\frac{c-Cs}{\sqrt{1-s^2}}u}+b\times\text{ \&c.}$$

. . . .

$$as^{n-1}+ds^{n-2}\sqrt{1-s^2}+bs^{n-3}\;\overline{1-s^2}+\text{ \&c.}$$

$$+nAs^{n-1}\times\overline{sw+\frac{c-Cs}{\sqrt{1-s^2}}}+b\times\text{ \&c.}$$

Fiant hæ tres coefficientes nihilo reſpective æquales, & pro coefficiente (C) ſubſtituatur ejus valor ($sc+\sqrt{1-s^2}\times\sqrt{1-c^2}$), & tres æquationes reſultantes ita reducantur in unam, ut exterminentur duæ incognitæ quantitates (c, C), & reſultat æquatio relationem inter abſciſſam (AN) & ordinatas (Nm) centralis curvæ quæſitæ exprimens.

Æquatio, relationem inter abſciſſam & ejus correſpondentes ordinatas curvæ exprimens, cujus locus eſt puncta interſectionum duarum diametrorum ſibi ipſis quam proximarum, e prædictis & incrementorum principiis facile erui poteſt.

PROB. IX.

Fig. 15. *Sit fluxio abſciſſæ* (AP=x) *invariabilis, & datâ algebraicâ æquatione relationem inter abſciſſam* (x) *& ejus correſpondentes ordinatas* (PM=y)

exprimente

exprimente, invenire ſummam ex ſingulis valoribus fluxionalis quantitatis, quæ ſit quantitatum (x & y) *functio, cujus fluens inveniri poteſt.*

RESOLUTIO.

Inveniatur fluens, & e lemmatibus primo & ſecundo & eorum corollariis inveniatur ſumma ex ſingulis valoribus hujuſce fluentis, hujuſce ſummæ fluxio &c. erit ſumma quæſita.

EXEMPLUM I.

Datâ abſciſſa, (x), & æquatione relationem inter abſciſſam (x) & ejus correſpondentes ordinatas (y) exprimente

$$y^n + \overline{a+bx}\,y^{n-1} + \overline{c+dx+ex^2}\,y^{n-2} + \&c. = 0.$$

ſummam ſubnormalium invenire.

Supponantur diverſi ordinatæ valores reſpective u, v, w, &c. & conſequenter diverſæ ſubnormales erunt reſpective $\frac{u\dot{u}}{\dot{x}}$, $\frac{v\dot{v}}{\dot{x}}$, $\frac{w\dot{w}}{\dot{x}}$, &c. ſit ($\dot{x}$) invariabilis quantitas, & harum quantitatum fluentes erunt $\frac{u^2}{2\dot{x}}$, $\frac{v^2}{2\dot{x}}$, $\frac{w^2}{2\dot{x}}$, &c.

Per lemma primum primi libri inveniri poteſt

$y^2 + v^2 + w^2 + \&c. = \overline{a+bx}^2 - 2 \times \overline{c+dx+ex^2}$, & conſequenter ſubnormalium ſumma quæſita erit

$$\frac{u\dot{u}+v\dot{v}+w\dot{w}+\&c.}{\dot{x}} = \frac{\overline{a+bx}\times b\dot{x} - \overline{d+2ex}\,\dot{x}}{\dot{x}} = ab - d + \overline{b^2-2e}\times x.$$

Si ordinatæ ſecant curvam in (n) punctis, & bis ſumma harum ſubnormalium nihilo æqualis eſt, tum vero, quæcunque ſit abſciſſa, nihilo æqualis erit.

Si curva habet generale centrum, & ejus abſciſſa per centrum tranſit, tum ſumma ſubnormalium erunt reſpective abſciſſis a centro incipientibus proportionales.

COR. I.

Hinc inveniri poſſunt innumeræ hujuſmodi propoſitiones. Aſſumatur enim quæcunque fluxionalis quantitas, abſciſſam, ordinatam &

& earum poteſtates & fluxiones involvens, ita vero ut ejus fluens inveniri poteſt.

Inveniantur lineæ, quæ datæ fluxionali quantitati ſunt æquales, deinde inveniatur ſumma ex ſingulis valoribus datæ fluxionalis quantitatis, & conſequenter invenitur ſumma ex ſingulis valoribus prædictarum linearum.

EXEMPLUM II.

Fig. 15. Aſſumatur fluxio $\frac{\ddot{y}}{\dot{x}^2}$, & ſit æquatio relationem inter abſciſſam ($AP=x$) & ejus correſpondentes ordinatas ($PM=y$)

$y^n + Ay^{n-1} + By^{n-2} +$ &c. $= o$, in quâ A & B &c. ſint rationales functiones abſciſſæ ($x=AP$).

Sit MQ chorda curvaturæ in directione (PM), & QR perpendiculum in tangentem demiſſum, & facile deduci poteſt $\frac{MQ}{QR^2} = \frac{\ddot{y}}{2\dot{x}^2}$.

Sed per vulgarem analyſin

$y + v + w +$ &c. $= A$ & conſequenter

$\frac{\ddot{y}}{\dot{x}^2} + \frac{\ddot{v}}{\dot{x}^2} + \frac{\ddot{w}}{\dot{x}^2} +$ &c. $= \frac{\ddot{A}}{\dot{x}^2}$ ergo ſumma quantitatum hujuſmodi $\frac{MQ}{QR^2}$ erit $\frac{\ddot{A}}{2\dot{x}^2}$.

EXEMPLUM.

Datâ eâdem æquatione

$y^n + Ay^{n-1} + By^{n-2} + Cy^{n-3}$ &c. $= o$.

invenire ſummam ex ſingulis valoribus ejus areæ.

Datâ abſciſsâ (x), ſint y, v, u, w, &c. diverſi ordinatæ (y) valores, &

$y + v + u + w +$ &c. $= A$ & conſequenter

$y\dot{x} + v\dot{x} + u\dot{x} + w\dot{x} +$ &c. $= A\dot{x}$, quarum fluentes ſunt areæ ipſæ, & earum ſumma quæſita æqualis eſt fluenti fluxionis ($A\dot{x}$).

COROLLARIUM

Nulla curva, quæ habet hyperbolicum crus conici generis, generaliter quadrari poteſt.

Ita transformetur enim data æquatio, ut reſultat æquatio relationem inter abſciſſam (quæ ſit hyperbolici cruris aſſymptoton) & ejus correſpondentes ordinatas exprimens; ex æquatione reſultante conſtat

ſtat ſummam ex ſingulis valoribus curvæ areæ i. e. fluentem fluxionis ($A\dot{x}$) haud generaliter quadrari poſſe.

Si vero curva ſit continua, eadem expreſſio vel æquatio, quæ generaliter quadrat unum curvæ crus, quadrabit etiam ſingula.

Et conſequenter ſi ſumma ex ſingulis valoribus areæ, haud inveniri poteſt, tum ab æquo ſequitur nullum curvæ crus quadrabile eſſe.

Et methodo haud multum diſſimili, ſequitur nullam eſſe algebraicam curvam, conſtantem ſolummodo ex ovalibus ſeſe in dato puncto vel punctis minime inter ſecantibus, quæ generaliter quadrari poteſt.

Ovales ſeſe ſecantes in datis punctis, nonnunquam generaliter quadrabiles eſſe docebit ſubſequens exemplum.

Sit æquatio curvæ

$$y = \overline{2a - 2x} \times \sqrt{2ax - x^2},$$ &

curva vel dici poteſt ovalis ſeſe interſecans in uno puncto, vel duæ ovales ſeſe interſecantes; ſed nullum habet crus in infinitum pergens.

Generaliter quadrari poteſt curva; & ejus area eſt in genere

$$\frac{2 \times \overline{2ax - x^2}^{1\frac{1}{2}}}{3}.$$

Sit curva algebraica, & conſtat ſummam ex ſingulis ejus areæ valoribus ſemper inveniri poſſe, conjunctim finitorum terminorum, circularium arcuum & logarithmorum ope.

Datâ abſciſsâ (x), & (n) & non plures ſunt diverſæ areæ ei correſpondentes, & conſequenter æquatio, cujus radix eſt area ipſa, (n) & non plures habere poteſt dimenſiones.

EXEMPLUM III.

Invenire ſunnmam ſolidorum a rotatione curvæ circa axem ſuum generatorum.

Summæ quæſitæ fluxio eſt $3{,}14159\&c. \times \overline{y^2 + v^2 + u^2 + w^2 + \&c.} \times \dot{x}$

$y^2 + v^2 + u^2 + w^2 + \&c. = A^2 - 2B$ & conſequenter $3{,}14159 \; \overline{A^2 - 2B} \times \dot{x}$ eſt fluxio ejuſdem ſummæ, & ejus fluens ſumma ſolidorum quæſita.

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EXEMPLUM IV.

Data abſciſſâ (x), invenire ſummam ex reciprocis ſubtangentibus.

Sit ultimus datæ æquationis terminus

$$px^{n}-qx^{n-1}+rx^{n-2}-sx^{n-3}+tx^{n-4}-\&c.$$

qui æqualis eſt contento ($y\times v\times u\times w\times$ &c.) ſub ſingulis ordinatæ (y) diverſis valoribus *i. e.*

$$yvuw\&c=px^{n}-qx^{n-1}+rx^{n-2}-sx^{n-3}+tx^{n-4}-\&c.$$

Inveniantur fluxiones ex utrâque æquationis parte, & oritur æquatio
$\dot{y}vuw\&c+\dot{v}yuw\&c.+\dot{u}vyw\&c+\dot{w}yvu\&c.=$
$npx^{n-1}-\overline{n-1}\,qx^{n-2}+\overline{n-2}\,rx^{n-3}-\overline{n-3}\,sx^{n-4}+\&c.$
dividatur prior æquationis pars per contentum ($yuvw$ &c.), poſterior vero per quantitatem $px^{n}-qx^{n-1}+rx^{n-2}-sx^{n-3}+\&c.$ ei contento æqualem, & reſultat æquatio

$$\frac{\dot{y}}{y\dot{x}}+\frac{\dot{v}}{v\dot{x}}+\frac{\dot{u}}{u\dot{x}}+\frac{\dot{w}}{w\dot{x}}+\&c.=\frac{npx^{n-1}-\overline{n-1}qx^{n-2}+\overline{n-2}rx^{n-3}-\overline{n-3}sx^{n-4}+\&c.}{px^{n}-qx^{n-1}+rx^{n-2}-sx^{n-3}+\&c.}$$

Illæ autem quantitates $\frac{\dot{y}}{y\dot{x}}, \frac{\dot{v}}{v\dot{x}}, \frac{\dot{u}}{u\dot{x}}, \frac{\dot{w}}{w\dot{x}}$, &c. ſunt reſpective reciprocæ ſubtangentes, & earum ſumma quæſita eſt

$$\frac{npx^{n-1}-\overline{n-1}qx^{n-2}+\overline{n-2}rx^{n-3}-\overline{n-3}sx^{n-4}+\&c.}{px^{n}-qx^{n-1}+rx^{n-2}-sx^{n-3}+\&c.}$$

COROLLARIUM.

Fig.16. Occurrat recta quævis, per punctum datum (P) ducta, curvæ algebraicæ in tot punctis (A, B, C &c.), quot ſunt ejus dimenſiones: rectæ (AK, BL, CM, &c.) curvam tangentes in his punctis, ſecent abſciſſam *i. e.* alteram rectam lineam (PLM) per idem datum punctum ductâm, in punctis (K, L, M, &c.) & erit ſumma reciprocarum ſubtangentium $\frac{1}{PK}+\frac{1}{PL}+\frac{1}{PM}+$ &c. data quantitas.

Summa reciprocarum ſubtangentium ſolummodo pendet ex ultimo datæ æquationis termino,

$$px^{n}-qx^{n-1}-rx^{n-2}-sx^{n-3}+\&c.$$

abſciſſâ igitur eadem manente, in eam mutetur vero inclinatio ordinatarum, & a problematis primi corollario primo conſtat haud mutari ultimum datæ æquationis terminum, & conſequenter ſi ordinatæ diverſas haben-

habentes ad abſciſſam inclinationes, per idem punctum (P) abſciſſæ tranſeant *i. e.* habeant eoſdem quantitatis (x) valores, & erunt ſummæ reciprocarum ſubtangentium reſpective inter ſe æquales.

PROB. X.

Data æquatione, relationem inter abſciſſam (x) *& ejus ordinatas* (y) *exprimente, & fluxionali quantitate, quæ eſt functio abſciſſæ* (x) *& ordinatæ* (y), *cujus fluens haud inveniri poteſt, invenire ſummam ex ſingulis hujus functionis valoribus.*

Ex datâ æquatione inveniatur prima fluxio ordinatæ (y), & ſic ſecunda, tertia, &c. fluxiones ordinatæ, quibus valoribus in datâ fluxionali quantitate ſubſtitutis, reſultat algebraica quantitas; cujus ſumma ex ſingulis valoribus erui poteſt ex libri primi lemmate primo, & ſecundo & ejus corollariis.

EXEMPLUM I.

Datâ æquatione curvæ $y^2 - Ay + B = 0$, in quâ A & B ſunt quæcunque abſciſſæ (x) functiones, invenire æquationem cujus radices ſunt ordinatæ (y) fluxiones.

Sint duæ ordinatæ (y) radices reſpective y & v, & ſumma requiſitæ æquationis radicum, erit $\dot{y} + \dot{v} = \dot{A}$; & productum ſub binis valoribus contentum, erit fractio

$$\dot{y}\dot{v} = \frac{\dot{A}y - \dot{B}}{2y - A} \times \frac{\dot{A}v - \dot{B}}{2v - A};$$

cujus denominator eſt

$$4yv - 2A \times \overline{y+v} + A^2 = 4B - A^2,$$

Numerator vero

$$\dot{A}^2 yv - \dot{A}\dot{B} \times \overline{y+v} + \dot{B}^2 = B\dot{A}^2 - A \times \dot{A}\dot{B} + \dot{B}^2$$

& conſequenter æquatio quæſita eſt

$$\dot{y}^2 - \dot{A}\dot{y} + \frac{B\dot{A}^2 - A\dot{A}\dot{B} + \dot{B}^2}{4B - A^2} = 0$$

EXEMPLUM II.

Invenire æquationem, cujus radices ſunt ſubtangentes curvæ, cujus æquatio eſt $y^2 - Ay + B = 0$.

Sint y & v duo ordinatæ (y) valores, & duæ ſubtangentes erunt $\frac{\overline{2y^2 - Ay} \times \dot{x}}{\dot{A}y - \dot{B}}$ & $\frac{\overline{2v^2 - Av} \times \dot{x}}{\dot{A}v - \dot{B}}$; quibus fractionibus ad communem denominatorem reductis, fit denominator

$$\dot{A}^2 yv - \dot{A}\dot{B} \times \overline{y+v} + \dot{B}^2 = B\dot{A}^2 - A \times \dot{A}\dot{B} + \dot{B}^2;$$

Numerator vero

$$\dot{x} \times \overline{2\dot{A}yv \times \overline{y+v} - 2A\dot{A}yv - 2\dot{B}\overline{y^2+v^2} + A\dot{B} \times \overline{y+v}} =$$

$$= \dot{x} \times \overline{2BA\dot{A} - 2BA\dot{A} - 2\overline{A^2 - 2B} \times \dot{B} + A^2\dot{B}},$$ & conſequenter ſumma ſubangentium erit

$$\frac{4B\dot{B} - A^2\dot{B}}{\dot{A}^2 B - \dot{B}\dot{A}A + \dot{B}^2}\dot{x}.$$

Fractionis, quæ ſit productum ſub duabus ſubtangentibus contentum, numerator erit

$$\dot{x}^2 \times \overline{4y^2v^2 - 2Ayv \times \overline{y+v} + A^2 \times yv} =$$

$\dot{x}^2 \times \overline{4B^2 - 2A^2B + A^2 \times B} = 4B^2 - A^2B$; & conſequenter æquatio, cujus radices ſunt ſubtangentes (w), erit

$$w^2 - \frac{\overline{4B\dot{B} - A^2\dot{B}} \times \dot{x}}{\dot{A}^2 B - \dot{B}\dot{A}A + \dot{B}^2} w + \frac{\overline{4\dot{B}^2 - A^2\dot{B}^2} \times \dot{x}^2}{\dot{A}^2 B - \dot{B}\dot{A}A + \dot{B}^2} = 0.$$

EXEMPLUM III.

Datâ æquatione curvæ $y^3 + By - C = 0$, in quâ (B) & (C) ſunt quæcunque abſciſſæ functiones, invenire æquationem, cujus radices ſunt ordinatæ (y) fluxiones.

Sint tres radices ordinatæ (y, v, w), tum ſumma earum fluxionum $\dot{y} + \dot{v} + \dot{w} = 0$. &

$$\dot{y}\dot{v} + \dot{v}\dot{w} + \dot{w}\dot{v} = \frac{\dot{C} - \dot{B}y}{3y^2 + B} \times \frac{\dot{C} - \dot{B}v}{3v^2 + B} + \frac{\dot{C} - \dot{B}y}{3y^2 + B} \times \frac{\dot{C} - \dot{B}w}{3w^2 + B} +$$

$\frac{\dot{C} - \dot{B}v}{3v^2 + B} \times \frac{\dot{C} - \dot{B}w}{3w^2 + B}$; reducantur hæ fractiones in communem denominatorem,

minatorem, & erit denominator

$\overline{3y^2+B}\times\overline{3v^2+B}\times\overline{3w^2+B}=27C^2+4B^3$; numerator vero

$\overline{3w^2+B}\times\overline{\dot{C}-\dot{B}y}\times\overline{\dot{C}-\dot{B}v}+\overline{3v^2+B}\times\overline{\dot{C}-\dot{B}y}\times\overline{\dot{C}-\dot{B}w}+\overline{3y^2+B}\times\overline{\dot{C}-\dot{B}v}\times\overline{\dot{C}-\dot{B}w}=$

$\dot{B}^2B^2-3B\dot{C}^2+9C\dot{B}\dot{C}$, ergo ſumma productorum ex quibuſque binis in ſeſe ductis, erit $\frac{\dot{B}^2\times B^2-3\dot{C}^2B+9C\dot{B}\dot{C}}{27C^2+4B^3}$; & contentum ſub tribus radicibus $(\dot{y}\dot{v}\dot{w})$ erit $\frac{\dot{C}^3-\dot{B}^3C+\dot{B}^2\dot{C}B}{27C^2+4B^3}$; unde æquatio quæſita eſt $\dot{y}^3+\frac{B^2\dot{B}^2-3\dot{C}^2B+9C\dot{B}\dot{C}}{27C^2+4B^3}\dot{y}-\frac{\dot{C}^3-\dot{B}^3C+\dot{B}^2\dot{C}B}{27C^2+4B^3}=0$.

COROLLARIUM.

Eodem modo deduci poteſt æquatio, cujus radices ſunt ſecundæ datæ æquationis ordinatæ (y) fluxiones, ex æquatione, cujus radices ſunt primæ datæ æquationis fluxiones, quo deducitur ea æquatio ex datâ; & ſic deinceps.

EXEMPLUM IV.

Invenire ſubtangentium curvæ, cujus æquatio eſt $y^3+By-C=0$, ſummam.

Sint (y, v, w) diverſi ordinatæ (y) valores, & ſubtangentes erunt reſpective

$$\frac{\overline{3y^3+By}\times\dot{x}}{\dot{C}-\dot{B}y},\ \frac{\overline{3v^3+Bv}\times\dot{x}}{\dot{C}-\dot{B}v},\ \frac{\overline{3w^3+Bw}\times\dot{x}}{\dot{C}-\dot{B}w}$$

quibus fractionibus ad communem denominatorem reductis, fit denominator.

$$-\dot{B}^3yvw+\dot{B}^2\dot{C}\times\overline{yv+yw+vw}-\dot{B}\dot{C}^2\times\overline{y+v+w}+\dot{C}^3$$

$=\dot{C}^3-\dot{B}^3C+\dot{B}^2\dot{C}B$; & eâdem methodo invenitur numerator $\overline{4B^2\dot{C}\dot{B}-5BC\dot{B}^2+9\dot{C}C^2}\times\dot{x}$, & conſequenter ſubtangentium ſumma quæſita eſt

$$\frac{\overline{-5\dot{B}^2CB+4\dot{B}\dot{C}B^2+9\dot{C}^2C}\times\dot{x}}{\dot{C}^3-\dot{B}^3C+\dot{B}^2\dot{C}B}.$$

His principiis poſitis, inveniri poſſunt infinitæ conſimiles curvarum proprietates.

Hoc problema multis in caſibus ſolvi poteſt e transformatione duarum vel plurium æquationum in unam, ita ut incognitæ quantitates exterminentur.

Hinc facile conſtat numerum perpendiculorum a dato puncto ad curvam (n) dimenſionum ductorum, non majorem eſſe quam n^2, & numerum tangentium a dato puncto ad curvam ductarum etiam non majorem eſſe quam (n^2).

Sæpe vero ſolutionem his plane diverſam, e lemmate primi libri ſecundo facile excudat aliquis.

PROB. XI.

Sint (n) *rectæ* (A B, C D &c.), n *etiam aliæ* (E F, G H &c.), n *aliæ* (LN, OP &c.) *& ſic deinceps; & a quocunque puncto* (M) *ad has rectas ducantur rectæ* $M\alpha$, $M\beta$, &c. $M\iota$, $M\zeta$, &c. $M\lambda$, $M\mu$, $M\nu$, &c. *in datis angulis vel directionibus, ſingulæ ad ſingulas; & ducatur contentum ſub* (n) *lineis* $M\alpha \times M\beta \times$ &c. & $M\iota \times M\zeta \times$ &c. $M\lambda \times M\mu \times$ &c. *reſpective in quaſlibet datas quantitates; quantitatum reſultantium ſumma nihilo æqualis fiat & locus punctorum* (M), *vel erit curva* (n) *dimenſionum, vel erunt curvæ vel rectæ, quarum dimenſiones ſimul ſumptæ erunt* (n).

Fig. 16. Aſſumatur enim recta (AP) pro abſciſſâ & PM ordinata (y), tum

$$M\alpha = \alpha y + \beta x + \gamma,\quad M\iota = \iota y + \zeta x + n,\quad M\lambda = \lambda y + \mu x + \nu$$

$$M\beta = \overset{1}{\alpha} y + \overset{1}{\beta} x + \overset{1}{\gamma},\quad M\zeta = \overset{1}{\iota} y + \overset{1}{\zeta} x + \overset{1}{n},\quad M\mu = \overset{1}{\lambda} y + \overset{1}{\mu} x + \overset{1}{\nu}$$

&c. &c. &c. &c. &c. &c.

Ducantur hæ quantitates ($M\alpha$, $M\beta$, &c. & $M\iota$, $M\zeta$, &c. $M\lambda$, $M\mu$, &c.) reſpective in ſeſe; & contenta multiplicentur reſpective in datas quantitates, fiant quantitates reſultantes inter ſe æquales, & reſultat æquatio relationem inter abſciſſam $(AP = x)$ & ejus correſpondentes ordinatas (PM, y) exprimens, (n) dimenſiones habens.

EXEMPLUM I.

Fig. 17. Sit Trapezium $(ABCD)$, & a puncto (M) ad ejus latera in datis angulis ducantur lineæ MQ, MR, MS, MT, ſingulæ ad ſingula; & ſit

ſit rectangulum ductarum ad oppoſita duo latera $MQ \times MS$ in datâ ratione $(m:1)$ ad rectangulum ductarum ad alia duo latera $MR \times MT$; tum punctum (M) vel erit punctum ad conicam ſectionem, vel ad duas rectas lineas.

Aſſumatur enim pro abſciſſa $AP\,(x)$, & pro (PM) ordinata (y), tum

$$MS = \alpha y + \beta x + \gamma \quad \& \quad MR = \delta y + \epsilon x + \zeta$$

$$\times \quad MQ = y \qquad\qquad MT = ny + \Theta x$$

$$\overline{MQ \times MS = \alpha y^2 + \beta xy + \gamma y} \quad = m \times MR \times MT = m\delta n y^2 + \overline{\delta\Theta + \epsilon n} \times x + n\zeta \times y + \Theta\epsilon x^2 + \zeta\Theta x,$$

quæ eſt æquatio ad conicam ſectionem, vel duas rectas lineas.

COROLLARIUM I.

Hinc conſtat curvam tranſire per angulum Trapezii (A), quoniam ſi abſciſſa (x) nihilo eſt æqualis, tum correſpondens ordinata (y) nihilo eſt æqualis, & ſic probari poteſt conicam ſectionem tranſire per puncta (B, C, D), & conſequenter Trapezium in conicâ ſectione inſcriptum eſſe.

COROLLARIUM II.

Sit conica ſectio, in quâ inſcribitur Trapezium $(ABCD)$, & M quodlibet peripheriæ punctum; in lineas AB, BC, CD, DA in datis angulis ducantur MQ, MR, MS, MT; & erit rectangulum $MQ \times MS$ ad rectangulum $MR \times MT$ in datâ ratione.

EXEMPLUM II.

Sint tres lineæ AB, CD, EF; & tres aliæ BG, DH, FI, & a puncto Fig. 18. (M) in datis angulis ad lineas prædictâs ducantur lineæ MQ, MR, MS, MV, MW, MZ reſpective; & ſit ſolidum $MQ \times MR \times MS$ ad ſolidum $MV \times MW \times MZ$ in datâ ratione; & punctum (M) ponitur ad curvam ſecundi generis.

Hoc demonſtrari poteſt eodem modo, quo prior exemplum.

COROLLARIUM I.

Puncta B, D, F, L, N &c. in quibus lineæ AB, CD, EF ſecant lineas, BG, DH, FI, erunt puncta ad curvam ſecundi generis.

COROLLARIUM II.

Sit curva ſecundi generis, & ducantur tres lineæ AB, CD, EF, & tres aliæ BG, DH, FI, has tres ſecantes in punctis curvæ $(B, D, F,$

$L,$

L, N, &c.) & a quolibet curvæ puncto in hæc latera ducantur lineæ in datis angulis MQ, MR, MS, MZ, MV, MW, reſpective; tum ſolidum $MQ \times MR \times MS$ erit ad ſolidum $MZ \times MV \times MW$ in datâ ratione.

Et ſic in quibuſdam caſibus de lineis quarti, quinti, ſexti, &c. ordinum.

Fig. 16. Sit curva ($m\overset{1}{m}\overset{11}{m}$ &c.) habens (m) dimenſiones, ſint etiam ($n-m$) rectæ (AB, CD &c.) & (n) etiam rectæ (EF, GH &c.); a quocuque puncto (M) in datâ directione (MP) ad curvam ducatur linea $Mm\overset{1}{m}\overset{11}{m}$ &c. eam ſecans in (m) punctis ($m, \overset{1}{m}, \overset{11}{m}$, &c.); & ſic agantur in datis angulis ad rectas (AB, CD &c, EF, GH &c.) lineæ $M\alpha$, $M\beta$, $M\gamma$, &c, $M\iota$, $M\zeta$, Mn &c. & ſit contentum ſub (n) lineis $Mm \times M\overset{1}{m} \times M\overset{11}{m} \times$ &c. $\times$ $M\alpha \times M\beta \times M\gamma$ ad contentum ſub (n) lineis $M\iota \times M\zeta \times Mn \times M\theta$ &c. in datâ ratione; tum punctum (M) erit punctum ad curvam (n) dimenſionum, vel ad curvas, quarum dimenſiones ſimul ſumptæ ſunt (n).

Aſſumatur enim recta (AP, x) pro abſciſſâ, & PM pro ordinata (y), ſit etiam æquatio relationem inter abſciſſam ($AP=x$) & datæ curvæ ordinatas (v) Pm, $P\overset{1}{m}$ &c. ubi m & $\overset{1}{m}$ in eâdem directione, *i.e.* linea cum M jacent, exprimens

$Av^m + \overline{a+bx}\,v^{m-1} + \overline{c+dx+ex^2}\,v^{m-2} + \&c. = 0$, & contentum ſub ($m$) lineis $Mm \times M\overset{1}{m} \times M\overset{11}{m} \times$ &c. erit

$$Ay^m + \overline{a+bx}\,y^{m-1} + \overline{c+dx+ex^2}\,y^{m-2} + \&c.$$

lineæ vero $M\alpha$, $M\beta$, $M\gamma$, &c. ſint

$M\alpha = \alpha y + \beta x + \gamma$ & ſic de lineis $M\iota$, $M\zeta$, Mn, &c.

$M\beta = \overset{1}{\alpha} y + \overset{1}{\beta} x + \overset{1}{\gamma}$ &c.

Mn, &c. tum contentum ſub (m) lineis ($Mm \times M\overset{1}{m} \times M\overset{11}{m} \times$ & ($\overline{n-m}$) $M\alpha$, $M\beta$, $M\gamma$ &c. *i.e.*

$$\overline{Ay^m + \overline{a+bx}\,y^{m-1} + \overline{c+dx+ex^2}\,y^{m-2} + \&c.} \times \overline{\alpha y + \beta x + \gamma} \times \overline{\overset{1}{\alpha} y + \overset{1}{\beta} x + \overset{1}{\gamma}}$$

erit in datâ ratione ad contentum ſub (n) lineis $M\iota \times M\zeta \times Mn \times M\theta$ &c. cujus proportionis extremis & mediis in ſeſe ductis, & inter ſe æquales eſſe ſuppoſitis, reſultat æquatio, in quâ (n) & non plures inveniuntur

untur dimenſiones incognitarum quantitatum (x & y) abſciſſæ & ordinatæ quæſitæ.

Sint plures curvæ & a quocunque puncto in eas ducantur lineæ in datis directionibus, & eadem erit ratiocinandi methodus de hujuſmodi punctorum loco.

Hinc facile multæ hujuſmodi proprietates deduci poſſunt, quæ omnibus algebraicis curvis competunt.

COROLLARIUM.

Sint quæcunque lineæ, quæ exprimantur terminis abſciſſæ & ordinatæ (x & y), &c. e priori analyſi inveniri poteſt, an data æquatio

$$Ay^n + \overline{a+bx}\,y^{n-1} + \overline{c+dx+ex^2}\,y^{n-2} + \&c. = 0$$

eſt integra harum linearum functio.

Ex. gr. Curvæ parium dimenſionum ſemper formari poſſunt e linearum in quæcunque puncta ductarum functionibus; nullam enim habet quadratum lineæ a dato puncto ad curvam ductæ, irrationalitatem: et infinitis modis variari poſſunt hæ functiones.

Hinc deduci poſſunt quam plurimæ conicarum ſectionum proprietates.

PROB. XII.

Deſcribere curvas, quarum æquationes relationem inter abſciſſam & ejus correſpondentes ordinatas dantur.

Inveniatur methodus conſtruendi generaliter datam æquationem, & fit.

COROLLARIUM I.

Hinc omnes curvæ trium & quatuor dimenſionum deſcribi poſſunt, datur enim methodus in genere conſtruendi æquationes trium & quatuor dimenſionum, circuli & conicæ parabolæ ope; datur igitur methodus deſcribendi curvas trium vel quatuor dimenſionum.

COROLLARIUM II.

Multæ ſunt curvæ ſuperiorum dimenſionum, quæ quarumcunque duarum curvarum inferiorum ordinum ope, haud generaliter deſcribi poſſunt.

Æquationes enim, quas conſtruere licet harum duarum curvarum ope, per octavi problematis corollarium primum libri ſecundi, haud ſæpe includunt omnes, quicunque ſit valor abſciſſæ, harum curvarum æquationes.

Hinc ſequitur in deſcribendi[illegible] [illegible] (x) [illegible] haud raro opus eſſe quam plurimis in [illegible]

CO-

COROLLARIUM III.

Si recta rotetur, ſemper tranſeuns per datum curvæ punctum, tum dimenſiones æquationis conſtruendæ ſolummodo erunt $(n-1)$.

Si per duplex punctum curvæ tranſit, tum dimenſiones ſolummodo erunt $(n-2)$; & ſic deinceps.

COROLLARIUM IV.

Curvam (n) dimenſionum, cujus $(n-1)$ crura ſeſe interſecant in dato puncto (A), deſcribere.

Fig. 20. Sit æquatio relationem inter abſciſſam (AP, x) a puncto (A) incipientem & ejus ordinatas (PM, y), in eam ad rectum angulum inſiſtentes

$$Ay^n + \overline{a+bx}\, y^{n-1} + \overline{c+dx+ex^2}\, y^{n-2} + \&c. = 0.$$

Ita transformetur hæc æquatio, ut exprimat relationem inter abſciſſam Ap, z & ordinatam (pM, v).

Sint q & s ſinus & coſinus anguli ApP; tum ſubſtituatur in datâ æquatione pro x, qz; & pro y, $v+sz$; & coefficientes terminorum (z^n, z^{n-1}) erunt reſpective

$$As^n + bs^{n-1}q + es^{n-2}q^2 + \&c.$$

$$\&\quad as^{n-1} + ds^{n-2}q + bs^{n-3}q^2 + \&c.$$

& conſequenter ſi in abſciſſâ (Ap) ſumatur linea $\left(Am = \dfrac{as^{n-1} + ds^{n-2}q + bs^{n-3}q^2 + \&c.}{As^n + bs^{n-1}q + es^{n-2}q^2 + \&c.}\right)$, erit (m) punctum in peripheriâ curvæ.

Hinc facile inveniri poteſt quilibet punctorum numerus, per quæ tranſit curva.

Hujus quantitatis e geometriæ principiis ſequitur conſtructio.

Si curva habet punctum, in quo $(n-2)$ crura ſeſe interſecant, tum inveniri poſſunt puncta, per quæ tranſit curva, quadraticæ æquationis ope, unde facile conſtrui poteſt curva. Et ſic deinceps.

Non opus eſt hic deſcribere curvas motu datorum angulorum vel curvarum circa data puncta, & ſic deinceps: e generalibus principiis facile deduci poſſunt.

Datâ methodo deſcriptionis, erui poſſunt curvæ dimenſiones, vel e numero punctorum, in quibus a rectâ lineâ ſecari poteſt; vel e dimenſionibus æquationis relationem inter abſciſſam & ejus correſpondentes ordinatas relationem exprimentis.

Ex.

Ex. gr. Sint (n) puncta (a, b, c, d, &c.) &ſit (m) punctum, a quo ad data puncta (a, b, c, d, &c.) ducuntur lineæ (am, bm, cm, &c.) & ſit ſumma linearum $am+bm+cm+dm+$ &c. data quantitas; & locus punctorum (m) nonpoteſt eſſe curva plurium quam (2^n) dimenſionum. Fig. 21.

Datis quibuſdam curvis, ex iis ſæpe deduci poſſunt aliæ.

E punctis datæ curvæ, quolibet modo deriventur alia puncta; & conſequenter e datâ curvâ ſequitur altera.

Ex. gr. Sit curva *MNS*, & a dato puncto *C* in tangentem curvæ ſemper demittatur perpendiculum (*CP*), tum locus hujuſmodi punctorum (*P*) erit deducta curva. Fig. 22.

Et ſic ducatur a puncto (c) perpendiculum ad lineam *CS*, jungentem punctum curvæ (*S*) & datum punctum (*C*); hoc vero perpendiculum incidat in lineam (*CS*) in puncto (n); & locus hujuſmodi punctorum erit nova curva. Fig. 22.

Et ſic a dato puncto (*C*) ducatur perpendiculum in curvæ perpendiculum, & formatur alia curva: ſed de his ſatis. Fig. 23.

De curvis per data puncta tranſeuntibus, vel datas lineas tangentibus.

1. Sit curva data, tum tria puncta, vel tres tangentes,&c. ſufficiunt ad determinandam ejus poſitionem.

2. Numerus tangentium vel punctorum, quæ ſufficiant ad determinandam curvam, pendet e numero incognitarum coefficientium, quas habet data æquatio relationem inter abſciſſam & ejus correſpondentes ordinatas exprimens.

Ex. gr. Si curva (n) dimenſionum generale habet centrum, quod ſit (*C*), duci poteſt curva per $n \times \frac{n+1}{2}$ puncta; tot enim habet æquatio incognitas coefficientes.

Data æquationis formula, relationem inter abſciſſam & ejus correſpondentes ordinatas exprimentis, invenire curvam, quæ tranſeuns per data puncta, tangit datas lineas, &c.

Aſſumatur linea (*AP*) pro abſciſsâ, alia vero (*PM*) pro ordinatarum directione; pro ſingulo dato puncto, datur & abſciſſa & correſpondens ordinata, quibus pro ſuis valoribus in datâ æquatione ſubſtitutis, reſultant tot æquationes, quot dantur puncta. Fig. 24.

Ducatur linea (nm) parallela ordinatæ (MP), ſecans tangentem (EF) in (n), e datâ æquatione inveniatur ratio, quam habent fluxiones abſciſſæ & ordinatæ, quæ eſt $Fm : nm$; inveniatur etiam ſubtangens, cui æqualis eſt $x + AF$; ita transformentur omnes hæ æquationes in unam, ut exterminentur omnes præter unam incognitam coefficientem, & reſultat æquatio, cujus incognita quantitas eſt ea coefficiens.

Fortaſſe vero requirit problema, ut prius transformetur abſciſſa in alteram, quod fit per problema primum.

E prædictis æquationibus conſtant dimenſiones reſultantis æquationis.

THEO.

Fig. 25. *Sit circulus* ABDEFG &c. P, *cujus centrum eſt* C, *& radius, unitas: ſint etiam arcus* AB, BD, DE, EF, FG, *&c. inter ſe æquales, pro chordâ circuli* (PB) *ſcribatur* (p), *& aſſumatur quadratica æquatio* $x^2 - px + 1 = 0$, *ſupponantur duæ hujuſce æquationis radices* α *&* β; *tum erunt reſpective chordæ*

$$BP = \alpha + \beta$$
$$PD = \alpha^2 + \beta^2$$
$$PE = \alpha^3 + \beta^3$$
$$PF = \alpha^4 + \beta^4 \quad \text{\& ſic deinceps.}$$

DEMONSTRATIO.

Sit $BP = p = a$. tum e trigonometriâ ſequitur

$$PD = pa - 2 = b, \text{ \&}$$
$$PE = pb - a = c,$$
$$PF = pc - b = d. \text{ \& ſic deinceps.}$$

Sit quadratica æquatio, $x^2 - px + 1 = 0$, tum per *Newtoni* regulam,

$$\alpha + \beta = p = BP$$
$$\alpha^2 + \beta^2 = pa - 2 = PD$$
$$\alpha^3 + \beta^3 = pb - a = PE$$
$$\alpha^4 + \beta^4 = pc - b = PF. \text{ \& ſic deinceps.}$$

COROLLARIUM I.

Sit arcus $AG = n \times AB$, & chorda $BP = p$; ſit etiam quadratica æquatio $(x^2 - px + 1 = 0)$, cujus radices ſunt α & β; tum chorda $(PG = \alpha^n + \beta^n)$ erit per lemma primum præcedentis libri

$$p^n - np^{n-2} + n \times \frac{\overline{n-3}}{2} p^{n-4} - n \times \frac{\overline{n-5}}{2} \times \frac{\overline{n-4}}{3} p^{n-6} + n \times \frac{\overline{n-7}}{2} \times \frac{\overline{n-6}}{3} \times \frac{\overline{n-5}}{4} p^{n-8} + \text{\&c.}$$

COROLLARIUM II.

Duæ radices (α & β) quadraticæ æquationis $x^2 - px + 1 = 0$ ſunt reſpective $\sqrt{\frac{1}{4}p^2 - 1} + \frac{1}{2}p$ & $\frac{1}{2}p - \sqrt{\frac{1}{4}p^2 - 1}$, & conſequenter $\overline{\frac{1}{2}p + \sqrt{\frac{1}{4}p^2 - 1}}^n + \overline{\frac{1}{2}p - \sqrt{\frac{1}{4}p^2 - 1}}^n = \alpha^n + \beta^n = p^n - np^{n-2} + n \times \frac{n-3}{2} p^{n-4} - n \times \frac{n-5}{2} \times \frac{n-4}{3} p^{n-6} + \&c. = 0.$

COROLLARIUM III.

Hinc etiam reduci poteſt in ſuos quadraticos diviſores, quantitas ($\pi^{2n} - \alpha\pi^n + 1 = 0$.)

Sit quadratica æquatio $x^2 - \overline{\frac{\pi^2 + 1}{\pi}}\,x + 1 = 0$, cujus radices erunt reſpective π & $\frac{1}{\pi}$; & aſſumantur duo arcus AB & AG, quorum AG ſit $n \times AB$; ſit etiam chorda $PG = \alpha$, $PB = \frac{\pi^2 + 1}{\pi}$, & radius unitas; tum erit $PG = \alpha = \pi^n + \frac{1}{\pi^n}$, & conſequenter $\pi^{2n} - \alpha\pi^n + 1 = 0$. Datâ chordâ ($\alpha$), invenitur arcus ($PG$), & ejus ſupplementum ($AG$); & deinde arcus $AB = \frac{AG}{n}$, vel $\frac{360 + AG}{n}$, vel $\frac{2 \times 360 + AG}{n}$, vel $\frac{3 \times 360 + AG}{n}$, vel $\frac{4 \times 360 + AG}{n}$, & ſic deinceps, dato igitur arcu (AB) invenitur ejus ſupplementi chorda (PB), pro quâ ſcribatur β, & erit $\frac{\pi^2 + 1}{\pi} = \beta$, quâ æquatione reductâ, reſultat quadraticus diviſor $\pi^2 - \beta\pi + 1 = 0$.

COROLLARIUM IV.

Cubica æquatio ($\pi^{3n} - \alpha\pi^{2n} + \beta\pi^n - 1 = 0$) in ſimplicem & quadraticam; biquadratica ($\pi^{4n} - \alpha\pi^{3n} + \beta\pi^{2n} - \gamma\pi^n + 1$) in duas quadraticas reduci poteſt per methodos in analyſi vulgariter notas.

COROLLARIUM V.

Æquationis in genere quadratici diviſores inveniri poſſunt, tranſformando æquationem in alteram, cujus radices ſunt ſummæ ex quibuſque

buſque duabus datæ æquationis radicibus; & e conſtructione æquationis reſultantis (i. e. ex ejus poſſibilibus radicibus) inveniri poſſunt quadratici diviſores quæſiti.

COROLLARIUM VI.

Sit (r) radius, & (a) cujuſlibet arcus $(180-A)$ chorda, & chordæ ſupplementorum arcuum

$$\frac{A}{n},\ \frac{360+A}{n},\ \frac{2\times 360+A}{n},\ \frac{3\times 360+A}{n},\ \&c.$$

erunt radices æquationis

$$x^n - nr^2x^{n-2} + n\times\frac{n-3}{2}r^4x^{n-4} - \&c. = a.$$

COROLLARIUM VII.

Dividatur circuli peripheria in (n) æquales partes $(AB, BD, DE, EF, FG,$ &c.) aſſumatur quodcunque punctum (P) in peripheriâ circuli, & ducantur chordæ $PA, PB, PD, PE,$ &c, & erunt hæ chordæ radices æquationis

$$x^n - nr^2x^{n-2} + n\times\frac{n-3}{2}r^4x^{n-4} - \&c. = a.$$

COROLLARIUM VIII.

Sit (m) impar numerus & minor quam (n), & ſumma (m) poteſtatum ex ſingulis his chordis, erit nihilo æqualis.

COROLLARIUM IX.

Sit (m) minor quam (n), & ſumma $(2m)$ poteſtatum ex ſingulis chordis, erit

$$\frac{1\cdot 3\cdot 5\cdot 7\ldots 2m-1}{1\cdot 2\cdot 3\ldots m}\times 2^m\times n\times r^{2m}$$

Conſtant hæc e lemmate primo libri primi, ſed quoniam quadratum chordæ per diametrum i.e. $(2r)$ diviſum, æquale eſt verſo ſinui; ſumma (m) poteſtatis ex ſingulis verſubus ſinubus, erit

$$\frac{1\cdot 3\cdot 5\cdot 7\ldots 2m-1}{1\cdot 2\cdot 3\ldots m}\times nr^m.$$

Fig. 26. Aſſumatur punctum (P) quod haud poſitum eſt in circuli peripheriâ, & dividatur eodem modo, quo prior, circumferentia in (n) æquales partes $(AB, BD, DE,$ &c.) a puncto (P) ad hæc puncta diviſionum ducantur

cantur lineæ $PB, PD, PE,$ &c. producatur linea (CP) jungens centrum & punctum (P), donec peripheriam ſecat in puncto (Q); ducantur lineæ QB, QD, QE, &c; pro lineis PB, PD, PE, &c, ſcribatur reſpective (z); & pro lineis QB, QD, QE, &c. reſpective (x); pro radio, unitas, & pro lineâ (CP), a; tum ſequitur $x^2 = \frac{z^2 + 2a - a^2 - 1}{a}$, qua quantitate pro ſuo valore in æquatione

$$x^n - nr^2x^{n-2} + n \times \frac{n-3}{2} r^4 x^{n-4} - \text{\&c.} = 0.$$

ſubſtitutâ, reſultat (ſi n eſt par numerus) æquatio

$$\begin{array}{llll} z^n + \frac{n}{2} \times \overline{2a - a^2 - 1}\, z^{n-2} & + \frac{n}{2} \times \frac{n-2}{4} & \times \overline{2a - a^2 - 1}^2 z^{n-4} & + \text{\&c.} = 0 \\ \quad - n \qquad a & - n \times \frac{n-2}{2} & \times a \times \overline{2a - a^2 - 1} & - \text{\&c.} \\ & + n \times \frac{n-3}{2} & \times \quad a^2 & + \text{\&c.} \end{array}$$

Facile e calculo conſtat ultimum hujuſce æquationis terminum eſſe $a^n \pm 1$, qui æqualis eſt producto ſub ſingulis (n) radicibus.

Si vero (n) eſt impar numerus, tum ſubſtituatur pro (x), $\sqrt{\frac{z^2 + 2a - a^2 - 1}{a}}$, & reſultantis æquationis ultimus terminus etiam erit $a^n \pm 1$.

PROB. XIII.

Datâ æquatione, relationem inter abſciſſam & ejus correſpondentes ordinatas, rectum angulum cum abſciſſa facientes, exprimente Fig. 27.

$$Ay^n + \overline{a + bx}\, y^{n-1} + \overline{c + dx + ex^2}\, y^{n-2} + \text{\&c.} = 0.$$

& circa abſciſſæ verticem (P) ducantur lineæ $(PA, PB, PC, PD$, &c.) æquales angulos inter ſe facientes, invenire æquationem, cujus radices ſunt reſpective lineæ $(PA, P\overset{1}{A}, PB, P\overset{1}{B}, PC, P\overset{1}{C}$, &c)

Sint l & r duorum arcuum $(\alpha$ & $\beta)$ coſinus, quorum prior eſt multiplex (m) poſterioris, & radius eſt unitas, tum

$$r = \frac{1}{2} \sqrt[m]{l + \sqrt{l^2 - 1}} + \frac{\frac{1}{2}}{\sqrt[m]{l + \sqrt{l^2 - 1}}}$$

&

& r erit reſpective coſinus arcuum $\frac{\alpha}{m}$, $\frac{360-\alpha}{m}$, $\frac{360+\alpha}{m}$, $\frac{720-\alpha}{m}$, & ſic deinceps, i. e. ſi modo peripheria circuli dividatur in $(2m)$ æquales partes, quarum unum punctum diviſionis eſt arcus $\left(\frac{\alpha}{m}\right)$ tum erit (r) coſinus horum arcuum a ſe ipſis æqualiter diſtantium.

Si vero (r) eſt coſinus, tum ſinus erit

$$\sqrt{1-r^2} = \sqrt{\frac{1}{2} - \frac{1}{4}\sqrt[m]{2l^2-1+2l\sqrt{l^2-1}} - \frac{\frac{1}{4}}{\sqrt[m]{2l^2-1+2l\sqrt{l^2-1}}}};$$

ſit z radix æquationis quæſitæ, & abſciſſa (x) erit

$$z\times\frac{1}{2}\sqrt[m]{l+\sqrt{l^2-1}} + \frac{\frac{1}{2}}{\sqrt[m]{l+\sqrt{l^2-1}}};$$ & ordinata (y)

$$z\times\sqrt{\frac{1}{2} - \frac{1}{4}\sqrt[m]{2l^2-1+2l\sqrt{l^2-1}} - \frac{\frac{1}{4}}{\sqrt[m]{2l^2-1+2l\sqrt{l^2-1}}}};$$ quibus quantitatibus pro ſuis valoribus in data æquatione ſubſtitutis, reſultat æquatio, cujus radix eſt (z).

COROLLARIUM I.

Exterminentur irrationales quantitates, & aſcendere poteſt æquatio reſultans ad $(2nm)$ dimenſiones, ſi vero omnes poteſtates ordinatæ (y) ſunt pares, tum deprimi poteſt æquatio in alteram (nm) dimenſiones habentem.

COROLLARIUM II.

Inventa æquatione, cujus radix eſt (z), invenitur ſumma radicum, contentorum ſub quibuſque duabus, tribus, &c.

Sed ſine ſurdarum quantitatum exterminatione, inveniri poteſt ſumma (n) radicum, contentorum, &c.

Facile eſt has proprietates in ellipſes circa centrum, & quaſdam alias curvas extendere, & contentum ſub ſingulis valoribus invenire.

In prioribus, e datâ æquatione relationem inter abſciſſam & ejus ordinatas exprimente, inventæ fuerunt proprietates; & vice versâ, e datis proprietatibus inveniri poteſt formula æquationis.

Et

Et ſic de innumeris aliis proprietatibus.

Prius quam huic capiti finem imponam, adjicere viſum eſt tres ſubſequentes propoſitiones.

THEO.

Sit cubica parabola, cujus æquatio relationem inter abſciſſam (*AP*$=x$) & ejus correſpondentes ordinatas (*PM*$=y$) exprimens, eſt $p^2x=y^3$. Fig. 28.

Ducatur triangulum *EFG*, cujus baſis (*EF*) æquatur ultimæ ordinatæ (*PM*), altitudo vero abſciſſæ dimidio ($\frac{1}{2}AP$), & angulus *EFG*= *APM*; biſecentur lineæ *AM* & *EF* reſpective in duas æquales partes *Ar* & *rM*; *Eα*, *αF*; per punctum (*r*) ducatur linea parallela axi, ſecans curvam in *a*, jungantur lineæ *Aa* & *Ma*, & ſic per punctum (*α*) ducatur linea (*αβ*) parallela lineæ (*FG*), & triangulum *AMa* erit æquale trapezio *FαβG*.

Biſecentur lineæ *Ar*, *rM*, *Eα* reſpective in duas æquales partes *As*, *sr*; *rt*, *tM*, *Eγ*, *γα*, & ducantur lineæ *sb*, *tc* per (*s* & *t*) parallelæ axi; & ſecantes curvam in punctis (*b* & *c*), & ſic ducatur linea *γδ* parallela lineæ (*FG*), ducantur etiam lineæ *Ab*, *ba*; *ac*, *cM*, & ſumma duorum triangulorum (*Aba*+*acM*) æqualis erit trapezio *αβγδ*, & ſic continuo biſecentur & lineæ (*AM*) partes, & baſis trianguli (*EF*,) *Eα*, *Eβ*, &c. & figura inſcripta ſemper erit æquale correſpondenti trapezio, & conſequenter triangulum *EFG* æquale eſt figuræ *AMcabA* & exinde figura *MPAbacM* etiam æqualis eſt iſti triangulo.

Conſtat igitur in cubica æque ac in conicâ parabolâ ſummam ex ſucceſſivis triangulis ex continuâ abſciſſæ (*AM*) partium biſectione ortis, eſſe inter ſe ſemper in ratione numeri quatuor ad unitatem.

Hanc ratiocinandi methodum in alias curvas promovere licet, ſed de hoc in alio loco.

Ejus demonſtratio hoc in loco nimis longa eſt inſerere.

PROB. XIV.

Inſcribere in data ellipſi (EFGH) *parallelogramma, dato parallellogrammati* (ABCD) *ſimile.* Fig. 29.

Inveniantur datæ ellipſeos duæ conjugatæ diametri (*EF*, *GH*), quæ angulum interſe æqualem parallellogrammatis angulo (*ABC*) faciunt.

Sit

Sit *C* centrum, & in lineâ (*EF*) capiatur *CP* = *AB*; per punctum (*P*) ducatur linea *PQ*=*BC*, & parallela diametro *HG*; jungatur linea *CQ* ſecans curvam in (*s*), & per (*s*) ducatur linea (*st*) parallela diametro *HG*, & erit parallellogrammatis quæſiti latus; ducantur lineæ *sm* & *tn* parallelæ diametro *E F*, & erunt duo reliqua parallellogrammatis latera.

COROLLARIUM I.

Si data ellipſis duas non habet conjugatas diametros angulum inter ſe dato parallellogrammatis angulo (*ABC*) æqualem facientes, tum haud admittit ſolutionem problema.

COROLLARIUM II.

Hinc conſtat methodus ducendi duas lineas per centrum datæ conicæ ſectionis, datam inter ſe habentes rationem & datum angulum ſecum facientes.

PROB. XV.

Fig. 30. *Inſcribere in datâ ellipſi* (EFGH) *trapezium, quod erit ſimile dato trapezio* (ABCD).

Ducantur duæ dati trapezii diagonales *AC* & *BD*; ſeſe ſecantes in puncto (*n*), & ſit (*nA*) major quam *Cn*; & *Bn* quam *nD*.

Secundum problema præcedens ducantur duæ lineæ (*LM* & *RS*) per ellipſeos centrum (*C*) angulum (*LCR*) inter ſe habentes, angulo(*AnB*), quem inter ſe faciunt diagonales *AC* & *BD*, æqualem, & ita ut $LM^2 : RS^2 :: An \times nC : Bn \times nD$.

Ducantur *NP* & *TV* conjugatæ diametrorum *LM* & *RS*, ſit etiam $2CP : Pp :: AC : nC$ & $2CT : Tt :: BD : nD$; & ſuper *CL* & *Cp* tanquam ſemiconjugatas diametros deſcribatur ellipſis *LpbPL*; & ſuper ſemiconjugatas diametros (*RC* & *Ct*) deſcribatur ellipſis (*RtbSR*); & interſecent hæ duæ ellipſes ſeſe in puncto (*b*), per (*b*) ducantur duæ lineæ (*bab* & *bcd*) reſpective parallelæ lineis *LM* & *RS*; & erunt quæſiti parallellogrammatis diagonales.

Angulus enim, quem inter ſe faciunt lineæ *bab* & *bcd* æqualis eſt angulo (*AnB*), & diagonalium ſegmenta *ab*, *bb*, *cb*, *bd*, erunt inter ſe reſpective in ratione linearum *An*, *nC*, *Bn*, *nC*: & ſic de aliis hujuſmodi propoſitionibus.

LEMMA.

LEMMA.

Omnia parallelogramma circa circulum deſcripta ſunt rhombi, & cujuſcunque parallelogrammi duæ diametri ſeſe interſecant in circuli centro, & rectum angulum inter ſe faciunt.

Et conſequenter duæ diagonales parallelogrammi circa ellipſin deſcripti erunt inter ſe conjugatæ diametri.

PROB. XVI.

Circa datam ellipſim (EFGH) *parallelogrammum deſcribere, quod parallelogrammo* (ABCD) *erit ſimile.* Fig. 31.

Ducantur duæ dati parallelogrammi diametri (*AC* & *BD*) ſeſe inter ſecantes in centro *N* ad datum angulum *ANB*.

Ellipſeos inveniantur duæ conjugatæ (*EF*, *GH*) diametri, quæ faciunt angulum inter ſe æqualem dato angulo *ANB*, & hæ conjugatæ diametri (*EF*, *GH*) juxta præcedens lemma erunt diametri parallellogrammi quæſiti.

Scribantur pro ſemiconjugatis datæ ellipſeos diametris (*EF*, *GH*) reſpective t, c; & pro dati parallelogrammi ſemidiametris (*AN*, *NB*) reſpective r, s, aſſumatur abſciſſa $CK = \sqrt{\frac{t^4 s^2}{t^2 s^2 + r^2 c^2}}$, per *K* ducatur ejus ordinata *KL*; ellipſeos punctum (*L*) ducatur linea *MN* tangens, & ſecans diametros *EF* & *GH* in *M* & *N*, & *MN* erit latus parallelogrammi quæſiti, quo dato facile etiam deduci poſſunt reliqua latera.

COROLLARIUM.

Si nullæ ſint datæ ellipſeos conjugatæ diametri, quæ angulum inter ſe faciunt æqualem angulo, quem inter ſe habent dati parallelogrammi diametri, tum haud circumſcribi poteſt datæ ſpeciei parallelogrammum.

CAP. II.

PROB. I.

Fig. 32. *Conceſſis curvilinearum quadraturis, rectificare curvoidem, ſeu curvam a puncto in curvâ ſuper rectam rotante generatam.*

Rotetur curva MPL ſuper rectam lineam HG, ſit M punctum, quod deſcribit curvam, cujus rectificatio requiritur, ducatur (PI) tangens ad curvam in puncto P, MI perpendiculum ad tangentem (PI) a dato puncto M; & agatur Lm perpendiculum in axem $= MI$, mr axi parallela $= PI$, & ſecans curvoidem in puncto (n); agatur linea Lr, cui parallela ſit linea (Qn); & erit (Qn) perpendicularis ad curvam in puncto (n).

Ex hypotheſi enim $Lm = MI$, $mr = IP$ & $\angle Lmr = \angle MIP$ & conſequenter $Lr = MP$, & quando tangit axim in puncto (Q) rotans curva (n) erit punctum curvoidem deſcribens, unde linea (Qn) erit perpendicularis ad curvoidem, & conſequenter e ſimilibus triangulis ſequitur curvoidis incrementum $\frac{MP \times \dot{MI}}{PI}$, cujus igitur fluens eſt curvoidis arcus.

EXEMPLUM I.

Fig. 33. Sit curva (MPL) rotans, conici generis parabola, & punctum (M) curvoidem deſcribens, ejus principalis vertex.

Scribantur p & x reſpective pro axis parametro & abſciſsâ a vertice incipiente; tum

$$MI = \frac{\sqrt{p}\,x}{\sqrt{p+4x}},\ MP = \sqrt{px+x^2},\ PI = \frac{\overline{p+2x} \times \sqrt{x}}{\sqrt{p+4x}}\ \&$$

$$\frac{MP \times \dot{MI}}{PI} = \frac{\sqrt{p+x} \times \sqrt{p} \times \dot{x}}{p+4x};$$ cujus fluens & ejus conſtructio dantur a viris doctiſſimis Moivræo & Coteſio.

EXEMPLUM II.

Fig. 34. Sit curva (MPI) rotans conici generis ellipſis, & punctum (M) ellipſoidem deſcribens, ejus principalis vertex. Pro ellipſeos majori axi, ejus parametro & abſciſsâ a puncto ellipſoidem deſcribente, incipiente, ſcri-

ſcribantur reſpective $2a, p, x$, &

$MP = \sqrt{\frac{2pax - \overline{p-2ax}^2}{2a}}$, $MI = \frac{p\frac{1}{2}ax}{\sqrt{2ax \times \overline{2ax - x^2} + p \times \overline{a-x}}}$,

& $PI = \sqrt{MP^2 - MI^2}$.

Hinc inveniri poteſt ellipſoidis incrementum $\frac{MP \times \dot{MI}}{PI}$. & conſequenter per curvarum quadraturas ejus fluens *i. e.* ellipſoidis arcus.

COR. I.

Iiſdem poſitis, invenire curvoidis aream. Fig. 52.

Linea $mn = mr + rn$ (LQ vel arcui LP), & incrementum areæ curvoidis eſt

$MI \times \text{flux}\, mn = MI \times \overline{\text{flux}\, PI + \text{flux}\, ar\, LP}$; cujus fluens eſt area ipſa.

EXEMPLUM.

Sit curva rotans parabola, & punctum generans, ejus principalis vertex, tum e prædictis conſtat $MI = \frac{\sqrt{px}}{\sqrt{p+4x}}$, & $PI = \frac{\overline{p+2x}\sqrt{x}}{\sqrt{p+4x}}$, Fig. 53.

cujus fluxio eſt $\frac{8x^2 + 3px + \frac{1}{2}p^2}{\overline{p-4x}^{1\frac{1}{2}} \times x^{\frac{1}{2}}}\dot{x}$.

& fluxio arcus eſt $\frac{\sqrt{p+4x}\;\dot{x}}{2\sqrt{x}}$; ducantur hæ fluxiones reſpective in

$MI = \frac{\sqrt{px}}{\sqrt{p+4x}}$, & reſultat

$$MI \times \overline{\text{flux}\, PI} = \sqrt{p} \times \frac{8x^3 + 3px^2 + \frac{1}{2}p^2x}{\overline{p+4x}^2 \times x^{\frac{1}{2}}}\dot{x},$$

& $MI \times \text{flux}\, arc\, LP = \frac{1}{2}\sqrt{p} \times \sqrt{x} \times \dot{x}$, & conſequenter

$$MI \times \text{flux}\, \overline{PI + \text{flux}\, ArLP} = \sqrt{p} \times \overline{\tfrac{1}{2}\sqrt{x} + \frac{8x^3 + 3px^2 + \frac{1}{2}p^2x}{\overline{p+4x}^2 \times x^{\frac{1}{2}}}} \times \dot{x}$$

cujus fluens innoteſcitur, & eſt area quæſita.

COR. II.

Invenire curvoidis in quocunque puncto (n) curvaturæ radium.

1. Sint

Fig. 32. 1. Sint $\dot{z}, \dot{x}, \dot{y}$ reſpective $\frac{MP \times \dot{MI}}{PI}$, $\dot{MI}$ & $\frac{MI \times \dot{MI}}{PI}$; tum radius curvaturæ, per vulgarium fluxionum principia, in puncto (n), erit $\left(\frac{\dot{z}^3}{\dot{y}\ddot{x} - \dot{x}\ddot{y}}\right)$.

2. Scribatur pro chorda curvaturæ in puncto P & directione MP, R, tum ſubſequens proportio invenit radium curvaturæ in puncto (n) quæſitum $R - 2\mathcal{Q}n\,(MP) : 2\mathcal{Q}n :: \mathcal{Q}n$: radium quæſitum.

COR. III.

Sit curva generans algebraica curva, & rectificari poteſt generaliter curvois; ſi curva generans rectificari poteſt; ſin aliter vero non.

In genere, utrum curva generans eſt algebraica, necne; rectificari generaliter poteſt curvois, ſi ſumma $PI + Ar\ MP$ exprimi poteſt in algebraicis terminis lineæ MI; ſin aliter vero non.

COR. IV.

Datâ curvâ revolvente, invenire, utrum ſunt quædam areæ curvoidis quadrabiles, vel quidam arcus, qui rectificari poſſunt.

Inveniatur (ſi modo poſſibile ſit) area vel arcus generaliter in algebraicis terminis abſciſſæ & ordinatæ, & earum circularium arcuum & Logarithmorum.

Fiant illi termini, ſi modo fieri poteſt, qui involvunt circulares arcus & Logarithmos, nihilo reſpective æquales, & reſultant areæ vel arcus, quæ quadrari, vel qui rectificari poſſunt.

Eodem fere modo inveniri poſſunt areæ vel arcus, qui datam inter ſe habent relationem.

Inveniantur generaliter duæ areæ vel duo arcus, infinitis terminis abſciſſæ & ordinatæ, & earum circularium arcuum & Logarithmorum, & fiant illi termini finiti, circulares arcus & Logarithmi reſpective inter ſe datam habere, ſi modo talem admittunt, relationem, & fit propoſitum.

Si area vel arcus haud inveniri poteſt finitorum terminorum, circularium arcuum vel Logarithmorum ope, tum inveniri poteſt ope infinitarum ſerierum, & e principiis in tractatu de fluxionibus poſtea tradendis erui poteſt problematis ſolutio.

PROB.

PROB. II.

Iiſdem poſitis, rectificare epicurvoidem, i. e. *curvam generatam a puncto in unâ curvâ ſuper alteram tanquam baſim rotante.* Fig. 35.

1. Sit baſis curva (LAS), curva revolvens vero LQP; rectificentur hæ curvæ inter ſe, & invenietur relatio inter lineas LP chordam curvæ revolventis; PQ perpendiculum a puncto (P) in tangentem (LQR) & LQ tangentem, LS ordinatam baſis, SR ejus ſubtangentem, & LR ejus tangentem; pro abſciſsâ (AS) ſcribatur (x), & pro

$x - \sqrt{LP^2 - \frac{\overline{PQ\times RS - SL\times LQ}^2}{LR^2}}$ ſcribatur (z), & epicurvoidis arcus fluxio erit $\frac{LP\times LR\times \dot{z}}{PQ\times RS - SL\times LQ}$, cujus fluens eſt epicurvoidis arcus.

2. Sit ($\dot{v}$) incrementum quantitatis $LS + \frac{PQ\times RS - SL\times LQ}{LR}$, & fluxio epicurvoidis arcus erit $\frac{LP\times LR\times \dot{v}}{\sqrt{LP^2\times LR^2 - \overline{PQ\times RS - LS\times LQ}^2}}$.

3. Datis ad ſingulum punctum (P) radiis curvaturæ & curvæ revolventis, & curvæ ſuper quam tanquam baſim progreditur ea curva. Fig. 36.

Ducantur hi circuli curvaturæ (LM & GH), cujus radii ſint reſpective PD & PC, ſit Q punctum, quod deſcribit epicurvoidem, ducantur etiam lineæ CQ, PS perpendiculum a puncto contactus (P) in lineam (CQ), & DQ, & cum radio DQ deſcribatur circulus ($QmN\tau$), ducantur linea (Pr) per punctum (P) parallela lineæ (DQ), & occurrens lineæ CQ in (r), linea Qv rectum angulum cum linea (PD) faciens, & Qn, Qm, Qo, cotemporanea incrementa epicurvoidis, arcus circuli ($QmN\tau$), & lineæ (CQ).

Facile conſtat lineam Qn eſſe ad rectum angulum ad lineam PQ, & lineam on ad lineam CQ; & exinde duo triangula Qon & PQS eſſe ſimilia, & conſequenter epicurvoidis incrementum $Qn = \frac{PQ\times Qo}{PS}$.

Epicurvoidis incrementum (Qn) etiam erit $\frac{Qm\times QP}{Qr}$; vel etiam (ſi modo

modo TN ſit diameter circuli $TQmN$, incrementum epicurvoidis (Qn) erit $\frac{TN\times CD\times QP\times \text{flux}: TQ}{\sqrt{TN^2-TQ^2}\times DQ\times CP}=\frac{2CD\times QP\times \text{flux}: TQ}{\sqrt{TN^2-TQ^2}\times CP}$.

Ex his formulis inveniatur generaliter ad ſingulum curvarum punctum epicurvoidis fluxio, & ejus fluens erit arcus epicurvoidis quæſitæ.

EXEMPLUM I.

Invenire epicycloidis longitudinem.

In poſtremâ formulâ incrementum epicycloidis fuit

$Qn=\frac{2CD\times QP\times T\dot{Q}}{\sqrt{TN^2-TQ^2}\times CP}$; ſed in hoc caſu $\sqrt{TN^2-TQ^2}=NQ=QP$

& conſequenter $Qn=\frac{2CD\times T\dot{Q}}{CP}$; ſed CD & CP ſunt in caſu propoſito datæ quantitates, unde incrementum epicycloidis eſt in datâ ratione ad incrementum lineæ QT, horum incrementorum inveniantur & corrigantur integrales; & fit.

EXEMPLUM II.

Fig. 37. Invenire curtatæ vel prolatæ epicycloidis longitudinem.

Scribantur pro lineis, ND, NP, CP, NV reſpective a, b, e, x; unde in formulâ $\frac{2CD\times PQ\times T\dot{Q}}{\sqrt{TN^2-TQ^2}\times CP}$ linea

$$PQ=\sqrt{\overline{b+x}^2\,(PV^2)+2ax-x^2\,(QV^2)}=\sqrt{b^2+2\times\overline{b+a}\times x}$$

& quantitates $\sqrt{TN^2-TQ^2}=QN=\sqrt{2ND\times NV}=\sqrt{2ax}$ & $TQ=\sqrt{TV\times TN}=\sqrt{\overline{2a-x}\times 2a}$, & conſequenter ejus fluxio $T\dot{Q}=\sqrt{2a}\times\frac{-\frac{1}{2}\dot{x}}{\sqrt{2a-x}}$, quibus quantitatibus pro ſuis valoribus in prædictâ formulâ ſubſtitutis, reſultat fluxio prolatæ vel curtatæ epicycloidis $\frac{\overline{a+b+e}\times\sqrt{b^2+2\times\overline{b+a}\,x}\times\dot{x}}{\sqrt{2ax-x^2}}$.

COR. I.

Epicurvoidis curvaturæ radius vel inveniri poteſt e duabus primis formulis, e quibus inveniri poſſunt ejus abſciſſæ & ordinatæ fluxiones;

nes; vel ſi S & R ſint reſpective radii curvaturæ & curvæ revolventis & baſis, ad punctum contactus, *i. e.* S & R ſint PD & CD, & $D =$ PQ diſtantiæ puncti curvam generantis a puncto contactus, C chordæ curvaturæ in directione PQ, tum radius curvaturæ epicurvoidis in puncto (Q) quæſitus, erit $\frac{2 \times \overline{R+S} \times D^2}{2 \times \overline{R+S} \times D - CR}$.

COR. II.

Invenire aream epicurvoidis.

Per problema inveniri poteſt abſciſſa & ejus correſpondens ordinata, & conſequenter productum ex ordinatâ in fluxionem abſciſſæ ductâ, cujus fluens erit area ipſa.

COR. III.

Invenire aream inter curvam, quæ ſit baſis, & epicurvoidem contentam. Fig. 38.

Ducantur duo circuli (LM & GH) ejuſdem curvaturæ ac curvæ revolventis, & baſis, ſuper quam progreditur; cum vero curva revolvens tangat baſim in puncto (P), ſit (a) punctum epicurvoidem deſcribens, rotetur curva a puncto P ad punctum (p), & per (p) ducatur linea ($p\overset{1}{a}$) parallela lineæ (Pa), & linea pb æqualis lineæ (Pa), & eundem angulum cum baſis radio (pC) faciens, quem linea (Pa) facit cum radio (PC) baſeos ad punctum contactus (P), cum vero curva revolvens ponitur ad punctum (p), ſit punctum (l) punctum curvam deſcribens; tum incrementum areæ epicurvoidis inter lineas Pa, Pp, pl, la contentæ, optime diſtingui poteſt in tres partes, una, parallelogrammum inter lineas aP & $p\overset{1}{a}$ contentum, ſecunda vero triangulum inter lineas $p\overset{1}{a}$, pb & $\overset{1}{a}b$ contentum, & tertia triangulum inter lineas pb, pl & bl contentum.

Parallelogrammum ($P a p \overset{1}{a}$) æquale eſt baſi Pp ($\dot{A}$) ductæ in perpendiculum an ($\dot{P}$) $= P\dot{A}$.

Triangulum vero $\overset{1}{a}pb$ æquale eſt fractioni $\frac{\overline{aP}^2}{2CP} \times Pp$ ($\dot{A}$), in quâ pro aP ſcribe D, & pro CP, R; & fit $\frac{D^2}{2R} \times \dot{A}$.

Duca-

Ducatur enim perpendiculum ($\dot{a}r$) a puncto ($\dot{\alpha}$) in lineam (pb); & duo triangula $\dot{a}pr$ & CpP ſunt ſimilia, angulus enim PCp æqualis eſt angulo $\dot{a}pr$, & conſequenter perpendiculum $\dot{\alpha}r = \frac{\dot{a}P}{CP} * Pp$, & area trianguli ($apb$) eſt $\frac{1}{2}\, ap \times ar = \frac{\overline{aP}^2}{2CP} \times Pp$. Q: E: I.

Triangulum vero pbl æquale eſt rectangulo $\frac{1}{2} pb \times bm$ perpendiculum a puncto (b) in lineam (pl); ducatur linea (lq) tangens internum circulum Ll in l, in tangentem lq ducatur perpendiculum (pq) a puncto (p), & erit $bm = \frac{pq \times bl}{pl}$, ſed $bl = \frac{Pp \times El}{PE}$, & conſequenter $\frac{1}{2} pb \times bm = \frac{pq \times El}{PE} \times Pp$, & exinde ſumma horum incrementorum erit $\overline{P + \frac{D^2}{2R} + \frac{pq \times El}{2PE}} \times \dot{A}$, cujus fluens eſt area quæſita.

COR. IV.

Inveniri poteſt, utrum ſint quædam epicurvoidum areæ, &c. quæ exprimi poſſunt in algebraicis terminis datarum linearum, ex principiis, quæ prius data fuerunt.

COROLLARIUM.

Fig. 35. Invenire, utrum epicurvois eſt algebraica curva, necne;

E prædictis conſtat (z) epicurvoidis abſciſſam eſſe

$LV - SA = \sqrt{LP^2 - \frac{\overline{PQ \times SR - LQ \times LS}^2}{LR^2}} - SA$, & ejus ordinatam ($y$) eſſe $LS + \frac{PQ \times SR - LQ \times LS}{LR}$;

Harum duarum quantitatum (z & y) termini vel ſunt algebraicæ vel fluxionales, ſi ſint algebraicæ tum datur inter eas (z & y) algebraica relatio, & curva eſt algebraica curva, ſi vero ſit fluxionalis, tum poſtea in libro de fluxionibus dabitur ſolutio.

Hinc conſtat, ſi curvæ ſint algebraicæ & rectificari poſſint, tum neceſſario epicurvois erit curva algebraica. Et ſic de reliquis proprietatibus.

PROB.

PROB. III.

Iiſdem poſitis, & datis quibuſlibet numeris curvarum (LM, MK, GF, &c.) *ſuper ſeſe revolventium, rectificare curvam a puncto in extremâ curvâ generatam.* Fig. 39.

Sint velocitates curvarum ſuper ſeſe rotantium reſpective, n, m, r, s, t &c.

Sit prima curva (LM) baſis, ſuper quam progreditur curva (HK), rectificentur hæ curvæ inter ſe, & ducantur circuli, quorum radii ſunt CP & PD reſpective, eandem curvaturam habentes ac datæ curvæ in puncto contactus, (P), ſit (Q) punctum contactus ſecundæ curvæ (HK) & tertiæ (GF), ducatur linea DQ jungens centrum curvaturæ ſecundæ curvæ (HK) & punctum (Q), & radio DQ deſcribatur circulus (QN &c.) jungatur etiam (P) & CQ, & ducatur linea (Pr) parallela lineæ (DQ); & ſint Qo, Qm, Qn cotemporanea incrementa lineæ CQ, arcûs circuli (QN), & epicurvoidis generatæ a puncto (Q) curvæ; tum propter ſimilia triangula Qmn & PrQ, erit

$Qn : Qm :: QP : Pr$; & quoniam incrementa (Qm & Pp) cotemporaneorum arcuum circulorum (QN &c. & HK) ſunt inter ſe in ratione ſuorum radiorum PD & DN, erit

$Qm : Pp :: DN : DP$; quibus rationibus compoſitis, reſultat proportio

$$Qn : Pp :: QP \times DN : Pr \times DP \quad \text{i. e.}$$
$$Qn : Pp :: QP \times CD : CP \times DP \quad \text{i. e.}$$

velocitas puncti Q deſcribentis epicurvoidem eſt ad velocitatem primæ curvæ progredientis ſuper ſecundam in ratione $QP \times CD : CP \times DP$; ſed per hypotheſin velocitas ſecundæ curvæ progredientis ſuper primam eſt ad velocitatem tertiæ curvæ progredientis ſuper ſecundam in ratione $m : n$; & conſequenter velocitas puncti (Q) deſcribentis epicurvoidem eſt ad velocitatem tertiæ curvæ ſuper ſecundam

$$:: m \times QP \times CD : n \times CP \times DP.$$

Sint SQ radius curvaturæ epicurvoidis generatæ a puncto (Q), & QT radius curvaturæ tertiæ curvæ ad punctum contactus (Q); aſſumatur linea $QZ = \dfrac{m \times QP \times CD \times SQ \times TQ - n \times SQ \times TQ \times CP \times DP}{n \times SQ \times CP \times DP + mQP \times CD \times TQ}$ cum diſtantiis TZ & SZ deſcribantur circuli (Qv & zw), & ſit punctum contactus tertiæ curvæ & quartæ, (k), ducatur linea Zk, & erit

* Q perpen-

diculum in curvam generatam a puncto (k) in eo puncto, & exinde conſtat curvæ peripheriæ incrementum.

Radius curvaturæ curvæ generatæ a puncto (k) eodem modo inveniri poteſt, quo radius curvaturæ curvæ generatæ a puncto (Q); & exinde inveniri poteſt incrementum peripheriæ curvæ generatæ a puncto in quartâ curvâ rotante ſuper tertiam, & ſic deinceps; & conſequenter inveniri poteſt incrementum peripheriæ curvæ generatæ a puncto in poſtremâ curvâ, & dato ejus incremento, erui poteſt curvæ peripheria.

COR. I.

Methodo ſupra datâ, inveniri poteſt area, &c.

Eadem etiam regula ſæpius repetita invenit radium curvaturæ in pluribus æque ac in duabus curvis ſuper ſeſe revolventibus.

COR. II.

Fig. 40. Datis omnibus præter unam curvis, earum velocitatibus, & curvâ generatâ ad ſingula puncta, & exinde deduci poteſt ea curva.

His enim datis, datur diſtantia inter duas curvas, inter quas interponitur quæſita curva, *i. e.* datur chorda quæſitæ curvæ, & incrementum arcus, unde inveniri poteſt curva quæſita.

Ex. Data curvoide, (QQ &c.) i. e. curvâ generatâ a rotatione curvæ ſuper rectam, invenire curvam rotantem.

Datâ curvoide, datur relatio inter abſciſſam & ejus correſpondentem ordinatam, & conſequenter inter Qn perpendiculum in curvam, & QF abſciſſam, & nF; ſit vero (MPL) curva quæſita, tum MP chorda quæſitæ curvæ, MI perpendiculum in tangentem, & PI erunt reſpective æquales lineis Qn, QF & nF; & conſequenter datur relatio inter chordam, perpendiculum in tangentem, & tangentem quæſitæ curvæ, unde inveniri poteſt quæſita curva.

Ex his principiis multæ hujuſmodi propoſitiones ſolutionem habere poſſunt.

PROB. IV.

Fig. 41. *Datâ curvâ* (LMN), *invenire alteram* (LRST), *cujus* (*ſi abſciſſa* (LP) *eadem ſit ac datæ curvæ abſciſſa*) *ordinatæ erunt proportionales areis* (LMLC) *datæ curvæ circa datum punctum* C *deſcriptæ*.

Scriba-

Scribatur pro abſciſſa (LP) x, pro ordinatâ PM (y), & pro LC, (a), & fluxio areæ circa datum punctum (C) deſcriptæ ſemper erit $y\dot{x} - x\dot{y} + a\dot{y}$; cui fluxioni proportionalis eſſe debet fluxio curvæ quæſitæ ordinatæ ($\dot{v}$), ſi modo ejus abſciſſa eſt (x); inveniatur igitur fluens hujus fluxionis, & erit proportionalis ordinatæ curvæ quæſitæ.

COROLLARIUM.

Sit data curva, circulus vel ellipſis, & erit $y\dot{x} - x\dot{y} + a\dot{y} = \sqrt{2rx - x^2}\, \dot{x} - \frac{\overline{rx - x^2} \times \dot{x}}{\sqrt{2rx - x^2}} + a\dot{y} = \dot{v} = \frac{rx \times \dot{x}}{\sqrt{2rx - x^2}} + a \times \dot{y}$; quæ eſt æquatio ad trochoidem, & conſequenter per trochoidem ſecari poteſt area ellipſeos proportionalis quibuſcunque datis quantitatibus, quam propoſitionem primum invenit Walliſius.

CAP. III.

PROB. I.

Fig. 42. *DATA æquatione relationem inter datæ curvæ* (Mm A, &c.) *abſciſſam* (AP) *& ejus correſpondentes ordinatas* (PM) *exprimente, invenire æquationem relationem exprimentem inter abſciſſam* (Ep) *& ejus correſpondentes ordinatas* (pm) *cujuſcunque ſectionis ſolidi e converſione dati ſolidi circa ejus axem generati.*

Sit $r:s::Ep\,(x):EP$ etiamque $r^2:t^2::\overline{Ep}^2\,(x^2):\overline{Pp}^2$ ſed $PM^2-\overline{Pp}^2$ æqualis eſt quadrato lineæ vel ordinatæ (y^2) per punctum (p) ductæ, unde erit datæ curvæ abſciſſa $AP=a\,(AE)+\frac{sx}{r}\,(EP)$ & ejus ordinatæ quadratum $PM^2=y^2+\frac{t^2x^2}{r^2},\,(\overline{Pp}^2)$.

Subſtituantur hæ quantitates pro ſuis valoribus in datâ æquatione & reſultat æquatio relationem inter abſciſſam ($Ep=x$) & ordinatas (y) requiſitæ ſectionis exprimens.

COR. I.

Sectio ſolidi generati a rotatione algebraicæ curvæ (n) dimenſionum circa axem, ſolummodo poteſt eſſe curva ($2n$) dimenſionum.

Si vero nullæ ſunt impares ordinatæ dimenſiones in datâ æquatione, tum ſectio non poteſt eſſe curva plurium quam datæ curvæ dimenſionum.

COR. II.

Sectiones ſolidi generati ex converſione conicæ ſectionis circa ejus axem, erunt conicæ ſectiones, ellipſeos, ellipſis; parabolæ, ellipſis vel parabola, prout axem ſecat, vel ei eſt parallela; hyperbolæ, Ellipſis, parabola vel hyperbola; prout ſecat duas aſymptotos ex eodem latere, parallela eſt uni aſymptoto, vel ſecat oppoſitas hyperbolas.

Et ſic de aliis hyperbolis & parabolis.

COR. III.

Sectiones ſolidi generati a rotatione conicæ hyperbolæ circa ejus axem erunt ſimiles ſectionibus coni, cujus latera ſunt hyperbolæ aſſymptoti, & cujus ſectiones eundem angulum cum axe, quem habent hyperbolæ ſectiones. Sectiones vero ſolidi generati a rotatione pa-

parabolæ circa ejus axem, erunt ſimiles ſectionibus cylindri, quæ eundem angulum cum lateribus cylindri, ac ſectiones paraboloidis cum ejus axe faciunt.

Generaliter vero, ſint duæ conicæ ſectiones (*ABCD* & *EFGH*), quarum duæ æquationes relationem inter utriuſque curvæ abſciſſas (z & w) ad axes & earum correſpondentes ordinatas, ſunt reſpective, Fig. 43.

$$v^2 = \frac{c^2}{t^2}\,\overline{t^2 \pm z^2} \;\&\; u^2 = \frac{C^2}{T^2}\,\overline{T^2 \pm w^2};$$

rotentur hæ curvæ circa ſuos axes, & ſolidi generati a rotatione curvæ (*ABCD*) ſit ſectio ($\alpha\beta\gamma\delta$) ſimilis ſectioni ($\epsilon\zeta\eta\theta$) ſolidi (*EFGH.*)

Prior vero ſectio cum axe ejus ſolidi faciat angulum, cujus ſinus eſt (S); poſterior vero cum ejus ſolidi axe angulum, cujus ſinus eſt (s), tum erit

$$S^2 = \frac{\overline{1 \pm \frac{C^2}{T^2}} \times s^2 \pm \frac{c^2}{t^2} \pm \frac{C^2}{T^2}}{1 \pm \frac{c^2}{t^2}}$$

ſubſtituantur enim pro (z & v^2) per problema in priori æquatione $\sqrt{1-S^2}\,x$ (ſectionis abſciſſam) $+a$, & y^2 ejus ordinatæ quadratum $+S^2 \times x^2$; in poſteriori æquatione pro (w & u^2) reſpective $\sqrt{1-s^2}\,\zeta$ (ſectionis abſciſſam) $+e$, & ξ^2 (ejus ordinatæ quadratum) $+s^2\zeta^2$, & reſultant duæ æquationes

$$y^2 + S^2x^2 = c^2 \pm \frac{c^2 \times \overline{1-S^2}}{t^2}\,x^2 + \&c.$$

$$\xi^2 + s^2\zeta^2 = C^2 \pm \frac{C^2 \times \overline{1-s^2}}{T^2}\,\zeta^2 + \&c.$$

ſint hæ duæ ſectiones inter ſe ſimiles, & coefficiens quadrati ordinatæ ad coefficientem quadrati ejus abſciſſæ in utraque æquatione eandem habet rationem, & conſequenter

$$S^2 \mp \frac{c^2 \times \overline{1-S^2}}{t^2} = s^2 \mp \frac{C^2 \times \overline{1-s^2}}{T^2},$$

reducatur hæc æquatio, & fit

$$S^2 = \frac{\overline{1 \pm \frac{C^2}{T^2}}\,s^2 \pm \frac{c^2}{t^2} \mp \frac{C^2}{T^2}.}{1 \pm \frac{c^2}{t^2}}$$

co-

COR. IV.

Fig. 44. Conica ſectio ABG circa ejus axem (BC) rotetur, & ſit ſectio ſolidi generati (DEF); ducatur plano (DEF) parallelum planum (HIK), tangens ſolidum in puncto (I); biſecetur axis ſectionis (DF) in L, & jungatur punctum contactus (I) & L, & hæc linea producta tranſit per omnia centra ſectionum parallelarum ſectioni (DEF).

Similiter ſit (Q) focus conicæ ſectionis, cujus rotatione generatur ſolidum, ſit DEF hujus ſolidi ſectio, cui ſit planum (HIK) tangens ſolidum parallelum; ſit linea (IN) diameter a puncto contactus (I) per centrum ſolidi tranſeuns; ſuper lineam tanquam axem per punctum (Q) deſcribatur conica ſectio, & hæc conica ſectio per focos omnium ſectionum parallelarum ſectioni (DEF) tranſibit.

Et ſic inveniri poſſunt centra, foci &c. ſectionum a planis circa data puncta rotantibus factarum, & conſequenter locos centrorum, focorum, &c.

PROB. II.

Fig. 45. *Datis duabus æquationibus, una, quæ exprimit relationem inter primam abſciſſam* (AP) *& ejus ordinatas in eodem plano inſiſtentes* (PM), *altera vero, quæ exprimit relationem inter ordinatam* (PM), *& abſciſſam* (AP), *partes ordinatæ* PM (Pp) *tanquam ſecundam abſciſſam, & ejus ordinatas* (pm) *per punctum abſciſſæ* (p) *tranſeuntes, invenire æquationem relationem inter abſciſſam & ejus correſpondentes ordinatas cujuſcunque ſectionis* (LMN) *hujus ſolidi exprimentem.*

Scribantur pro ſinubus angulorum, quos ſectio facit cum axe AP (pLP), cum ejus ordinatis PM (LpP), & ordinatæ faciunt cum ſuo axe (MPL), reſpective, t, r, s; & ſubſtituatur pro diſtantiâ inter punctum, in quo ſectio ſecat axem, & abſciſſæ axis verticem (AL) a; & pro abſciſſa (Lp) & ejus ordinatâ (pm) ſectionis quæſitæ reſpective x & y; tum erit

Sin $\angle pPL$ (s) = ſin PpL (r) : : Lp (x) : LP, & exinde $LP = \frac{rx}{s}$;

& axis abſciſſa ($AP = \frac{rx}{s} + a$); etiamque

Sin $\angle pPL$ (s) : ſin $\angle pLP$ (t) : : Lp (x) : Pp; unde ſecunda abſciſſa (pP) $= \frac{tx}{s}$.

Sub-

Subſtituantur hæ quantitates $\left(\frac{rx}{s}+a \text{ \& } \frac{tx}{s}\right)$ pro ſuis valoribus (AP & Pp) in datâ æquatione, & reſultant duæ æquationes tres incognitas quantitates (x, y & PM) habentes, reducantur hæ duæ æquationes in unam, ita ut exterminetur incognita quantitas (PM), & reſultat æquatio relationem inter incognitas quantitates (x & y) abſciſſam & correſpondentes ordinatas ſectionis quæſitæ.

COR. I.

Nonnunquam prius neceſſe eſt mutare inclinationem ordinatarum (pm) ad ſecundam abſciſſam (ap), & deinde prædictâ methodo progrediendum eſt.

COR. II.

Hujus ſolidi ſectio non poteſt habere plures dimenſiones, quam habet productum ſub dimenſionibus duarum datarum æquationum vel curvarum in ſeſe ductis.

COR. III.

Fiat curva MMM, &c. rectalinea, & ſectiones erunt curvæ, quarum dimenſiones ſunt æquales datæ curvæ dimenſionibus, ergo curva ſemper projiciat curvas dimenſionum ſibi ipſi æqualium.

COR. IV.

Sit curva MM, cujus æquatio, relationem inter abſciſſam (Pp) & ejus correſpondentes ordinatas (pm) per punctum (p) tranſeuntes exprimens, datur; & A punctum lucidum, invenire projectionem datæ curvæ in datum planum LpM. Fig. 46.

1mo. Sin $\angle LPp\,(s)$: ſin. $PpL\,(r)$: : $pL\,(x)$: PL; ergo $PL=\frac{rxx}{s}$ & $PL+LA=AP=\frac{rx}{s}+a$; & eodem modo ſin $\angle LPp\,(s)$: ſin $\angle PLp\,(t)$: : $pL\,(x)$: Pp, & conſequenter abſciſſa $Pp=\frac{tx}{s}$; ſit vero $p\times AP=PM$, & exinde $PM=\frac{prx}{s}+pa$; ſubſtituantur hæ quantitates

tates $\left(\frac{rx}{s}+a, \frac{tx}{s}, \frac{prx}{s}+pa\right)$ pro ſuis valoribus AP, Pp & PM, & reſultat æquatio relationem inter abſciſſam ($Lp=x$) & correſpondentes ordinatas (pm) quæſitæ ſectionis exprimens.

Parallelæ ſectiones hujus generis ſolidi erunt inter ſe ſimiles.

In nonnullis caſibus primum mutare ordinatarum inclinationem neceſſe eſt.

COR. V.

Hinc conſtat curvas, quarum æquationes relationem inter abſciſſam (x) & ejus correſpondentes ordinatas (y) exprimentes, ſunt

$$y^{2n}+\overline{c+dx+ex^2}\,y^{2n-1}+\&c.=0$$

projicere poſſet omnes curvas ($2n+1$) dimenſionum.

Eodem modo invenire licet curvas, quas projicere poſſunt quæcunque curvæ, & vice versâ.

Ex æquatione etiam inveniri poteſt, annon ſolidum dividi poteſt in duas vel plures æquales partes.

Ex iiſdem principiis, quibus uſus fui in hujus libri capite primo inveniri poſſunt æquationes inter quaſcunque alias lineas, ad hujuſmodi ſolidi abſciſſas & ordinatas quamcunque aſſignabilem relationem habentes.

PROB. III.

Fig. 47. *Dato puncto lucido* (Q) *& duabus æquationibus, e quibus una exprimit relationem inter abſciſſam* (AP) *& ejus correſpondentes in eodem plano ordinatas* (PM), *altera vero exprimit relationem inter ordinatam* (PM), *abſciſſam* (AP), *& partes* (Pp) *ordinatæ* PM *tanquam ſecundam abſciſſam, & ejus ordinatas* (pm) *per punctum ſecundæ abſciſſæ* (p) *tranſeuntes, invenire ejus projectionem in datum planum.*

Datum planum tranſeat per axem AP in directione ordinatarum ad ſecundum axem, & ita transformetur ordinatarum (PM) inclinatio ad primum axem, vel quod idem eſt ſecundæ abſciſſæ (Pp) planum, ut fiat idem ac planum linearum jungentium punctum lucidum (Q) & puncta axis (AP).

Ducatur QH parallela ordinatis (PM), occurrens axi (AP) in H, ſit (L) quodlibet axis (AP) punctum ducatur QL, ducantur ad partem

tes lineæ QL correſpondentes ordinatæ (pm, &c.) ducatur linea Qm tangens ſolidum in (m) & ſecans datum planum per axem tranſeuns in o, per punctum (p) ducatur linea (pP) parallela ordinatis (PM) vel lineæ QH, & a puncto (Q) perpendiculum in axem (Qb).

Scribantur pro lineis $QH, AH, Qb, bH, HL, Lo, Lp, pm, Pp, PH,$ reſpective, $a, b, c, d, x, y, X, v, z, w$; in duabus datis æquationibus ſubſtituantur pro ($HP = w$ & $Pp = z$) earum reſpectivi valores

$$-\frac{\sqrt{c^2 + \overline{x-d}^2} \times \overline{x+b} - xX}{\sqrt{c^2 + \overline{x-d}^2}}, \text{ \& } \frac{aX}{\sqrt{c^2 + \overline{x-d}^2}};$$

& reſultant duæ æquationes, quibus in unam reductis ita ut exterminetur incognita quantitas (PM), reſultat æquatio relationem inter abſciſſam Lp (X) & ejus ordinatas pm (v) exprimens; inveniatur ſubtangens hujus curvæ, quæ fiat æqualis quantitati $\sqrt{c^2 + \overline{x-d}^2} - X$; & in æquatione reſultanti pro $pm = v$ ſubſtituatur ejus valor $\frac{y \times \sqrt{c^2 + \overline{x-d}^2} - X}{\sqrt{c^2 + \overline{x-d}^2}}$, & reſultant duæ æquationes tres incognitas quantitates (x, y, X) habentes, quibus in unam reductis, ita ut exterminetur incognita quantitas (X), reſultat æquatio relationem inter abſciſſam (x) & ejus correſpondentes ordinatas ſolidi projectionis in datum planum exprimens.

EXEMPLUM I.

Sit ſolidum cylindricum, cujus baſis eſt quæcunque curva, & ejus laterum projectio in datum planum erit rectæ lineæ.

EXEMPLUM II.

Sit ſolidum conus, cujus baſis eſt quæcunque curva, & ejus laterum projectio in datum planum erit rectæ lineæ.

Facile conſtant hæc etiam ex vulgari trigonometriâ & fluxionibus.

COROLLARIUM

Si planum non tranſit per axem, & haud eandem inclinationem in axem cum ejus ordinatis habet, e præcedenti problemate & ejus corollariis facile deduci poteſt ejus projectio.

PROB. IV.

Fig. 48. *Datis, puncto lucido* (Q) *& duabus æquationibus ut in duabus præcedentibus propoſitionibus, & tertiâ relationem inter abſciſſam* ($\alpha\pi$) *& ejus correſpondentes ordinatas* ($\pi\mu$) *datæ curvæ exprimente, invenire projectionem datæ curvæ in datum ſolidum.*

Transformetur abſciſſa curvæ vel axis ſolidi, ita ut fiat projectio abſciſſæ curvæ in dati ſolidi axem, & ita transformetur ſecunda abſciſſa, ut ſit in eodem plano cum curvæ & ſolidi axe, & ſecundæ abſciſſæ ordinatæ (pm), ita ut fiant parallelæ plano datæ curvæ; vel ordinatæ datæ curvæ ($\pi\mu$), ut fiant parallelæ ordinatis (pm) ſecundæ abſciſſæ ſolidi.

Ducatur linea QH parallela ordinatis (PM), & occurrens abſciſſæ curvæ in (r) & axi ſolidi in (H).

Ducatur per punctum curvæ (μ) linea ($Q\mu$) occurrens ſolido in puncto (m), & per ejus correſpondens punctum abſciſſæ (π) linea Qp.

Scribantur pro lineis $Qr, QH, \alpha r, HA, AP, Pp, pm, \alpha\pi, \pi\mu$, reſpective $a, b, c, d, x, z, v, X, Y$; tum propter duo ſimilia triangula $Q\pi\mu$ & Qpm, erit $Qr : QH - Pp :: \pi\mu : pm$, i. e. $a : b - z :: Y : v$, & ex iiſdem principiis etiam ſequitur ſubſequens proportio

$$Qr : QH - Pp :: \alpha\pi - \alpha r : AP - AH, \text{ i. e. } a : b - z :: X - c : x - d;$$

e primâ proportione ſequitur $\frac{av}{b-z} = Y$, e ſecundâ vero

$X = \frac{ax - ad + bc - zc}{b - z}$; quibus quantitatibus pro ſuis valoribus in æquatione relationem inter abſciſſam curvæ (X) & ejus ordinatas (Y) exprimente, reſultant tres æquationes, quatuor incognitas quantitates (x, z, v, PM) habentes, quibus in unam reductis, ita ut extermineter incognita quantitas (PM) & una e reliquis (x, z, v), reſultat æquatio, relationem inter duas reliquas exprimens, quæſita.

COR. I.

Dimenſiones æquationis reſultantis non plures eſſe poſſunt, quam contentum ſub dimenſionibus trium datarum æquationum, i. e. ſint, m, n & r datarum æquationum dimenſiones, & reſultans æquatio non plures quam $n \times m \times r$ dimenſiones habere poteſt.

COR.

COR. II.

Si vero axis ſolidi non ſit projectio abſciſſæ curvæ ſit, & ordinatæ ſecundæ abſciſſæ ſolidi ordinatis haud ſint parallelæ, & ſecunda abſciſſa ſolidi haud ſit in eodem plano cum axe ſolidi & abſciſsâ curvæ, tum ex hoc problemate & capite primo petenda eſt ſubſtitutio.

COR. III.

Solidi illuminati contentum inveniri poteſt e principiis fluxionum, inveſtigando aream curvæ contentam inter lineas (*mpm*) & arcum curvæ *mRSm*; & ducendo hanc aream in fluxionem abſciſſæ (*AP*), & erit productum fluxio ſolidi fruſtri quæſiti.

COR. IV.

Arcûs curvæ umbram terminantis longitudo inveniri poteſt, inveſtigando fluentem fluxionis $\sqrt{\dot{z}^2+\dot{w}^2+\dot{v}^2}$, in quâ $\dot{z}, \dot{w}, \dot{v}$ reſpective denotant fluxiones abſciſſæ, ordinatæ & diſtantiæ a plano, inter ſe rectum angulum facientes.

Ex iiſdem etiam principiis inveniri poſſunt generaliter curvarum ſuperficies.

Hic inſerere licet curvoidalem trigonometriam, quæ in ſuo calculo multum habet laborem, ſed ejus principia ſatis perſpicua ſunt; etiamque multas alias propoſitiones de ſolidis verſantes: ſed ſatis quoad noſtrum propoſitum de his dictum eſt.

CAP. IV.

DEF. I.

LINEÆ fluxionales, ſecundum ordines fluxionum æquationis, quâ relatio inter abſciſſas & ordinatas definitur, diſtinguuntur in ordines; ea linea eſt primi fluxionalis ordinis, in cujus æquatione relationem inter ejus abſciſſas & ordinatas exprimente, continentur ſolummodo fluxiones primi ordinis, & ſic deinceps.

DEF. II.

Fluxionales lineæ ejuſdem ordinis, ſecundum dimenſiones æquationis, quâ exprimitur relatio inter abſciſſas (x) & ordinatas (y), ſi modo pro fluxione abſciſſæ ($\dot{x}$), vel ordinatæ ($\dot{y}$), ſcribatur x & y reſpectivè, & pro ($\ddot{x}$) vel ($\ddot{y}$), ſcribatur x^2 vel y^2; & ſic deinceps, diſtingui poſſunt in dimenſiones.

PROB. I.

Fig. 50. *Datâ fluxionali æquatione relationem inter abſciſſam* (AP, x) *& ejus ordinatas* (PM, y) *exprimente, eam in alteram transformare, relationem inter abſciſſam* (ap, z) *& ejus correſpondentes ordinatas* (pm, v) *exprimentem.*

Subſtituantur pro abſciſsâ (x), & ejus ordinatis (y), eædem quantitates, quæ in problemate primo ſubſtitutæ fuerunt, & pro abſciſſæ (x) & ordinatarum (y) fluxionibus ſubſtituantur prædictarum quantitatum correſpondentes fluxiones; & transformatur æquatio in alteram, relationem inter abſciſſam (ap) & ejus ordinatas (pm) exprimentem.

THEO.

Harum curvarum multæ ſunt proprietates algebraicarum curvarum proprietatibus analogæ, quæ per methodum convergentium ſerierum, erui & demonſtrari poſſunt.

EXEMPLUM I.

Sit fluxionalis curva (n) dimenſionum, & ducantur duæ rectæ, ſibi ipſis parallelæ, & curvam ſecantes in (n) punctis; linea, quæ ita ſe-

cat

cat has parallelas, ut ſumma partium ex uno ſecantis latere, æqualis ſit ſummæ partium ex altero ſecantis latere, eodem modo ſecabit omnes alias his parallelas.

COR. I.

Rejiciantur omnes datæ fluxionalis æquationis termini, iis exceptis, in quibus inveniuntur n & $n-1$ dimenſiones; æquationis reſultantis inveniatur fluens, quæ ſit algebraica curva, & diameter hujus algebraicæ curvæ erit etiam fluxionalis curvæ diameter, & vice verſa.

COR. II.

Sint (n) aſymptoti fluxionalis curvæ, cujus numerus dimenſionum eſt (n); & agatur recta linea, curvam in (n) punctis ſecans; tum iſtius lineæ ſumma partium, inter crura curvæ & eorum reſpectivas aſymptotos interceptarum, ex uno diametri latere, æqualis eſt ſummæ partium inter curvæ crura, & eorum reſpectivas aſymptotos inter ceptarum, ex altero diametri latere.

EXEMPLUM II.

Datâ fluxionali curvâ, in quâ nulli inveniuntur termini $(n-1)$ dimenſionum, & generale centrum habet curva.

Eodem modo progredi licet in curvas, omnes ſuas diametros parallelas habentes.

EXEMPLUM III.

Sint omnes ordinatæ dimenſiones in datâ fluxionali æquatione, pares; & correſpondentes ordinatæ valores, & affirmativi & negativi, erunt inter ſe æquales.

EXEMPLUM IV.

Facile e methodo convergentium ſerierum erui poteſt, quot ſunt crura parabolica, hyperbolicave; pendent enim e terminis datæ æquationis, quorum dimenſiones ſunt maximæ, & inveniri poſſunt eâdem methodo, quâ parabolica vel hyperbolica algebraicæ curvæ crura.

PROB. II.

Datâ fluxionali æquatione relationem inter abſciſſam (x), *& ejus correſpondentes ordinatas* (y) *exprimente, invenire abſciſſas & ordinatas, quando duo valores ordinatæ fiunt inter ſe æquales.*

In-

Inveniatur datæ æquationis fluxio, & fiat ordinatæ fluxio, vel nihilo, vel infinito, æqualis, & reſultant duæ fluxionales æquationes, duas incognitas quantitates (x & y) & earum fluxiones habentes, quæ reduci poſſunt in unam algebraicam æquationem, unam ſolummodo incognitam quantitatem (x vel y) habentem, ut poſtea docebitur in noſtro de fluxionibus tractatu.

COR. I.

Hinc inveniri poſſunt ovales, puncta conjugata, &c. ex iiſdem principiis, quæ in capite primo de inveniendis ovalibus, punctis conjugatis, &c. in algebraicis curvis, tradita fuerunt.

COR. II.

Hinc erui poteſt fluxionalium curvarum, quarum dimenſiones dantur, enumeratio.

Datur enim methodus inveniendi curvæ parabolica, hyperbolicave crura, & eorum aſymptotos, etiamque e præcedentibus principiis inveniri poſſunt puncta, in quibus aſymptotos ſecat curva.

Datur etiam methodus inveniendi ovales, puncta conjugata, &c. & conſequenter erui poteſt fluxionalium curvarum, quarum dimenſiones dantur, enumeratio.

PROB. III.

Curvæ primi fluxionalis ordinis conſtrui poſſunt, curvarum generatarum ex rotatione algebraicarum ſuper rectas, i. e. curvoidum & algebraicarum ope.

Curvæ ſecundi fluxionalis ordinis conſtrui poſſunt, curvarum ex rotatione algebraicarum curvarum ſuper algebraicas generatarum, i. e. epicurvoidum, curvoidum & algebraicarum curvarm ope, & ſic deinceps.

Hic adjicere liceret proprietates fluxionalium curvarum, fere omnibus, vel algebraicis vel fluxionalibus algebraicarum curvarum proprietatibus, analogas.

Sed plura adjiciam de his fluxionalibus curvis in tractatu de fluxionibus: progredior autem ad tractandas curvarum proprietates, quarum relatio inter abſciſſam & ejus correſpondentes ordinatas haud deſignatur per æquationem, vel fluxionalem, vel algebraicam; quæ-

pro-

proprietates in unâ curvâ pendent ex ejus concavitate vel convexitate, vel quibuſdam datis vel cognitis proprietatibus; in pluribus autem curvis, e datis æquationibus relationem inter quaſdam lineas in unâ, & quaſdam in aliis curvis, definientibus: ſed priuſquam harum curvarum nonnullas proprietates inveſtigo, neceſſe eſt ſubſequentia lemmata præmittere.

LEMMA I.

Dividatur ſingulum trianguli (*EFG*) latus in duas æquales partes, $EA = AG$, $GB = BF$ & $FC = CE$; jungantur puncta biſectionum A, B, C; & erit circumſcriptum triangulum ad inſcriptum in ratione numeri quaternarii ad unitatem. Fig. 51.

LEMMA II.

Dividatur ſingulum latus trapezii *EFGH* in duas æquales partes *EA*, *AF*; *FB*, *BG*; *GC*, *CH*; *HD*, *DE*; jungantur puncta biſectionum *ABCDA*, & circumſcriptum trapezium erit ad inſcriptum in ratione numeri binarii ad unitatem. Fig. 52.

DEMONSTRATIO.

Ducantur diagonales *EG*, *FH* & ex hypotheſi $AE = AF$, & $FB = BG$, & conſequenter triangulum (*AFB*) erit quarta pars trianguli (*FEG*); & ſic triangulum (*DHC*) erit quarta pars trianguli (*HEG*), & ſumma duorum triangulorum ($FAB + DHC$) erit quarta pars dati trapezii (*EFGH*); & ſic ſumma duorum triangulorum ($AED + BGC$) erit quarta pars dati trapezii; ergo ſumma quatuor triangulorum ($AFB + BGC + CHD + DEA$), & exinde trapezium (*ABCD*) erit ad datum trapezium (*EFGH*) in ratione unitatis ad numerum binarium.

COROLLARIUM.

Lineæ (*AB* & *DC*) ab *Euclide* demonſtrantur eſſe parallelæ diagonali (*EG*), & conſequenter parallelæ ſunt inter ſe; & ſimiliter demonſtrantur lineæ (*BC* & *AD*) inter ſe parallelæ: ergo inſcriptum trapezium (*ABCD*) eſt parallelogrammum.

In omnibus polygonis hâc methodo inſcriptis, area polygoni inſcripti erit ac aream dati polygoni circumſcripti, in majore ratione quam unitatis ad numerum binarium.

LEMMA

LMMA III.

Fig. 53. Dato quocunque trapezio (*ABCD*), cujus diagonales ſunt lineæ (*AC* & *BD*), per puncta (*A*, *B*, *C*, *D*) ducantur lineæ diagonalibus (*AC* & *BD*) parallelæ, & erit figura exinde formata (*EFGH*) parallelogrammum.

Lineæ enim (*FG* & *EH*) ducuntur parallelæ lineæ (*AC*), & igitur ſunt parallelæ inter ſe, & eodem modo probatur lineas (*EF* & *GH*) parallelas eſſe inter ſe, & conſequenter trapezium (*EFGH*) eſſe parallelogrammum.

COROLLARIUM.

Circumſcriptum parallelogrammum (*EFGA*), duplum eſt dati trapezii *ABCD*).

Parallelogrammum enim (*AG*) duplum eſt trianguli (*ABC*), & parallelogrammum (*AH*) duplum eſt trianguli (*DAC*), ergo totum parallelogrammum (*EG*) duplum eſt trapezii (*ABCD*).

Conſtat polygonum hâc methodo circumſcriptum, non habere ad datum polygonum inſcriptum, majorem rationem, quam numeri binarii ad unitatem.

LEMMA IV.

Fig. 54. Sit polygonum (*EFGHI*, &c.) & ducantur lineæ *AB*, *BC*; *CD*, *DE*; &c. ita ut angulus *ABE* æqualis ſit angulo *CBF*; angulus *BCF* ſit æqualis angulo *DCG*, $\angle CDG = \angle \overset{\text{I}}{D}DH$, &c.

1. Sit numerus polygoni laterum impar, & ſcribantur pro angulis dati polygoni *EFG*, *FGH*, *GHI*, &c. reſpective *a*, *b*, *c*, *d*, *e*, &c. tum erit angulus

$$CBF = \frac{180 - a + b - c + d - e + \&c.}{2}; \text{ \& ſic de}$$

reliquis.

DEMONSTRATIO.

Scribantur pro angulis CBF, BCF, CDG, $D\overset{\text{I}}{D}H$, $\overset{\text{I}}{D}\overset{\text{II}}{D}I$, &c. reſpective x, y, z, v, w, &c. & reſultant æquationes

$$\begin{aligned}
a + x + y &= 180^\circ \\
b + z + y &= 180 \\
c + v + z &= 180 \\
d + v + w &= 180 \\
e + x + w &= 180
\end{aligned}$$

&c. &c. ita reducantur hæ æquationes in unam

unam, ut exterminentur omnes incognitæ quantitates præter unam, quæ ſit (x), & ſit

$$x = \frac{180 - a + b - c + d - e + \&c.}{2}$$

Si quicunque angulus inventus, ſit negativus, tum tale polygonum inſcribi nequit.

2. Sit numerus polygoni laterum par, & hujuſce generis polygonum inſcribi nequit, ni

$$a - b + c - d + e - \&c. = 0.$$

COROLLARIUM I.

Scribantur pro lineis EF, FG, GH, HI, IK, &c. reſpective P, Q, R, S, T, &c. pro linea EA ſcribatur (x), ſit $EA : EB :: 1 : o$; $BF : FC :: 1 : n$; $CG : GD :: 1 : m$; $DH : \dot{H}D :: 1 : l$; &c. tum

$$EB = ox$$

$$FC = n \times \overline{EF - EB} = nP - nox$$

$$GD = m \times \overline{FG - FC} = mQ - mnP + mnox$$

$$\dot{H}D = l \times \overline{GH - GD} = lR - lmQ + lmnP - lmnox$$

& ſic deinceps.

Et exinde e ſimplici æquatione ſequitur valor incognitæ quantitatis (x), quo ſubſtituto, reſultant reliquarum incognitarum quantitatum valores.

COROLLARIUM II.

Sit polygonum (n) laterum triangulum (EFG), & ſumma laterum ($EF + FG + GH$) non habet ad ſummam linearum ($AB + AC + BC$) majorem rationem quam 2 : 1. Fig. 50.

Sit polygonum trapezium, & ſumma laterum ($EF + FG + GH + HE$) non habet ad ſummam linearum ($AB + BC + CD + DA$) majorem rationem quam $\sqrt{2} : 1$, & conſequenter ſumma laterum polygonorum plura latera habentium minorem habent ad ſummam prædictarum linearum rationem, quam $\sqrt{2} : 1$. Fig. 51.

LEMMA V.

Sit polygonum (n) laterum ($EFGH$, &c.) & ita biſecetur ſingulum latus in punctis (A, B, C, &c.) ut ejus ſegmenta ſint inter ſe in Fig. 54.

reciprocâ ratione ſummæ ſecantis & tangentis complementorum adjacentium angulorum, i. e. ſit

$EB : BF : \overline{Sec + Tan}$ Com : $\angle EFG$: $\overline{Sec + Tan}$ Com : $\angle FEH$

& $FC : CG : \overline{Sec + Tan}$ Com : $\angle FGH$: $\overline{Sec + Tan}$ Com : $\angle GFE$

& ſic deinceps.

Jungantur puncta contactus A & B, B & C, C & D, &c. & circumſcripti trapezii vel polygoni ſumma laterum $EF+FG+GH+$&c. non majorem habet rationem ad inſcripti trapezii vel polygoni ſummam laterum $AB+BC+CD+$ &c. quam $\sqrt{2} : 1$.

Sit datum polygonum triangulum, & circumſcripti trianguli laterum ſumma, non poteſt habere ad inſcripti trianguli laterum ſummam, majorem rationem, quam numerus binarius ad unitatem.

Et ſic de quam plurimis hujuſmodi propoſitionibus.

PROB. IV.

Invenire curvarum proprietates, quarum nulla datur æquatio relationem inter abſciſſam & ejus correſpondentes ordinatas deſignans.

Proprietates ejuſdem curvæ, vel ad unum ſolummodo curvæ punctum, vel ad plura, vel ad omne punctum referuntur.

1. Si ad unum ſolummodo punctum referuntur lineæ, inter quas requiruntur æquationes relationem exprimentes.

Hæ proprietates e vulgaris analyſeos & fluxionum principiis petendæ ſunt.

EXEMPLUM.

Ita transformare æquationem fluxionalem, in quâ ſolummodo continentur duæ variabiles quantitates (x & y), quarum (x) fluit uniformiter, ut fluat uniformiter (y).

Fig. 55. Sit quæcunque curva (MRS), cujus abſciſſa ($x=AP$) fluit uniformiter, ſit ejus fluxio, (Pp), & ſit fluxio ordinatæ ($y=PM$), Mb; ſecunda vero fluxio $2mf$; nunc fluat uniformiter ordinata (PM), ſit ejus fluxio (Mb), tum prima abſciſſæ (AP) fluxio erit be, & ſecunda ejus fluxio $2me$: duo triangula Mbe, fme ſunt ſimilia; ergo eb ($\dot{x}$) : bM ($\dot{y}$) : : $2em$ ($\ddot{x}$) ſecunda abſciſſæ fluxio, cum fluat uniformiter (y) : $2mf$ ($\ddot{y}$) ſecundam ordinatæ fluxionem, cum fluat uniformiter

miter (x); & conſequenter $\dot{y} = \frac{\dot{y}\ddot{x}}{\dot{x}}$, quo valore pro ($\dot{y}$) in datâ fluxionali æquatione ſubſtituto, & ejus fluxione pro $\ddot{y}$, & ſic deinceps; & ita transformabitur æquatio, ut fluat uniformiter (y).

Hinc ſequitur omnis hujuſce generis transformatio.

2. Si vero ad plura puncta referuntur lineæ, inter quas requiruntur æquationes relationem exprimentes.

Data relatio vel exprimitur per algebraicas vel fluxionales æquationes, ſi vero per algebraicas, tum ex vulgaris analyſeos & geometriæ principiis petenda eſt ſolutio.

Si vero per fluxionales æquationes exprimitur data relatio, tum inveniendæ ſunt lineæ, quæ inter ſe eaſdem habent rationes, quas habent inter ſe fluxiones, quibus pro fluxionibus reſpective ſubſtitutis, transformantur fluxionales æquationes in algebraicas.

Sequuntur hujus caſus quædam exempla, de inveniendis, quibuſdam areis, lineis, &c. quando maximæ vel minimæ ſint.

EXEMPLUM I.

Sit ovalis, nullum habens contrariæ flexuræ punctum, & ducantur (n) lineæ (EF, FG, GH, HI, &c.) ovalem tangentes, in (n) punctis (A, B, C, D, &c.) ſint etiam hæ lineæ ad contactuum puncta (A, B, C, D, &c.) in duas æquales partes diviſæ; i.e. Fig. 56.
$EA = AF$, $FB = BG$, $GC = CH$, $HD = DI$, &c. tum minus erit polygonum ($EFGH$, &c.) quam quodvis aliud, idem contactûs punctum (A) habens, & circa datam ovalem deſcriptum.

Si enim circumſcriptum polygonum, cujus latera EF, FG, GH, &c. in duas æquales partes, in punctis contactuum (A, B, C, D, &c.) dividuntur, non ſit minimum, quod circa ovalem deſcribi poteſt, & cujus punctum contactûs eſt (A); ſit polygonum ($LMNR$, &c.) miminium, habens latus (MN) a puncto contactûs (a) in duas inæquales partes aM & aN diviſum, quarum (aM) major ſit quam aN.

Ducatur linea $\alpha\beta$, lineas LM & NR in punctis α & β ſecans, & in duas æquales partes ($\alpha\gamma = \gamma\beta$) ad punctum ($\gamma$) contactûs diviſa: tum polygonum ($L\alpha\beta NR$, &c.) minus eſt quam polygonum ($LMNR$, &c.) quoniam triangulum $Mb\alpha$ minus eſt quam triangu-

lum $Nb\beta$; & conſequenter polygonum ($LMNR$, &c.) non eſt minimum quod circumſcribi poteſt, niſi ejus ſingulum latus in duas æquales partes *a* puncto ejus contactus dividitur.

EXEMPLUM II.

Fig. 57. *Sit eadem ovalis, & in eâ inſcribatur polygonum* (n) *laterum* (A,B,C,D), &c. per ſucceſſiva puncta (A, B, C, D, &c.) *ducantur tangentes* (EF, FG GH, *&c.*) *reſpective, & ſit linea* AC *parallela lineæ tangenti ovalem in puncto* (B), *linea* BD *parallela lineæ ovalem tangenti in puncto* (C), *& ſic deinceps: tum polygonum* (ABCD, *&c.*) *erit maximum polygonum* (n) *laterum, quod in datâ ovali inſcribi poteſt, & cujus punctum contactus eſt* (A).

Si enim lineæ AC, BD, CE, &c. non ſunt reſpective parallelæ lineis tangentibus ovalem in punctis (A, B, C, D, &c.) ſit BD haud parallela tangenti in puncto (C), & lineæ (BD) ducatur linea $\alpha\gamma$ parallela, tangens circulum in puncto (α); tum triangulum $B\alpha D$ majus erit quam triangulum BCD; & conſequenter polygonum $ABCD$, &c.) non eſt maximum polygonum (n) laterum, quod in datâ ovali inſcribi poteſt, habens ejus verticem punctum (A), ni prædictæ lineæ ſunt parallelæ reſpectivis tangentibus.

EXEMPLUM III.

Fig. 57. *Iiſdem poſitis, inſcribatur polygonum* (n) *laterum* (ABCDE, &c.) *& per ſucceſſiva puncta* (ABCDE, &c.) *ducantur lineæ* EF, FG, GH, HI, *&c. ovalem tangentes, in punctis* (A, B, C, D, *&c.*) *& ſit angulus* ABF $= \angle$CBG, *& angulus* BCG $= \angle$DCH, *&* CDH $=$ EDI, *&c. i. e. anguli, quos latera polygoni utrinque cum reſpectivis tangentibus ad contactuum puncta faciunt, ſint inter ſe æquales.*

Et ſumma polygoni (n) laterum erit major, quam ſumma, cujuſcunque alius polygoni (n) laterum in ovali inſcripti, & cujus vertex eſt punctum (A).

EXEMPLUM IV.

Fig. 57. *Deſcribatur circa ovalem polygonum* n *laterum* (EFGH, *&c.*) *cujus punctum contactus eſt* (A), *ſint puncta contactuum reſpective* (A, B, C, D, *&c.*) *& ſit*

A

$AE : AF :: \overline{\text{Tan} + \text{Sec}}\ \text{Com} : \angle AFG : \overline{\text{Tan} + \text{Sec}}\ \text{Com} : \angle AEP$

$FB : BG :: \overline{\text{Tan} + \text{Sec}}\ \text{Com} : < BGC : \overline{\text{Tan} + \text{Sec}}\ \text{Com} : < \overline{BFA}$

$GC : CH :: \overline{\text{Tan} + \text{Sec}}\ \text{Com} : < CHD : \overline{\text{Tan} + \text{Sec}}\ \text{Com} : < CGB$

& ſic deinceps, tum.

Summa (*n*) laterum polygoni, minor erit quam ſumma (*n*) laterum cujuſcunque alius polygoni, circa ovalem deſcripti, & cujus punctum contactus eſt (*A*).

EXEMPLUM V.

Deſcribatur circa ovalem, polygonum (n) *laterum* (EFGH, &c.) *cujus puncta contactuum ſint* (A, B, C, D, *&c.*) *in ſingulam tangentem ducantur duo perpendicula a duobus proximis angulis, contactus puncto utrinque jacentibus, i. e. ducantur* AP & CQ *perpendicula in tangentem* EBF; BR & DS *perpendicula in tangentem* FBG, *& ſic deinceps.* Fig. 57.

Sit etiam

$$AB^{m-2} : BC^{m-2} :: BQ : BP$$

& $BC^{m-1} : DC^{m-2} :: CS : CR$ & ſic de reliquis, tum ſumma $AB^m + BC^m + CD^m + DE^m +$ &c. major erit quam ſumma (*m*) poteſtatis ex ſingulis (*n*) lateribus cujuſcnnque alius polygoni in ovali inſcripti, cujus punctum contactus eſt (*A*).

EXEMPLUM VI.

Iiſdem poſitis, ſit

$$AB^2 : BC^2 : m \times PB : o \times BQ \ \&$$ Fig. 57*.

$$BC^2 : CD^2 :: o \times CR : p \times CS \ \&$$ ſic

deinceps, tum contentum

$$AB^m \times BC^o \times CD^p \times \&c.$$

majus erit, quam quodcunque aliud ſimile contentum, cujus punctum contactus eſt (*A*).

Et ſic de infinitis ſimilibus exemplis.

3. Si autem ad omne curvæ punctum referuntur lineæ, curvæ, &c. inter quas requiruntur æquationes relationem exprimentes.

Inveniantur linearum, arearum, &c. incrementa; & exinde erui poſſunt æquationes quæſitam relationem inter eas lineas, areas, &c. exprimentes.

EXEMPLUM I.

Circa ovalem, nullum habentem contrariæ fluxuræ punctum, deſcribantur duo polygona (n) *laterum* EFGH, &c. & PQRS, &c. & *ita ut utri-* Fig. 58.

uſque

uſque polygoni latera ſecentur in duas æquales partes in punctis contactuum A, B, C, D, &c; & α, β, γ, δ, &c; *reſpective; tum polygonum* (E F G H) &c. *erit æquale polygono* (P Q R S &c.)

DEMONSTRATIO.

Hæc polygona ſunt minima, quæ a reſpectivis punctis (*A* & *α*) incipiunt, & circa ovalem deſcribi poſſunt.

Deſcribantur etiam polygona, incipientia a punctis inter (*A* & *α*) poſitis, & ita ut ſint minima, quæ a reſpectivis punctis incipiunt.

Tum, quoniam polygonum perpetuo eſt minimum, quod deſcribi poteſt circa ovalem, nullum habet incrementum, & conſequenter inter ſe æqualia perpetuo manent polygona, & exinde ſequitur polygonum (*EFGH* &c.) æquale eſſe polygono (*PQRS* &c).

COR. I.

Fig. 59. Deſcribatur circa ovalem triangulum (*EFG*), cujus latera (*EF*, *FG*, *EG*) reſpective dividuntur in æquales partes ($AE=AF$, $BF=BG$, $CG=CE$) in punctis contactuum (*A*, *B*, *C*), *i.e.* ſit triangulum (*EFG*) minimum, quod circa ovalem deſcribi poteſt, & jungantur puncta contactuum (*A*, *B*, *C*), & triangulum (*ABC*) erit maximum, quod in datâ ovali inſcribi poteſt.

Facile enim conſtat lineam (*AB*) parallelam eſſe tangenti (*EG*), lineam (*BC*) parallelam tangenti (*EF*), & lineam (*AC*) tangenti (*FG*); & conſequenter triangulum (*ABC*) maximum eſſe, quod in datâ ovali deſcribi poteſt.

E lemmate primo conſtat maximum triangulum, quod in datâ ovali inſcribi poteſt, eſſe ad minimum triangulum, quod circumſcribi poteſt, in ratione unitatis ad numerum quaternarium.

COR. II.

Fig. 60. Deſcribatur circa ovalem trapezium (*EFGH*), cujus latera in æquales partes per ſucceſſiva contactus puncta (*A*, *B*, *C*, *D*) dividuntur, jungantur hæc contactus puncta, & Trapezium (*ABCD*) exinde formatum erit parallelogrammum, & ejus area dimidium areæ circumſcripti Trapezii.

Hoc corollarium ſequitur e lemmate ſecundo.

CO-

COR. III.

Sit ovalis, conici generis ellipſis, & circa conjugatas ejus diametros (*AB* & *CD*) deſcribatur parallellogrammum (*EFGH*); hujus parallelogrammi latera in punctis contactuum (*A*, *B*, *C*, *D*) biſecantur in duas æquales partes, & conſequenter minimum eſt trapezium, quod circa ovalem deſcribi poteſt, unde omnia parallellogramma circa conjugatas diametros ellipſeos deſcripta, erunt inter ſe æqualia. Fig. 60.

Et ſic circumſcribantur circa ellipſim polygona (*EFGH* &c. *PQRS* &c.) (*n*) laterum, a diverſis contactus punctis (*A* & *a*) incipientia, & ſint reſpective minima, quæ circa ellipſim deſcribi poſſunt, a punctis (*A* & *a*) incipientia; tum polygonum (*EFGH* &c.) erit æquale polygono (*PQRS* &c.) & minimum circumſcriptum polygonum (*EFGH* &c.) habet ad maximum inſcriptum eandem rationem, quam habet quadratum radii ad quadratum coſinus arcus $\left(\frac{360}{2n}\right)$. Fig. 58.

EXEMPLUM II.

Iiſdem poſitis, in ovali inſcribantur duo (*n* laterum) polygona (*ABCDEF*, &c. & $\alpha\beta\gamma\delta\epsilon$, &c.) ita ut diagonalis *AC* parallela ſit lineæ tangenti ovalem in puncto contactus (*B*), & ſic diagonales *BD*, *CE*, *DF*, &c. parallelæ ſint reſpective lineis curvam tangentibus in punctis contactuum *C*, *D*, *E*, &c. & ſimiliter ſit diagonalis $\alpha\gamma$ parallela lineæ tangenti ovalem in puncto (β), & lineæ $\beta\delta$, $\gamma\epsilon$, &c. ſint parallelæ lineis tangentibus ovalem in punctis (γ, δ, &c.). Fig. 61.

Hæc duo polygona *ABCDE*, &c. $\alpha\beta\gamma\delta\epsilon$, &c. erunt inter ſe æqualia.

Hoc exemplum eadem methodo, quâ præcedens, demonſtrari poteſt.

COROLLARIUM I.

Sit polygonum, triangulum (*ABC*), circumſcribatur triangulum (*EFG*), cujus puncta contactus ſunt (*A*, *B*, *C*), & triangulum (*EFG*) erit minimum, quod circumſcribi poteſt, & inſcriptum erit ad circumſcriptum: in ratione unitatis ad numerum quaternarium. Fig. 59.

COROLLARIUM II.

Sit polygonum trapezium (*ABCD*), & circumſcriptum trapezium, cujus puncta contactuum ſint (*A*, *B*, *C*, *D*), erit parallelogrammum, & habet Fig. 60.

habet ad inſcriptum ($ABCD$), eandem rationem, quam habet numerus binarius ad unitatem.

COROLLARIUM III.

Sint duo trapezia ($ABCD$, & $\alpha\beta\gamma\delta$) maxima, quæ in datâ ovali inſcribi poſſunt, a punctis (A & α) incipientia, & ſit trapezium $ABCD$ parallelogrammum, tum etiam erit trapezium $\alpha\beta\gamma\delta$ parallelogrammum. Et ſic de circumſcriptis.

COROLLARIUM IV.

Sit ovalis ellipſis conici generis, circumſcribatur circa eam polygonum (n) laterum, cujus puncta contactuum ſunt reſpective ($ABCD$, &c.) & erit circumſcriptum minimum, quod circa ellipſim deſcribi poteſt.

EXEMPLUM III.

Fig. 61. *Iiſdem poſitis, in ovali duo polygona* (ABCDE, *&c.* & $\alpha\beta\gamma\delta$*&c.*) *inſcribantur, ita ut lineæ* AB, BC; CD, DE, *&c. æquales angulos cum lineis ovalem tangentibus in punctis contactuum* (A, B, C, D, *&c.*) *utrinque faciant, & ſic lineæ* $\alpha\beta$, $\beta\gamma$; $\gamma\delta$, $\delta\epsilon$, *&c. æquales angulos cum lineis tangentibus ovalem in punctis contactuum* $\alpha, \beta, \gamma, \delta, \epsilon$, *&c. tum ſummæ polygonorum laterum erunt reſpective inter ſe æquales, i. e.*

$$AB+BC+CD+DE, \&c. = \alpha\beta+\beta\gamma+\gamma\delta+\delta\epsilon+ \&c.$$

COROLLARIUM I.

Fig. 59. Sit polygonum triangulum (ABC), & per puncta contactuum (A, B, C,) ducantur tangentes (EF, FG, GE), & ſumma laterum inſcripti trianguli (ABC) non habet ad ſummam laterum circumſcripti trianguli (EFG) minorem rationem quam 1 : 2.

COR. II.

Fig. 60. Sit polygonum, trapezium ($ABCD$), & ducantur methodo prædicta tangentes (EF, FG, GH, HE), & trapezium circumſcriptum $EFGH$ non habet ad trapezium inſcriptum $ABCD$ majorem rationem quam $\sqrt{2} : 1$.

COR. III.

Circa trapezium ($EFGH$) ſemper deſcribi poteſt circulus, qui tranſit per puncta E, F, G, H reſpective.

COR.

COROLLARIUM IV.

Sit ellipſis conici generis $ABCD$, cujus duo foci ſunt F & H; & lineæ a duobus focis ductæ ad quodlibet peripheriæ punctum (A), eundem angulum utrinque cum tangente, cujus punctum contactus eſt (A), faciunt; & conſequenter per hoc exemplum ſemper eadem manebit harum duarum linearum ($HA+AF$) ſumma; i.e. ſit B quodlibet aliud ellipſeos peripheriæ punctum, & Fig. 62.

$AH+AF=BH+BF$; & vicem verſam æquo jure concludere licet.

EXEMPLUM IV.

Sint duo polygona (EFGHI, *&c.* & PQRS, *&c.*) (n) *laterum, circa ovalem deſcripta, & ſint puncta contactus prioris polygoni reſpective* A, B, C, D, *&c. puncta vero poſterioris* $\alpha, \beta, \gamma, \delta$, *&c.* & Fig. 63.

$$FA : AE :: \overline{\text{Tan} + \text{Sec}}\ \text{Com} \angle AEp : \overline{\text{Tan} + \text{Sec}}\ \text{Com} \angle AFG$$

$$FB : GB :: \overline{\text{Tan} + \text{Sec}}\ \text{Com} \angle BGC : \overline{\text{Tan} + \text{Sec}}\ \text{Com} \angle BFA$$

& ſic deinceps.

Et eodem modo ſecentur polygoni $PQRST$, &c. latera in punctis contactuum $\alpha, \beta, \gamma, \delta$, &c. & ſumma laterum

$$FE+FG+GH+\&c. = PQ+QR+RS+ST+\&c.$$

COROLLARIUM I.

Sit polygonum triangulum (EFG), & jungantur puncta contactuum A, B, C; & ſumma laterum trianguli ABC, i.e. $AB+BC+AC$ non poteſt habere ad ſummam laterum trianguli (EFG), $EF+FG+EG$ minorem rationem quam 1 : 2.

Sit polygonum trapezium $EFGH$, & jungantur puncta contactus A, B, C, D; & non poteſt habere ſumma laterum $EF+FG+GH+HE$ ad ſummam laterum $AB+BC+CD+DA$ majorem rationem quam $\sqrt{2} : 1$.

EXEMPLUM V.

Inſcribantur duo polygona n *laterum in ovali* (ABCD, *&c.* $\alpha\beta\gamma\delta$, *&c.*) *ita ut contentum ſub ſingulis* (n) *lateribus* $AB \times BC \times CD \times$ *&c. majus ſit quam contentum ſub* (n) *lateribus cujuſcunque alius polygoni, cujus vertex eſt* (A); *& contentum* $\alpha\beta \times \beta\gamma \times \gamma\delta \times$ *&c. majus ſit quam contentum ſub*

(n) *lateribus omnis alius polygoni, cujus vertex eſt* (α); *tum hæc duo contenta erunt inter ſe æqualia; i. e.* $AB\times BC\times CD\times$ *&c.* $=\alpha\beta\times\beta\gamma\times\gamma\delta\times$ *&c.*

EXEMPLUM VI.

Sit ſumma (m) *poteſtatis ex ſingulis* (n) *lateribus polygoni,* (ABCD, &c.) *major quam ſumma* (m) *poteſtatis ex ſingulis* (n) *lateribus cujuſcunque alius polygoni eundem verticem habentis; & ſimiliter ſit ſumma* (m) *poteſtatis ex ſingulis* (n) *lateribus polygoni* (αβγδ, *&c.*) *major quam ſumma* (m) *poteſtatis ex ſingulis* (n) *lateribus omnis reliqui polygoni eundem verticem habentis, tum hæ duæ ſummæ erunt inter ſe æquales, i. e.*

$$AB^m+BC^m+CD^m+ \text{\&c.} = \alpha\beta^m+\beta\gamma^m+\gamma\delta^m+ \text{\&c.}$$

Et ſic progredi licet ad polygona hujuſmodi circumſcripta.

EXEMPLUM VII.

Fig. 64. *Sit ſegmentum curvæ* ACB, *& ducatur tangens* ECF *ad ſegmentum in quolibet puncto* C, *ſit angulus* BCF *ſemper æqualis angulo in alterno ſegmento* BAC, *tum erunt anguli in ſegmento contenti* BCA *vel* BDA *ſemper inter ſe æquales.*

Si enim angulus *BCF* æqualis ſit angulo *BAC* in alterno ſegmento, tum erit angulus in ſegmento (*ACB*) maximus; & conſequenter ejus incrementum nihilo eſt æquale, ſi vero incrementum anguli in dato ſegmento contenti continuo nihilo eſt æquale, tum iſte angulus ſemper idem manet i. e. $\angle BAC=\angle BDC$.

Et ſic de infinitis hujuſmodi proportionibus.

EXEMPLUM VIII.

Inſcribantur duæ ellipſes in datâ ovali, quæ nullum habet contrariæ flexuræ punctum, & incipientes a quibuſcunque diverſis ovalis punctis, & ſint ellipſes maximæ, quæ ab iſtis punctis inſcribi poſſunt; tum hæ duæ ellipſes erunt æquales inter ſe.

Et ſic de pleriſque aliis ovalibus.

EXEMPLUM IX.

Sit ſolidum, in ſeſe omni parte concavum, & inſcribantur coni, quorum vertices ſunt quæcunque dati ſolidi puncta, & ſint hi coni maximi, qui pro

pro verticibus ſuis reſpectiva puncta habent; tunc omnes hi coni erunt inter ſe æquales.

Omnes hæ propoſitiones æque applicari poſſunt in curvas, quæ ſibi ipſis in omni parte ſunt convexæ.

In genere inſcribantur vel circumſcribantur lineæ, ita ut ſolida, vel areæ, vel lineæ ſint maxima, vel quandam inter ſe habeant datam relationem; & mutetur punctum contactus, & continuo ſint ſolida, vel areæ, vel lineæ, maxima, vel habeant inter ſe relationem, quam exigat problema; tum ſolida, areæ, vel lineæ, utrum ab hoc, vel ab iſto puncto incipiunt, erunt inter ſe æqualia, vel requiſitam habent inter ſe relationem.

Ex hypotheſi enim ſolida, areæ, vel lineæ, ſemper ſunt maxima, &c. & conſequenter eorum incrementa nihilo ſunt æqualia, &c. ergo ſolida, areæ, vel lineæ continuo inter ſe manent æqualia, &c.

Facile conſtabit quomodo propoſitiones hujuſce generis applicari poſſunt in curvas, quæ in aliâ parte ſunt concavæ, in aliâ vero convexæ.

4. Invenire relationem inter lineas diverſarum curvarum.

Datis æquationibus, relationem inter quaſdam datarum curvarum lineas, exprimentibus, inveniatur relatio inter has lineas & lineas, inter quas requiritur relatio, quibus æquationibus reductis, reſultant æquationes relationem quæſitam exprimentes.

Hujus caſus facile eſt adjicere exempla.

CAP. V.

OMNES curvæ tertii generis reduci poſſunt ad duodecim ſubſequentes æquationum caſus.

CAS. I.

Sit æquatio $y = Ax^4 + Bx^3 + Cx^2 + Dx + E$, & hæc æquatio deſignat biquadraticam parabolam diametrum habentem, quæ eſt prima ſpecies.

CAS. 2.

Sit æquatio $y^3 + \overline{c + dx + ex^2} \times y = Ax^4 + Bx^3 + Cx + Dx + E$, & hæc æquatio deſignat parabolam tertii generis.

Curvarum ſpecies haud plures ſunt, quam vel pura parabola; vel parabola cum ovali, vel puncto, intus vel extra parabolam, poſitis; vel cum duabus ovalibus; vel ovali & puncto; vel ovali nodatâ, vel cuſpidatâ; vel duobus punctis; vel cum tribus ovalibus; vel duabus ovalibus & puncto, vel nodo, vel cuſpide; vel unâ ovali & duobus punctis, vel nodo & puncto, vel duobus nodis, vel cuſpide & puncto, vel cuſpide & nodo, vel duabus cuſpidibus; vel parabola cum tribus punctis; vel nodo, qui intus vel extra parabolam flectitur; vel parabola nodum vel cuſpidem habens, & cum ovali, vel puncto, vel duabus ovalibus, vel ovali & puncto; vel ovali nodata; vel cuſpidatâ; vel duobus punctis; vel parabola habet duos nodos ad idem punctum, vel duas cuſpides, vel nodum & cuſpidem ad idem vel diverſa curvæ puncta, & prætere a habere poteſt ovalem vel punctum; vel parabola habet duos nodos ad diverſa curvæ puncta, vel nodum & duas cuſpides, quarum una vel ponitur ad nodum, necne; vel duos nodos ad idem curvæ punctum cum cuſpide vel ad nodum, necne; vel tres nodos ad idem punctum, vel tres cuſpides; vel parabola habet ovalem curvam interſecantem in duobus punctis; vel punctum triplex; ergo ſpecierum numerus haud major eſt quam 59.

Sit æquatio $y^4 + \overline{a + bx}\, y^2 = Ax^3 + Bx^2 + Cx + D$, & denotat parabolas tertii generis diametrum habentes.

Earum ſpecies ſunt, vel pura parabola, vel parabola cum ovali vel puncto vel nodo intus vel extra parabolam poſitis, vel cuſpide, vel duabus

duabus ovalibus, vel ovali nodum habente, vel duobus punctis, vel duobus nodis ad curvam, vel triplici nodo, vel duabus cuſpidibus, vel ovali vel puncto vel ad curvam nodo vel cuſpide cum duabus ovalibus vel punctis, vel puncto triplici; ergo totus ſpecierum hujus caſus numerus haud major eſt, quam $59+23=82$.

CAS. 3.

1. Sit æquatio
$y^3+\overline{a+bx+cx^2}\times y^2+\overline{d+ex+fx^2+gx^3}\times y=Ax^4+Bx^3+Cx^2+Dx+E$, in quâ radices æquationis $cy^2+gy=A$, ſunt impoſſibiles, & denotat curvas unam parabolam conici generis habentes.

Numerus ſpecierum curvarum hujuſce caſus ſine diametro non eſt major, ut conſtat e præcedenti caſu, quam 59.

2. Sit æquatio $y^4+\overline{cx^2+dx+e}\times y^2=Ax^3+Bx^2+Cx+D$ & denotat parabolas conici generis, diametrum habentes.

Earum ſpecies ſunt, vel pura parabola, vel parabola cum ovali vel puncto vel nodo intus vel extra parabolam poſitis, vel cuſpide, vel duabus ovalibus, vel ovali nodum habente, vel duobus punctis, vel duobus nodis ad curvam, vel triplici nodo, vel duabus cuſpidibus; vel cum tribus ovalibus, vel duabus ovalibus vel punctis, & puncto vel ad curvam nodo vel cuſpide; vel ovali & duobus punctis vel duobus nodis, vel ad curvam duobus nodis vel cuſpidibus; vel duobus nodis vel cuſpidibus ad curvam cum puncto; vel puncto triplici; ergo numerus ſpecierum haud major eſt quam $59+28=87$.

CAS. 4.

1. Sit æquatio $y^2+\overline{a+bx+cx^2}\times y=Ax^4+Bz^3+Cx^2+Dx+E$; & ſint radices æquationis $y^2+cy=A$ poſſibiles; & æquatio deſignat curvas duas parabolas conici generis habentes.

Harum curvarum ſpecies ſunt; vel duæ puræ curuæ, quæ duas conſtituunt ſpecies, vel duæ curvæ ſeſe inter ſecant figurâ cruciformi, vel habent ovalem vel punctum vel nodum vel cuſpidem vel duos nodos curvas jungentes, quarum ſpecierum numerus eſt 8, ſed crura in infinitum eâdem directione pergentia, vel ſunt convexa vel concavo-convexa inter ſe; ergo numerus ſpecierum eſt 16.

Si deſint termini $\overline{a+bx+cx^2}\times y$, curvæ habent diametrum, & numerus ſpecierum, ut conſtat e prædictis, eſt 8.

Sit

Sit æquatio $y^2+\overline{a+cx^2}\times y=Ax^4+Cx^2+E$, & curvæ habent diametrum; earum ſpecies ſunt vel duæ puræ curvæ, quæ duas conſtituunt ſpecies, vel ſeſe interſecant figurâ cruciformi, vel hæ duæ curvæ habent inter eas ovalem vel puuctum vel duos nodos curvas jungentes, quarum numerus eſt 6, ſed crura eâdem directione in infinitum pergentia ſunt convexa vel concavo-convexa inter ſe; ergo numerus ſpecierum eſt 12.

Sit æquatio $y^2=Ax^4+Cx^2+E$, & curvæ habent duas diametros; & ſpecies ſunt, vel duæ puræ curvæ, quæ duas conſtituunt ſpecies, vel hæ duæ curvæ ſeſe interſecant figurâ cruciformi, vel habent ovalem vel punctum, quarum ſpecierum numerus eſt 5, & totus numerus $16+8+12+5=41$.

2. Sit æquatio

$$y^3+\overline{a+bx+c^2x^2}\times y^2+\overline{d+ex+fx^2\pm 2cgx^3}\times y=-g^2x^4+Bx^3+Cx^2+Dx+E$$

& haud plures poſſunt eſſe ſpecies, quam vel duæ puræ parabolæ conici generis, vel duæ parabolæ cum ovali vel puncto, vel duabus ovalibus, vel ovali & puncto vel nodo vel cuſpide; vel duobus punctis; vel cum tribus ovalibus, vel duabus ovalibus & puncto vel nodo vel cuſpide; vel ovali cum duobus nodis, vel nodo & puncto, vel duobus punctis, vel nodo vel puncto & cuſpide, vel duabus cuſpidibus, vel tribus punctis; vel parabola habet nodum vel cuſpidem, vel habet nodum vel cuſpidem cum ovali vel puncto, vel duabus ovalibus, vel ovali & puncto, vel ovali nodatâ vel cuſpidatâ, vel duobus punctis; vel parabola habet duos nodos ad idem punctum, vel nodum & cuſpidem ad eandem vel diverſas curvas, vel duas cuſpides ad diverſas curvas, vel etiam cum ovali vel puncto; vel duos nodos ad diverſas curvas, vel duos nodos, vel nodum & cuſpidem ad unam; & nodum vel cuſpidem ad alteram; harum ſpecierum numerus eſt 52; ſed ſingula ex his ſpeciebus diſtingui poteſt in duas alias, prout duæ parabolæ ſunt disjunctæ, vel unius crus adjungitur alterius cruri; & conſequenter ſpecierum numerus eſt 104; ſed etiam hæ parabolæ ſeſe interſecare poſſunt cruciformiter, vel cruciformiter cum ovali, vel puncto, vel nodo, vel cuſpide, vel cum duabus ovalibus, vel ovali & puncto vel nodo vel cuſpide; vel duobus punctis, vel cum nodo vel cuſpide ad curvam etiamque cum ovali vel puncto vel ad hanc

hanc vel iſtam curvam nodo vel cuſpide, quarum ſpecierum numerus eſt 36, vel curvæ bis ſeſe interſecare poſſunt, vel bis, cum ovali, vel puncto, vel nodo vel cuſpide ad curvam, vel ter ſeſe interſecare poſſunt; unde totus ſpecierum numerus eſt 193.

CAS. 5.

1. Sit æquatio $\overline{x^2+2ax+a^2+b^2}\times y^2+\overline{2cx^3+dx^2+ex+f}\times y=-\overline{c^2+A^2}x^4+Bx^3+Cx^2+Dx+E$; & hæc æquatio deſignat curvas, quarum nullum crus in infinitum progreditur.

Earum ſpecies ſunt; vel ovalis; vel duæ ovales; vel ovalis & punctum; vel ovalis nodum vel cuſpidem habens; vel tres ovales, vel duæ ovales & punctum vel nodus; vel duæ ovales, quarum una habet cuſpidem; vel ovalis cum duobus punctis, vel nodo & puncto, vel duobus nodis; vel ovalis nodum vel punctum & cuſpidem habens; vel ovalis cum duabus cuſpidibus, vel puncto triplici; quarum ſpecierum numerus eſt 16.

2. Sit æquatio $y^4+\overline{c+dx+ex^2}\times y^2+\overline{f+gx+hx^2+kx^3}\times y=Ax^4+Bx^3+Cx^2+Dx+E$, in quâ omnes radices æquationis $y^4+ey^2+ky=A$ ſunt impoſſibiles; & æquatio deſignat curvas, quarum nullum crus in infinitum progreditur, & quarum ſpecies eædem ſunt ac præcedentes cum quatuor ſubſequentibus; vel una ovali in altera incluſa, vel ovali punctum vel nodum vel duos nodos in ſe continente; quarum ſpecierum numerus eſt 20.

Sit æquatio $P\overline{x^2+a^2}\times y^2=-A^2x^4+Bx^3+Cx^2+Dx+E$, & deſignat curvas diametrum habentes.

Earum ſpecies ſunt, vel una ovalis; vel duæ; vel punctum & ovalis; vel ovalis cum nodo vel cuſpide.

Sit æquatio $P\times\overline{x^2+a^2}\times y^2=-A^2x^4+Cx^2+E$, & deſignat curvas duas diametros habentes, quarum ſpecies ſunt, vel una ovalis; vel duæ, vel ovalis nodata.

Sit æquatio $y^4+\overline{a+2cx^2}\times y^2=-\overline{A^2+c^2}+x^4+Cx^2+E$, & deſignat curvas duas diametros habentes. quarum ſpecies ſunt eædem ac præcedentes cum tribus ſubſequentibus, vel ovali continente in ſe ovalem vel punctum; vel duabus ovalibus ſeſe in duobus punctis ſecantibus.

Sit

Sit æquatio $y^4 + \overline{a + bx + cx^2} \times y^2 = -\overline{A^2 + c^2} \times x^4 + Bx^3 + Cx^2 + Dx + E$, & deſignat curvas unam diametrum habentes, quarum ſpecies ſunt, vel ovalis, vel duæ ovales, vel ovalis & punctum, vel ovalis nodum vel cuſpidem habens; vel tres ovales, vel duæ ovales cum puncto, vel nodo; vel ovalis nodum & punctum, vel duos nodos, vel duas cuſpides, vel punctum triplex, vel duo puncta, habens; vel etiam ovalis habens in ſe ovalem, vel punctum, vel ovalis intus ſemel vel bis nodata, ergo totus ſpecierum numerus hujus caſus eſt $20+6+17=43$.

CAS. 6.

Sit æquatio $x^2y^2 + \overline{ax^2 + bx + c} \times y = -Ax^4 + Bx^3 + Cx^2 + Dx + E$; & deſignat circumſcriptam hyperbolam unam aſymptoton habentem.

Si $-Ax^6 + Bx^5 + Cx^4 + Dx^3 + Ex^2 + \frac{1}{4} \times \overline{ax^2 + bx + c}^2 = 0$; tum duo ordinatæ ($y$) valores erunt inter ſe æquales.

Et exinde innoteſcunt ſpecies; quæ ſunt vel pura hyperbola; vel hyperbola cum ovali; vel puncto; vel nodo; vel cuſpide; vel cum duabus ovalibus; vel ovali & puncto; vel ovali & nodo; vel ovali nodum vel cuſpidem habente; vel cuſpide & ovali vel puncto; vel duobus punctis; vel puncto & nodo; vel duobus nodis ad idem vel diverſa curvæ puncta; vel duabus cuſpidibus; vel nodo & cuſpide ex eadem vel diverſa aſymptoti parte; quarum ſpecierum numerus eſt 19.

Si terminus (cy) deeſt; tum figura conſtat ex duabus hyperbolis ex eadem vel diverſis aſymptotos partibus utrinque in infinitum progredientibus; & ſpecies erunt, vel duæ puræ hyperbolæ, vel duæ hyperbolæ cum ovali; vel puncto; vel nodo; vel cuſpide; quæ ſunt 10 ſpecies.

Si $x^2y^2 + \overline{ax^2 + c} \times y = -Ax^4 + cx^2 + E$; æquatio deſignat circumſcriptam hyperbolam diametrum habentem; ejus ſpecies ſunt, vel pura circumſcripta hyperbola; vel hyperbola cum duabus ovalibus; vel duobus punctis; vel duobus nodis; vel duabus cuſpidibus; quæ ſunt 5 ſpecies.

Si terminus (cy) deſit, tum duæ ſunt hyperbolæ circa eandem aſymptoton utrinque in infinitum pergentes.

Si

Si $y^2x^2 = -Ax^4 + Bx^3 + Cx^2 + Dx + E$, deſignat duas hyperbolas diametrum habentes circa eandem aſymptoton circumſcriptas; & earum ſpecies ſunt, vel duæ puræ hyperbolæ; vel duæ hyperbolæ cum ovali, vel puncto, vel nodo, vel cuſpide; quæ ſunt 5 ſpecies.

Sit æquatio $x^2y^2 = -Ax^4 + Cx^2 + E$, & duæ hyperbolæ duas habent diametros; ergo totus numerus ſpecierum hujuſce caſus eſt 41.

CAS. 7.

1. Sit æquatio $xy = Ax^4 + Bx^3 + Cx^2 + Dx + E$; & figura conſtat ex duabus curvis, quæ partim hyperbolicæ, partim cubico-parabolicæ ſunt.

2. Sit æquatio $y^2 + \overline{a + bx + cx^2 + dx^3} \times y = Ax^2 + Bx + C$, & figura vel conſtat e duabus curvis partim hyperbolicis & partim cubico-parabolicis, quarum hyperbola & parabola continuo ductu vel in oppoſitas partes progrediuntur, vel in eandem redit, & hyperbolicum crus vel eſt concavum vel convexum in parabolicum verſus eandem partem pergens, quæ quatuor conſtituunt ſpecies; vel conſtat e duabus hyperbolo-parabolicis curvis, quarum hyperbolica & parabolica crura in eandem partem pergentia vel ſunt convexa vel concavo-convexa inter ſe cum figurâ cruciformi; vel ovali; vel puncto; vel nodo; vel cuſpide; vel duabus ovalibus; vel ovali nodum habente; vel ovali & nodo ad curvam; vel ovali & puncto vel cuſpide ad ovalem vel curvam; vel cuſpide ad unam curvam & nodo ad alteram; vel cuſpide & nodo ad eandem; vel cuſpide & puncto; vel duobus nodis ad eandem vel diverſas curvas; vel puncto & nodo ad curvam; vel duobus punctis; vel duobus vel tribus nodis duas curvas connectentibus; vel duabus cuſpidibus; ſi terminus Ax^2 deeſt, tum crura hyperbolica ex eodem aſymptoti latere ad infinitam diſtantiam jacent, & conſtat ſpecierum numerum minorem eſſe per unitatem quam numerum curvarum, quarum hyperbolica crura verſus eandem partem progredientia ſunt convexa vel concavo-convexa inter ſe; quoniam non poteſt habere duas cuſpides curva, cujus æquatio eſt prædictæ formulæ, majorem vero per 9, quoniam duæ curvæ diverſæ ſunt ſpeciei inter ſe; ergo totus numerus ſpecierum hujus caſus eſt 77.

CAS. 8.

1. Sit æquatio $xy^2 + ay = Ax^4 + Bx^3 + Cx^2 + Dx + E$ & deſignat ſemicubicam parabolam cum hyperbolâ.

Curvarum hujus æquationis ſpecies ſunt, vel duæ curvæ, quæ partim parabolicæ & partim hyperbolicæ ſunt; vel hæ duæ curvæ cum ovali vel puncto vel nodo vel cuſpide, vel duabus ovalibus, vel ovali & puncto; vel ovali & ad curvam nodo vel cuſpide; vel ovali nodum vel cuſpidem habente; vel duobus punctis, vel puncto & ad curvam nodo vel cuſpide; vel ad curvam duobus nodis, vel nodo & cuſpide: ſpecies etiam erunt vel ſemicubica parabola cum anguineâ hyperbolâ, vel ſemicubica parabola & anguinea hyperbola cum ovali, vel puncto, quæ vel ponuntur inter parabolam & hyperbolam, necne; vel nodo vel cuſpide ad parabolam; vel ad hyperbolam nodo vel cuſpide ex hac vel illâ aſymptoti parte; vel parabola & hyperbola ſeſe interſecant, vel ſeſe interſecant & habent ovalem vel punctum, vel ad hyperbolam nodum vel cuſpidem, vel hyperbola & parabola bis ſeſe interſecant, ergo ſpecierum numerus eſt 33.

Si terminus (ay) deſit, tum curvæ habent diametrum, & ſpecies ſunt vel duæ puræ curvæ partim parabolicæ & partim hyperbolicæ, vel hæ duæ curvæ cum ovali; vel puncto; vel duabus ovalibus; vel ovali & puncto; vel ovali nodum vel cuſpidem habente; vel duobus punctis; vel ſint parabola & conchoidalis hyperbola ex eâdem vel diversâ aſymptoti parte, vel hæ duæ curvæ cum ovali vel puncto inter eas, necne, vel ad parabolam vel hyperbolam nodo vel cuſpide, vel parabola & hyperbola ſeſe interſecant, vel ſeſe interſecant & habent ovalem vel punctum, vel parabola & hyperbola bis ſeſe interſecant, ergo harum ſpecierum numerus eſt 30.

2. Sit æquatio $y^3 + \overline{a + bx + cx^2 + dx^3} \times y = Ax^2 + Bx + C$, & deſignat duas curvas partim ſemicubicas parabolas & partim hyperbolas; in quibus hyperbolicum crus & ejus correſpondens parabolicum in infinitum eâdem directione pergentia vel ſunt convexa vel concavo-convexa inter ſe, & ex eâdem vel diversâ aſymptoti parte in infinitum progrediuntur hyperbolica crura. E caſus quarti parte ſecundâ conſtat ſpecies hujus æquationis haud plures eſſe quam 4×19 ſine nodo vel cuſpide ad curvam vel interſectione duarum curvarum;

rum; haud plures eſſe, quam 4×8 nodum vel cuſpidem ad hanc vel illam curvam habentes, haud plures quam $4 \times 2 \times 3$ duos nodos vel nodum & cuſpidem ad idem punctum curvæ, vel 4×3 duos nodos ad diverſas curvas poſitos, vel 36 nodum & cuſpidem vel duas cuſpides habentes, vel 4×2 tres nodos ad idem punctum curvæ vel duos nodos & cuſpidem, vel 4×2 nodum vel cuſpidem ad unam curvam & nodum & cuſpidem vel duos nodos ad alteram, vel plures quam 4×28, quarum duæ curvæ ſeſe interſecant; vel 4×7 in quibus curvæ bis ſeſe interſecant; vel 4 in quibus curvæ ter ſeſe interſecant ergo totus numerus ſpecierum hujus caſus haud major eſt quam 492.

CAS. 9.

1. Sit æquatio $xx \times \overline{x+a} \times y^2 = -Ax^4 + Bx^3 + Cx^2 + Dx + E$, & deſignat hyperbolas duas parallelas aſymptotos & diametrum habentes.

Earum ſpecies ſunt, vel duæ puræ inſcriptæ hyperbolæ inter aſymptotos jacentes, vel hæ duæ hyperbolæ cum ovali vel puncto, vel duabus ovalibus, vel ovali & puncto, vel ovali nodum vel cuſpidem habente, vel duobus punctis, vel ex unâ aſymptotum parte ovali vel puncto & ovali vel puncto ex alterâ aſymptotum parte; vel duæ conchoidales hyperbolæ extra aſymptotos, vel hæ duæ hyperbolæ cum ovali vel puncto inter hyperbolas, necne, vel nodo vel cuſpide; vel una conchoidalis inter aſymptotos, altera vero extra eas, vel hæ duæ hyperbolæ cum ovali vel puncto inter aſymptotos vel ex hâc vel illâ aſymptotum parte, vel cum nodo vel cuſpide ad hanc vel illam hyperbolam; vel duæ conchoidales hyperbolæ inter aſymptotos, vel hæ cum ovali vel puncto inter vel extra eas, vel nodo vel cuſpide, vel hæ duæ hyperbolæ ſeſe interſecant figurâ cruciformi, vel ſeſe interſecant & habent ovalem vel punctum, vel bis ſeſe interſecant, ergo numerus ſpecierum eſt 40.

2. Sit æquatio
$xx \times \overline{x+a}\, y^2 + \overline{a+bx+cx^2+dx^3} \times y = Ax^4 + Bx^3 + Cx^2 + Dx + E$, & radices æquationis $y^2 + dy = A$ impoſſibiles, & deſignat curvas duas parallelas aſymptotos habentes.

 Earum

Earum ſpecies ſunt, vel duæ anguineæ hyperbolæ quarum duæ ſunt ſpecies; vel hæ hyperbolæ cum ovali vel puncto vel nodo vel cuſpide inter vel extra aſymptotos; vel cruciformes fiant anguineæ, vel habent etiam ovalem vel punctum vel nodum vel cuſpidem, vel bis cruciformiter ſeſe interſecant; vel duæ ſunt ambigenæ hyperbolæ, vel hæ hyperbolæ cum ovali vel puncto vel nodo vel cuſpide vel duabus ovalibus, vel ovali & puncto, vel ovali nodum vel cuſpidem habente, vel duobus punctis, vel nodo vel cuſpide ad curvam cum ovali vel puncto; vel ad curvam duobus nodis vel nodo & cuſpide; vel curva habet ovalem vel punctum, vel nodum, vel cuſpidem ex unâ aſymptotum parte, & ovalem, vel punctum, vel nodum vel cuſpidem, ex alterâ aſymptotum parte; vel una eſt hyperbola inſcripta & altera circumſcripta, vel hæ cum ovali vel puncto, vel duabus ovalibus, vel ovali & puncto, vel ovali nodum vel cuſpidem habente, vel duobus punctis, vel cum nodo ad curvam etiamque cum ovali vel puncto vel nodo vel cuſpide ex hâc vel illâ aſymptotum parte; & ſic de cuſpide ad curvam; vel eſt una curva inſcripta & altera ambigena; vel hæ hyperbolæ habent etiam ovalem vel punctum ex hâc vel illâ aſymptotum parte vel nodum vel cuſpidem, vel duas ovales, vel ovalem & punctum, vel duo puncta ex hâc vel illâ ex eâdem vel diverſâ aſymptotum parte, vel ovalem nodum vel cuſpidem habentem ex hâc vel illâ aſymptotum parte, vel ambigena curva habet nodum vel cuſpidem & ovalem vel punctum ex eâdem vel diverſâ aſymptotum parte, vel habet duos nodos vel nodum & cuſpidem; vel eſt conchoidalis hyperbola, quæ ponitur vel inter vel extra aſymptotos & anguinea hyperbola, vel hæ habent inter eas ovalem vel punctum vel ad anguineam vel conchoidalem nodum vel cuſpidem, vel extra aſymptotos ovalem vel punctum ex hâc vel illâ aſymptotum parte, vel ad anguineam vel conchoidalem nodum vel cuſpidem, vel curvæ fiant cruciformes, vel ſint cruciformes & habeant extra eas ovalem vel punctum ex hâc vel illâ aſymptotum parte, vel ad anguineam nodum vel cuſpidem; vel bis fiant cruciformes; his denique ſpeciebus adjiciendæ ſunt tot ſpecies, quot habet prior æquatio, & erit numerus ſpecierum $40+147=187$.

3. Sit

3. Sit æquatio $x^2y^2+a\overline{+bx}\times y=Ax+B$; & deſignat hyperbolas duas aſymptotos habentes; curvarum ſpecies ſunt vel, duæ hyperbolæ, una inſcripta & altera circumſcripta circa eundem angulum, vel hæ duæ hyperbolæ cum ovali vel puncto vel nodo vel cuſpide, vel duæ convergentes hyperbolæ; vel hæ duæ hyperbolæ ſeſe interſecant figurâ cruciformi.

Si terminus (Ax) deeſt, tum quatuor hyperbolæ ſunt circa duas aſymptotos deſcriptæ.

Sit æquatio $x^2y^2=Ax+B$; & curva habet diametrum, quarum ſpecierum numerus eſt 9.

4. Sit æquatio

$$xy^3+\overline{a+bx+cx^2}\times y^2+\overline{e+fx+gx^2+px^3}\times y=Ax^2+Bx+C$$

in quâ radices æquationis $y^2+cy+b=0$ ſunt impoſſibiles, & deſignat duas hyperbolas.

Hæ autem hyperbolæ ſunt vel una inſcripta, ambigena vel circumſcripta, & altera etiam vel inſcripta ambigena vel circumſcripta.

Et ſpecierum numerus haud major eſſe poteſt, quam ut conſtat e parte ſecundâ caſus 4ti 9×23 ſine nodo, cuſpide vel ad curvam cruciformi, non plures quam $15\times\overline{9+8}$ unum nodum vel cuſpidem ſolummodo habentes, non plures quam 15×3 duos nodos vel nodum & cuſpidem ad idem vel diverſum curvæ punctum, vel duas cuſpides; vel 15 tres cuſpides, vel nodum & duas cuſpides, quarum una vel ponitur ad idem vel diverſum punctum curvæ; vel duos nodos ad idem vel diverſum curvæ punctum & cuſpidem ad idem vel diverſum punctum; non plures quam 9×3 nodum vel cuſpidem ad utraſque curvas habentes, non plures quam 15×3 nodum ad unam & cuſpidem ad alteram curvam habentes, vel 15 nodum & cuſpidem ad unam & cuſpidem ad alteram, non plures ſunt quam 270, curvæ, quarum crura ſemel ſolummodo ſeſe ſecant cruciformiter; non plures quam 87, in quibus curvæ bis ſecant ſeſe cruciformiter, non plures quam 9 in quibus ter ſeſe ſecant cruciformiter, ergo totus numerus ſpecierum hujus caſus haud major eſt quam $1254+9+187=1450$.

CAS.

CAS. 10.

Sit $xx\overline{x+a}xy^2+\overline{ex+f}xy=Ax+B$, & hyperbolas deſignat tres aſymptotos habentes, quarum duæ ſunt inter ſe parallelæ.

Earum ſpecies ſunt vel inſcripta, ambigena & anguinea hyperbola, cujus duæ ſunt ſpecies, vel ambigena & anguinea ſeſe interſecant, vel una inſcripta cum duabus ambigenis; vel hæ cum ovali vel puncto vel nodo vel cuſpide; vel una hyperbola ſerpens circa ejus aſymptoton in infinitum cum duabus ambigenis, vel ſerpens & ambigena ſeſe interſecant, vel duæ inſcriptæ hyperbolæ, quarum una ponitur inter duas parallelas aſymptotos, & tertia circumſcripta; vel hæ habent ovalem vel punctum vel ad circumſcriptum nodum vel cuſpidem.

Si terminus (fy) deſit, tum vel datur hyperbola in infinitum circa ejus aſymptoton ſerpens, & inſcripta & ambigena, quarum duæ ſunt ſpecies, vel ambigena & ſerpens hyperbola ſeſe interſecant, vel ſint duæ inſcriptæ hyperbolæ cum anguineâ, vel inſcripta ambigena & conchoidalis hyperbola inter vel extra aſymptotos, ſi ponitur conchoidalis inter aſymptotos, ambigena & conchoidalis ſeſe interſecare poſſunt; vel ſint duæ inſcriptæ & ambigena hyperbola, vel hæ cum ovali vel puncto.

Sit $xx\overline{x+a}xy^2+\overline{ex+f}xy=B$; & vel duæ ſunt inſcriptæ hyperbolæ & duæ ambigenæ, quarum duæ ſunt ſpecies, vel duæ ambigenæ ſeſe interſecant, vel tres inſcriptæ & una ambigena hyperbola, & idem erit numerus ſpecierum, ſi utræque radices æquationis $y^2+ey=B$ ſunt vel affirmativæ vel negativæ.

Sit $xx\overline{x+a}xy^2=Ax+B$, & curva habet diametrum, ſpecies earum ſunt, vel duæ inſcriptæ hyperbolæ cum conchoidali vel inter vel extra aſymptotos, vel una hyperbola ſerpens circa ejus aſymptoton in infinitum cum duabus inter parallelas aſymptotos inſcriptis; vel quatuor inſcriptæ hyperbolæ, ergo numerus ſpecierum eſt 38.

Sit æquatio $x^2y^2+\overline{x^3+bx^2+cx+d}xy=Ax^2+Bx+C$, & denotat hyperbolas tres aſymptotos habentes.

Earum ſpecies ſunt vel una hyperbola inſcripta, ambigena, vel circumſcripta; & hyperbola, quæ ei opponitur, vel inſcripta, ambigena, vel circumſcripta; & tertia convergens hyperbola; vel hæ cum figurâ cruciformi, ex unâ, alterâ, vel utrâque parte vel etiam ovali, vel

vel puncto vel nodo vel cuſpide, &c. ſi terminus (dy) deſit, tum transformatur convergens hyperbola vel in duas conchoidales, circa eandem aſymptoton hyperbolas, vel in duas, quarum reſpectiva crura eodem aſymptoti latere & eadem directione in infinitum ſerpunt, harum ſpecierum numerus eſt 216.

Si nullæ ſint oppoſitæ hyperbolæ, tum hyperbolæ vel deſcribuntur ad angulos vel latera aſymptotum, & ſpecies ſunt vel una inſcripta vel ambigena, altera ambigena vel circumſcripta & tertia inſcripta ambigena vel circumſcripta hyperbola, vel hæ cum ovali vel puncto; vel nodo vel cuſpide ad hanc vel illam curvam; vel cum duabus ovalibus, vel ovali & puncto vel nodo vel cuſpide, vel duobus punctis, vel nodo vel cuſpide ad curvam cum ovali vel puncto; vel duobus nodis vel nodo & cuſpide ad eandem vel diverſam curvam, vel duabus cuſpidibus; vel ſemel bis vel ter cruciformes fiant eædem hyperbolæ, vel ſit etiam conchoidalis hyperbola, & curvæ habent ovalem vel punctum vel nodum vel cuſpidem, vel unam vel duas interſectiones, vel quatuor ſunt hyperbolæ, ergo totus numerus ſpecierum hujus caſus eſt 1263.

CAS. II.

Sit æquatio $xy^2 + \overline{ax^3 + bx^2 + cx + d}\,xy = Ax^2 + Bx + C$, & deſignat curvas duas hyperbolas & parabolam conici generis habentes.

Earum ſpecies ſunt vel duæ curvæ partim parabolicæ & partim hyperbolicæ cum anguinea vel conchoidali hyperbola, vel hæ curvæ cruciformiter ſeſe interſecant ex una vel utrâque parte; vel habeant ovalem vel punctum vel ad unam e curvis nodum vel cuſpidem, vel duplicem interſectionem, vel duplicem interſectionem ad unam curvam & interſectionem ad alteram; quarum ſpecierum numerus eſt 134; vel duæ curvæ partim parabolicæ & partim hyperbolicæ cum ambigenâ vel inſcriptâ vel circumſcriptâ hyperbolâ, vel hæ ſeſe interſecant, vel habent ovalem vel punctum vel duas ovales, vel ovalem & punctum vel ovalem nodatam vel cuſpidatam vel duo puncta, vel nodum vel cuſpidem ad unam e curvis, vel nodum vel cuſpidem cum ovali vel puncto, vel duos nodos vel nodum & cuſpidem ad eandem vel diverſas curvas, vel duas cuſpides, vel unam vel duas vel tres interſectiones, quarum ſpecierum numerus eſt 290.

2. Sit

2. Sit æquatio

$$xy^3+\overline{2ax^2+bx+c}\times y^2+\overline{a^2x^3+dx^2+ex+f}\times y=Ax^2+Bx+C$$

& denotare poteſt curvas duas hyperbolas & parabolam conici generis habentes.

Earum ſpecies ſunt vel parabola & duæ hyperbolæ, vel duæ curvæ partim parabolicæ & partim hyperbolicæ cum inſcriptâ vel ambigena vel circumſcriptâ hyperbolâ; numerus autem harum curvarum ſpecierum ſine nodo, cuſpide vel cruciformi ad curvam non major erit quam 19×33; numerus autem cum nodo vel cuſpide, nodis vel cuſpidibus ſine cruciformi non major erit, quam 120×33, cum vero cruciformi non major erit quam 195×33, ergo totus numerus curvarum hujus caſus non major erit quam 8342.

CAS. 12.

1. Sit æquatio

$$xy^3+\overline{a+bx+cx^2+ex^2}\times y^2+\overline{d+ex+fx^3+gx^3}\times y=Ax^2+Bx+C$$

& deſignare poteſt curvas quatuor aſymptotos habentes.

Hyperbolæ vel jacent ad duo latera trapezii ab aſymptotis, comprehenſi & duos ejus angulos, vel ad tria latera & unum angulum.

Et poſſunt eſſe quatuor hyperbolæ vel inſcriptæ, ambigenæ vel circumſcriptæ, quarum ſpecierum numerus eſt 2×49=98.

Numerus curvarum ſine nodo, cuſpide vel cruciformi non major eſt quam 19×98=1862.

Numerus curvarum cum nodo vel cuſpide vel duobus nodis, vel nodo & cuſpide, vel duabus cuſpidibus, vel tribus nodis, vel duobus nodis & cuſpide, vel nodo & duabus cuſpidibus, vel tribus cuſpidibus ſine cruciformi, haud major erit, quam 98×212=20776.

Numerus curvarum cum cruciformi vel duabus vel tribus cruciformibus & nodo vel nodis, cuſpide vel cuſpidibus haud major eſt, quam 98×376=36848.

Vel una e quatuor hyperbolis continere poteſt in ſe alteram, vel tres ſint hyperbolæ & anguinea vel conchoidalis; & numerus harum curvarum ſine nodo vel cuſpide, &c. non major eſt, quam 864.

Numerus curvarum cum nodo vel cuſpide, &c. ſine cruciformi non major erit, quam 1728.

Numerus curvarum ſeſe interſecantium cruciformiter non major erit, quam 864.

2. Si

2. Si curvæ habent generale centrum, i. e. quatuor aſymptoti ſeſe interſecant in eodem puncto; numerus curvarum non major eſt quam 2500.

3. Sit æquatio $y^4+\overline{a+bx+cx^2}\times y^2=Ax^4+Bx^3+Cx^2+Dx+E$, & deſignat curvas diametrum habentes.

Curvæ collocantur ad angulos & ad latera figuræ ab interſectione aſymptotum formatæ ut in caſu priori, & ſunt vel duæ inſcriptæ, ambigenæ, vel circumſcriptæ hyperbolæ & duæ aliæ etiam vel inſcriptæ, ambigenæ, vel circumſcriptæ hyperbolæ, harum ſpecierum numerus non major eſt quam 2392. Si deſint termini bxy^2 & Bx^3 curvæ habent generale centrum, & earum numerus ut conſtat e præcedenti non major erit quam 132.

Si $y^4+\overline{a+bx^2}\times y^2=Ax^4+Cx^2+E$; & curvæ habent duas vel quatuor diametros, & numerus earum ſpecierum non major erit quam 42.

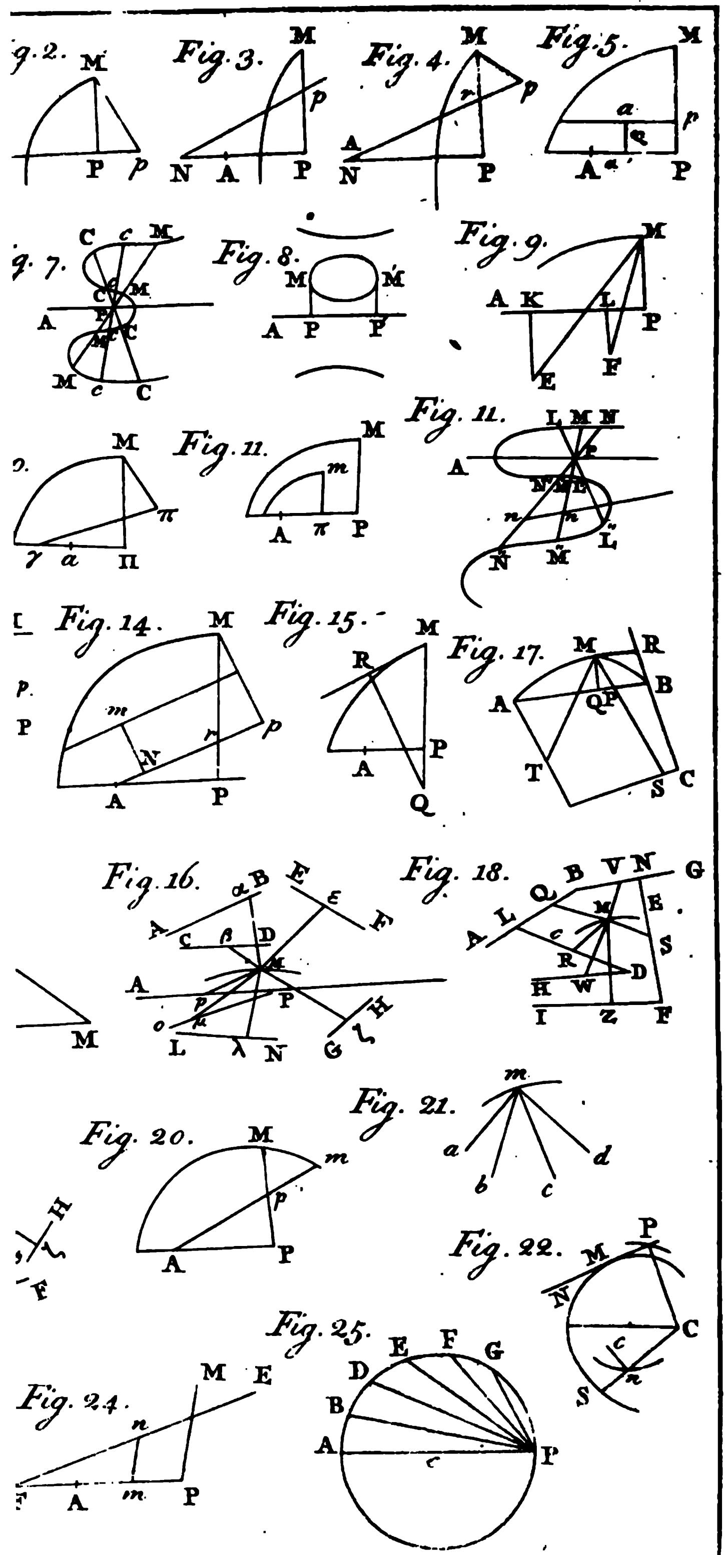
4. Sit æquatio $x\times\overline{x+a}\times y^2+\overline{ax^3+bx^2+cx+d}\times y=Ax^2+Bx+C$, & denotat hyperbolas quatuor aſymptotos habentes, quarum duæ ſunt inter ſe parallelæ, quarum ſpecies ſunt vel una inſcripta, ambigena, vel circumſcripta; & altera etiam vel inſcripta ambigena vel circumſcripta cum duabus ambigenis vel anguineis vel unâ inſcriptâ & alterâ ambigenâ, vel unâ conchoidali & alterâ anguinea, &c. quarum ſpecierum numerus erit 4097.

Sit æquatio $x\times\overline{x+a}\times y^2=+Ax^4+Bx^3+Cx^2+Dx+E$, & denotat hyperbolas diametrum & quatuor aſymptotos habentes, quarum duæ ſunt inter ſe parallelæ, & numerus harum curvarum ut facile colligi poteſt, erit 163. & numerus curvarum duas aſymptotos inter ſe parallelas & duas alias inter ſe parallelas habentium ſine diametro non major erit quam 193, cum vero unâ diametro eſt 6, & duabus diametris 3.

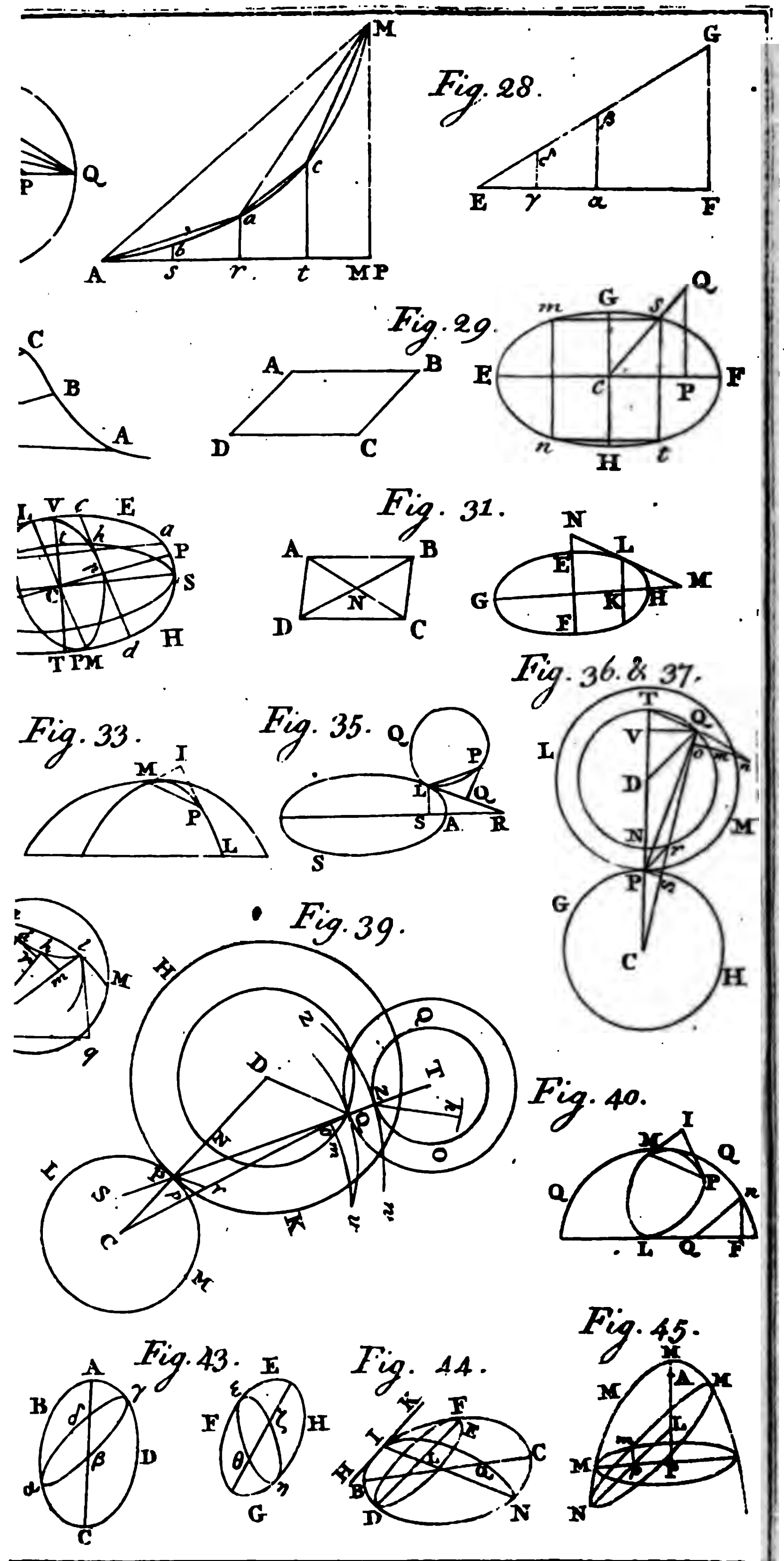
5. Sit æquatio $\overline{x^3+ax^2+bx+c}\times y=Ax^2+Bx+C$, & deſignare poteſt decem curvas quatuor aſymptotos habentes, quarum tres ſunt inter ſe parallelæ; ergo totus numerus curvarum hujus caſus non erit major quam 72480. & non plures ſunt quam 84551. curvæ tertii ordinis.

Si abſciſſa incipit a puncto duplici, tum deſunt termini $\alpha x + \beta y + K$; ſi incipit a puncto triplici, tum deſunt termini $\gamma x^2 + \delta xy + \epsilon y^2 + \alpha x + \beta y + K$; ſi incipit a puncto quadruplici, tum deſunt termini $Zx^3 + nx^2y + \Theta xy^3 + \kappa y^3 + \gamma x^2 + \delta xy + \epsilon y^2 + \alpha x + \beta y + K$; & ſic deinceps.

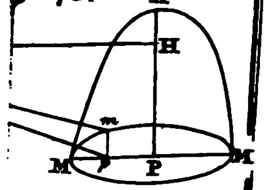
FINIS.


Fig. 3.
Fig. 4.
Fig. 5.
Fig. 8.
Fig. 9.
Fig. 11.
Fig. 11.
Fig. 14.
Fig. 15.
Fig. 17.
Fig. 16.
Fig. 18.
Fig. 21.
Fig. 20.
Fig. 22.
Fig. 25.
Fig. 24.

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Fig. 28.
Fig. 29.
Fig. 31.
Fig. 33.
Fig. 35.
Fig. 36. & 37.
Fig. 39.
Fig. 40.
Fig. 43.
Fig. 44.
Fig. 45.

q. 48.



?. 52.

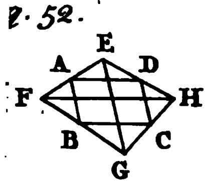
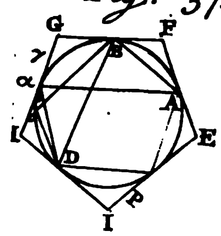
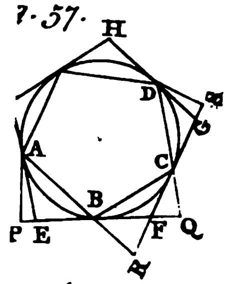


Fig. 57.



?. 57.



q. 60.

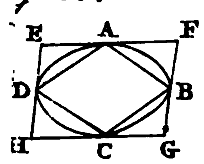
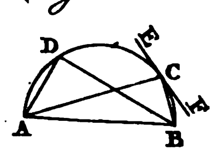


Fig. 64.



OF
NOX AND
DATION

CORRIGENDA.

PAG. 6. l. 10. pro cujus, lege quorum; pag. 18. l. 23. pro nullum, lege nullam; pag. 23. l. 16. pro multiplicatur, lege multiplicetur; pag. 33. l. 18. dele tentando vel; pag. 37. l. 18. pro q, r, s, t, v ſcribe p, q, r, s, t reſpective; pag. 39. l. 7. pro quod & æquale, lege qui & æqualis; pag. 44. pro $\sqrt{-b^2}$, lege $\sqrt{b}$, & pro $\overline{a^2-b^2}^{\frac{1}{n}}$, lege $\overline{a^2-b}^{\frac{1}{n}}$; pag. 52. l. 14. pro ſit data æquatio, lege ſint duæ datæ æquationes; pag. 59. l. 7. pro ſupponantur, lege ſupponatur; pag. 60. l. 12. pro quantitates, lege quantitatis; pag. 62. l. 2. delè & novæ quantitatis (v); pag. 63. l. 3. pro quibus, lege quarum; pag. 68. l. 4. pro $\frac{C^2}{s^2}$, lege $\frac{C^2}{s^2} \times \frac{S}{s}$; pag. 70. l. ult. pro major, lege major vel æqualis; pag. 71. l. 2. pro eorum, lege earum; pag. 71. cor. 8. extirpandum; ſumma diametrorum non major eſſe poteſt quam (n), &c. ſed de hoc in alio loco; pag. 71. l. 14. lege exprimens; pag. 72. l. 18. pro cujus, lege quarum; pag. 78. l. 72. pro pro præcedentes, lege prædictæ; pag. 80. l. 15. pro incognitæ quantitatis, lege incognitarum quantitatum; l. 16. pro z & y, lege z, & v. l. 18. pro x & v, lege z & v; pag. 84. l. 22. pro datum, lege datam; pag. 86. l. 1. pro æqualis, lege æquale; pag. 87. l. 21. pro duæ, lege duo; pag. 92. pro y ſcribatur $\frac{\sqrt{1-s^2} \times \overline{Y-sX}}{c}$ & pro X, $\frac{Cy}{c} + \frac{\overline{c-Cs} \times X}{c\sqrt{1-s^2}}$, pag. 94. l. 9. adjice exprimens, pag. 101. l. 28. pro prior, lege prius; pag. 104. l. 11. ſcribe exprimens; pag. 108. l. 27. pro prior, lege prius; pag. 111. pro parallellogramma & parallellogrammati, lege parallellogrammum & parallellogrammo &; pag. 117. l. 15. pro cujus, lege quorum; pag. 119. l. 23. pro una, lege quarum una eſt; pag. 121. l. 12. pro CQ, lege Q; pag. 122. l. 33. pro $LMLC$, lege $LMCL$; pag. 121. l. 1. pro datis quibuſlibet numeris, lege dato quolibet numero; pag. 137. l. 4. pro negativus, lege major quam internus angulus, & pro fig. 50. & 51. lege 51. & 52; pag. 139. pro triangulum Mba minus, lege triangulum Mba majus; pag. 141. l. 13. pro FBG, lege FCG; pag. 146. l. 21. pro $BAC = BDC$, lege $BDA = BCA$; pag. 156. l. 24. pro eſt, lege ſunt; pag. 158. l. 1. pro circumſcriptum, lege circumſcriptam, & l. 5. pro cujus, lege quarum; pag. 158. l. 26. pro curva habet, lege curvæ habent. pag. 139, 140, 141, &c. in exemplis, lege ſi modo talia polygona continuo inſcribi vel circumſcribi poſſunt.

(I.c)

MEDITATIONES ALGEBRAICÆ,

AB

EDVARDO WARING,

MATHESEOS PROFESSORE LUCASIANO, CANTAB.

REGIÆ SOCIETATIS,

ET

ACADEMIÆ BONONIENSIS INSTITUTI SOCIO.

EDITIO TERTIA RECENSITA ET AUCTA.

CANTABRIGIÆ,

TYPIS ACADEMICIS excudebat J. ARCHDEACON;

Veneunt apud J. NICHOLSON, Cantabrigiæ; J. C. & F. RIVINGTON, S. CROWDER,
H. GARDNER, & S. HAYES, Londini; & J. FLETCHER, Oxonii.

MDCCLXXXII.

PRÆFATIO.

DE incrementis iis, quæ gradatim res cœperit algebraica, narrationem hic contexui brevem, ut ſua inventoribus deferatur gloria; atque ut iis ſimul, qui progreſſus artium inveſtigant curioſius, aliquâ ſit ex parte ſatisfactum. Ex hiſtoriis clar. virorum Walliſii *&* Monteclu *quædam mutuatus ſum, quorum alter* Harriotto *noſtrati nimis favet, alter quidem gallicis ſcriptoribus, ſed humanum eſt ſic errare. Agnoſcendum eſt hiſtoriam clariſſimi* Monteclu *omnibus aliis & elegantiâ & rei philoſophicæ prudentiâ præſtare; ſed rerum mathematicarum ſtudioſi pleniorem adhuc recentiorum nonnullorum inventorum hiſtoriam deſiderant. Controverſiæ ſerram de procurſibus, quos in hâc diſciplinâ fecerint antiqui, reciprocare vix operæ pretium erit: vulgares certe arithmeticæ operationes, æquationumque tum ſimplicium tum quadraticarum reſolutiones probe calluerunt.* Diophantus Alexandrinus, *qui floruit circa annum domini* 360, *algebraiſtarum, quorum opera ſuperſunt, eſt antiquiſſimus. Multas æquationes, quæ duas incognitas quantitates involvunt, ita reduxit, ut quantitates innoteſcerent incognitæ; arteſque quibus utitur in valoribus rationalibus inveniendis adeo concinnæ & elegantes ſunt, ut a recentioribus raro ſuperentur. Figuræ numerariæ* 0, 1, 2, 3, *&c. ab Indis inventæ ad Arabes primo tranſierunt, & deinde inter annos* 1000 *&* 1100 *vel forſan antea in uſum apud Europæos venerunt. Arithmetica decimalis ſecundum* Walliſium *ab* Arzachel *&* Regiomontano *circiter annum* 1400 *inchoata eſt, & a* Purbach *quoque, referente* Monteclu: *radium quidem in partes* 60000000 *diviſerunt; ſed fractiones inveniſſe decimales vix dici poſſunt. Ad umbilicum fere perducta eſt decimalium methodus a* Petro Ramo *in arithmeticâ ſuâ circa annum* 1560 *editâ, ubi docet in extrahendâ radice quadraticâ cubicâve (ſi quid reſiduum ſit poſt integrorum operationem) proſequi rem ut in integris, quo habeatur radix in integris cum adjunctis partibus decimalibus. Idem jam ante factum eſſe a noſtrate* Gulielmo Buclæo

Collegii Regalis apud Cantabrigienſes alumno aſſerit Walliſius, *cujus arithmeticam memorativam (verſu conſcriptam) ſubjunctam vidit* Setoni *logicæ editæ Cantabrigiæ anno* 1631*:* Buclæus *autem mortuus eſt circa annum* 1550 *(regnante* Edvardo *ſexto, qui obiit anno* 1553*).*

Noſtras Briggius *invenit* (n) *differentias numerorum* 1^n, 2^n, 3^n, 4^n, 5^n, *& eſſe æquales, & differentias iſtas eſſe* $1 \cdot 2 \cdot 3 \cdot 4 \ldots \times (n-1) \times n$; *& exinde interpolavit logarithmos inter alios datos; in hoc tractatu de hâc re non agitur, attamen datur methodus inveniendi functionem quantitatum* (x, y, z, *&c.*), *quæ evadet* S, S′, S″, *&c.*; *cum quantitates* (x, y, z, *&c.*) *evadant reſpective* α, β, γ, *&c.*; α', β', γ', *&c.*; *&c.*

Mohammed Ben Muſa & Thebit Ben Corah *algebraiſtarum apud Arabes vetuſtiſſimi ſunt, quorum primus, auctore* Cardano, *æquationum quadraticarum reſolutionem invenit: dicitur quoque* Omar Ben Ibrahim *cubicas reduxiſſe æquationes: Arabica autem* MSS. *non ſatis ſcrutati ſunt recentiores.* Leonardum de Piſa *collocat* Voſſius *circa annum* 1400 *aut citius, eumque dicit (ex* Blancano) *primum fuiſſe ex hodiernis, qui de alegebrâ ſcripſit. Primus autem, qui tractatum de algebrâ typis exhibuit, erat* Lucas de Burgo; *omnia, quouſque æquationes pertingunt quadraticæ, tradidit, earumque duas eſſe radices oſtendit: æquationis cubicæ reſolutionem* Scipioni Ferreo *attribuit* Cardanus, *methodum vero ſuam ſtudioſe celavit* Ferreus; *eandem interim ſuo marte invenit* Tartalea, & Cardano *roganti, & ſe arcanum fore pollicenti, communicavit; fidem vero fefellit* Cardanus, *methodumque* Tartaleæ *auctam tamen atque promotam edidit; ſecundum æquationis cubicæ terminum abſtulit* Cardanus, *caſumque irreducibilem tres vel negativas vel affirmativas habere radices, oſtendit. Æquationis biquadraticæ reſolutionem invenit* Ludovicus Ferrarius, Bombelliumque *ſuam methodum ex* Ludovico Ferrario *hauſiſſe dicit* Walliſius *in epiſtolâ ad* Morlandum *cap.* 53. *algeb.* — *Quoad notationem* Buteo *in ejus logiſticâ anno* 1559 *editâ (ſub finem libri tertii in ſuâ regulâ quantitatis dictâ) varias quæſitas quantitates literis* A, B, C, *&c. deſignavit.* Vieta *primum literas pro incognitis quantitatibus ſubſtituit, atque coefficientem ſecundi æquationis termini ſummam eſſe radicum, & ſimiliter omnium coefficientium conſtitutionem docuit, etiamque radices datæ*

æqua-

æquationis per datas quantitates auxit & diminuit, multiplicavit & diviſit. Dedit etiam methodos haud inelegantes inveniendi rationales valores incognitarum quantitatum quarundam quadraticarum æquationum correſpondentes. De quantitate binomiali $(a + b)^n$ *ad poteſtatem quamlibet evectâ verba facit, dicitque literalia contenta eſſe* a^n, $a^{n-1}b$, $a^{n-2}b^2$, *&c.; etiamque coefficientem termini ſecundi unum fore e numeris triangularibus, coefficientem tertii termini unum fore e numeris pyramidalibus, & ſic deinceps.* Paſcall *de numeris triangularibus, pyramidalibus, &c. inveniendis agit; deinde* Neutonus *invenit binomiale theorema, viz.*

$$(a + b)^n = a^n + n a^{n-1} b + n \cdot \frac{n - 1}{2} a^{n-2} b^2 + \&c.$$

Quantitates aa, aaa, *&c. aliquando* a^2, a^3, *&c. pro quantitatibus* Aq, Ac, *&c. ſubſtituit* Harriottus, *æquationem integram ad unam partem revocat, & nihilo facit æqualem, & exinde veram originem æquationum ſuperiorum ex compoſitione lateralium & coefficientium conſtitutionem oſtendit; ad detegendas impoſſibiles radices cubicæ æquationis uſus eſt iiſdem principiis, ex quibus ulterius promotis acquiri poſſunt* Neutoni *aliorumque conſimiles regulæ. Hunc inſecutus eſt* Carteſius, *ultimas literas alphabetæ* x, y, z, *&c. pro incognitis quantitatibus, primas vero* a, b, c, *&c. pro cognitis adhibet; tot eſſe radices affirmativas, quot ſint mutationes ſignorum de* + *in* —, *&* — *in* +; *cæteras vero negativas eſſe, ſi modo omnes æquationis radices* (x) *ſint poſſibiles, aſſerit* Carteſius, *cujus regulæ demonſtrationem ex multiplicatione datæ æquationis in* x — a *&* x + a, *cum ejus radices ſint omnes poſſibiles, primus tradidit* Gua: *tot ſaltem mutationes ſignorum in omni æquatione eſſe de* + *in* —, *&* — *in* +; *quot ſunt affirmativæ radices; & tot continuos progreſſus de* + *in* + *&* — *in* —, *quot ſunt negativæ radices, invenit* Segner: *primum in primâ editione hujuſce operis edocta fuit methodus detegendi, annon omnes radices datæ æquationis ſint poſſibiles, & exinde utrum hæc regula applicari poteſt, necne: hìc forſan haud indignum eſt obſervatu, quod in æquationibus ſuperiorum dimenſionum fortuito aſſumptis perraro occurrunt æquationes, quarum omnes radices ſunt poſſibiles; igitur in hâc re deſideranda eſt regula, quæ primum inveniat numerum impoſſibilium datæ æquationis radicum; deinde dato impoſſibilium radi-*

cum

cum numero quærenda eſt regula, quæ inveniat numerum affirmativarum & negativarum radicum: duo problemata ob faſtidium calculi tam operoſi in æquationibus ſuperiorum dimenſionum vix unquam ſuperanda; de his autem quædam in hoc opere continentur: methodum diverſam a prædictâ Ferrarii *methodo biquadraticas æquationes reſolvendi invenit etiam* Carteſius: *diverſas methodos idem problema reſolvendi dedere* Eulerus, Bezout, Vaudremonde, *&c.; nova etiam methodus in hoc opere continetur: ſed animadvertendum eſt, quod omnes methodi idem problema reſolvendi ex primis principiis deductæ, ſemper erunt eædem: deprehendit etiam* Carteſius *omnem radicem, qui eſt integer numerus, eſſe diviſorem* (a) *ultimi datæ æquationis* ($x^n - px^{n-1} + qx^{n-2} - \&c. = 0$) *termini; & ex diviſione datæ æquationis per* $x - a$ *invenit, utrum* (a) *ſit radix datæ æquationis, necne: hæc methodus ob ſimplicitatem & in pleriſque caſibus ob facilitatem methodo* Waeſſanaeri *minime poſthabenda eſt, quamvis multi algebraici ſcriptores de eâ vix mentionem faciunt: methodum etiamque dedit amicabiles numeros detegendi. Ex augendo & diminuendo datæ æquationis radices per aſſumptam quantitatem, ita ut ejus coefficientes ex negativis evadant affirmativæ, & vice versâ ex affirmativis evadant negativæ, invenit limites, inter quos ponuntur quædam datæ æquationis radices.* — *Methodum extrahendi radicem quamlibet* (n) *e quantitate binomiali* $a + \sqrt{(b)}$, *ſi modo radix poſſit exprimi per quantitatem* $\frac{x + \sqrt{(y)}}{\sqrt[n]{(Q)}}$, *dedit* Schootenius. 1mo. *detegit numerum* N, *qui in datam quantitatem* $a + \sqrt{(b)}$ *ductus, facit* $N^2(a^2 - b)$ *numerum, cujus radix* (n) *poteſtatis* $(N^2(a^2 - b))^{\frac{1}{n}} = \pi$ *extrahi poteſt; deinde invenit rationalem quantitatem* (v) *majorem quam* $(N(a + \sqrt{(b)})^{\frac{1}{n}}$, *ita ut differentia non major ſit quam* $\frac{1}{2}$; *tum quæſivit proximum integrum* (z) *vel majorem vel minorem, &c. quantitate* $v \pm \frac{\pi}{v}$; *exinde* $\frac{\frac{1}{2}z + \sqrt{(\frac{1}{4}z^2 - \pi)}}{\sqrt[n]{(N)}}$ *erit radix quæſita.*

Neutonus

Neutonus *prædictum Principium a* Schootenio *ſumit, viz. ducit datam quantitatem* $a + \sqrt{(b)}$ *vel* $\sqrt{(a)} + \sqrt{(b)} = A + B$, *ubi* A *major eſt quam* B *in* N, *ita ut radix poteſtatis* (n) *quantitatis* $(N \times (A^2 - B^2))^{\frac{1}{n}} = \pi$ *extrahi poſſit; computat* $(N^{\frac{1}{2}}(A + B))^{\frac{1}{n}}$ *in numeris integris proximis, qui ſit* r; *dividit* $A\sqrt{(N)}$ *per maximum rationalem diviſorem, cujus quotus ſit* s, & *ſit* $\frac{r + \frac{\pi}{r}}{2s}$ *in proximis integris* t, *erit* $\frac{ts + \sqrt{(t^2 s^2 - \pi)}}{\sqrt[2n]{(N)}}$ *radix quæſita:* Eulerus *aſſumit* $\frac{x + y}{\sqrt[2n]{(N)}} = \sqrt[n]{(A + B)}$, *unde* $x^2 - y^2 = \sqrt[n]{(\overline{A^2 - B^2}\,N)} = r^n$, *ubi* x, y & p *ſunt integri numeri; invenit* $x^2 + y^2 = \frac{1}{2}\sqrt{(\overline{A + B}^2 \times N)} + \frac{1}{2}\sqrt[n]{(\overline{A - B}^2 \times N)}$, & *exinde* x^2 & y^2, &c. *animadvertebat, quod prædictæ* Schootenii & Neutoni *regulæ aliquando fallunt: primus, quantum ſcio, rationem prædicti* Schootenii *principii demonſtravi; etiamque probavi* $a^n + n \cdot \frac{n-1}{2} a^{n-2} b + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} a^{n-4} b^2 +$ &c., & $n a^{n-1} + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} a^{n-3} b +$ &c., *ubi* a & b *ſunt inter ſe primi numeri, nullum habere communem diviſorem; ni* $2, 2^2, 2^3, \ldots 2^n$; *exinde ſequitur, ſi modo ducatur data quantitas* $A + B$ *in* 2^n, *ſemper extrahi poſſe ejus radicem prædictam vel per* Schootenii *vel per* Neutoni *regulam; quædam etiam adjiciuntur regulæ ex principio magis generali quam iſto a* Schootenio *poſito: datæ æquationis ſimplices diviſores invenire ſubſtituendo* $-1, 0, 1,$ &c. *pro incognitâ quantitate docuit* Jacobus *a* Waeſſanaer; *methodum* Waeſſaneri *ad diviſores quadraticos, cubicos,* &c. *inveniendos promovit* Neutonus: *in hoc opere methodus* Carteſii *ad conſimiles diviſores detegendos promovetur, etiamque datur methodus inveniendi integros correſpondentes valores duarum incognitarum quantitatum in duabus æquationibus contentarum; eadem principia ad plures æquationes plures incognitas quantitates ha-*

bentes, applicari poſſunt. Huddenius *ſubſequentem obſervationem protulit, viz. ſi modo detur æquatio, quæ plures incognitas quantitates recipiat, & una vel duæ vel plures incognitæ quantitates nihilo vel quibuſdam datis quantitatibus aſſumantur æquales; & exinde reſultet æquatio, quæ diviſores haud admittat, tum datam æquationem diviſores etiam haud recipere: docuit methodum inveniendi diviſorem datæ æquationis, cujus unus vel plures termini vel deficiunt vel dantur; terminos datæ æquationis ſucceſſivos per arithmeticam ſeriem multiplicat, & exinde detegit ejus radices æquales, ſi modo tales in datâ æquatione contineantur: aſſumit* $x^6 + qx^4 + rx^3 + sx^2 + tx + v = 0$, *cujus diviſor ſit* $x^2 + yx + h = 0$; *invenit æquationem, cujus radix eſt* y. *Duarum æquationum duarum vel plurium dimenſionum reductionem ad unam, ita ut incognita quantitas exterminetur per vulgarem methodum nunc in uſu docuit* Eraſmus Bartholinus *in ſuo Capite* 16, "*De methodo generali, quoad uſum ſecundarum radicum ad tollenda ſigna radicalia ex æquatione propoſita:*" *cujus methodi æquationes reducendi ita ut incognita quantitas exterminetur, & tollendi radicalia ſigna ſunt eædem ac eæ, quæ poſtea traduntur in Arithmet. Univer.* Neutoni. *Fractionem, uti loquuntur, continuam primus invenit vicecomes* Brounker; *& ex dato uno integro valore quantitatis* x *vel* y *in æquatione* $ax^2 + 1 = y^2$, *modum plures inveniendi deduxit. Ejus generalis lex per ſeriem in hoc libro docetur. Hanc methodum ad æquationem* $ax^2 + bx + c = y^2$ *promovit* Eulerus; *de caſibus, in quibus haud recipiunt integros valores quantitates* x & y, *quædam adjecit* Le Grange: *methodum reducendi fractiones & rationes ad minores terminos, ſervato quam fieri poteſt proximo valore, dedit* Walliſius; *methodumque approximandi ad valores integros incognitarum quantitatum in æquatione* $ax^2 + 1 = v^2$ *& ſimilibus æquationibus.* Bachet *dicit omnem numerum eſſe ſummam quatuor quadratorum, cujus demonſtrationem in fractionalibus quantitatibus primus dedit* Eulerus; *in integris autem* Le Grange: *primus etiam detexit generalem methodum inveniendi integros correſpondentes valores æquationis* $ax + by = P$; *ubi* a, b & P *integros denotant affirmativos numeros;* Eulerus *idem perfecit dividendo maximam coefficientem* (a) *per minimam* (b), *& diviſorem per reſiduum, &c. uſque donec ad aliquod reſiduum perveniatur, quod dividit*

dit numerum P; *deinde ex quotientibus inveſtigavit unum valorem ſingularum quantitatum* x & y; Simpſon *poſtea idem perfecit ex reſiduis*; *in primâ editione hujuſce operis ad regiam ſocietatem anno* 1757 *miſsâ*, *deinde anno* 1760 *impreſsâ*, & *inter quoſdam amicos communicatâ*, & *anno* 1762 *cum proprietatibus algebraicarum curvarum editâ*, *primùm traditur methodus inveniendi generales expreſſiones pro integris valoribus ſingularum incognitarum quantitatum* x, y, z, &c. *in æquatione* $ax + by + cz + dv + \&c. = A$, *cum tres vel plures incognitæ quantitates* x, y, z, v, &c. *in datâ æquatione contineantur*, *in terminis quantitatum* u, w, &c.; *ubi* a, b, c, d, &c. *ſunt integri*; & u, w, &c. *quicunque numeri ad libitum aſſumendi*: *in ſecundâ editione hujuſce operis annis* 1768 & 1769 *impreſsâ*, & *ad initium anni* 1770 *editâ*, *datur etiam methodus reducendi duas æquationes* $ax + by + \&c. = P$ & $a'x + b'y + \&c. = P$, &c. (*ducendo ambas æquationes in* m & n *reſpective*, & *inveniendo integrum valorem correſpondentem quantitatum* m & n *in æquatione* $ma - na' = 1$; *vel inveniendo integros generales valores quantitatis* x *in terminis prædictarum quantitatum* u, w, &c. *ex unâ datâ æquatione*, & *ſubſtituendo in alterâ vel reliquis*) *in unam*, *ita ut incognita quantitas exterminetur*, & *ſi incognitæ quantitates in reſultantibus æquationibus ſint integri*, *tum exterminata etiam erit integer numerus*: *aſſeritur quantitatem* $ax + by$, *ubi* a & b *ſunt inter ſe primi*, & x & y *integri numeri*, *poſſe conficere omnem integrum numerum*, *qui ſuperat numerum* $a \times b - a - b$: *in primâ editione hujuſce operis anno* 1762 *editâ*, *docetur ſubſequens reductio duarum vel plurium* (m) *æquationum*, *vel* (m + 1) *vel plures incognitas quantitates habentium in unam*, &c., *ita ut exterminetur una vel plures incognitæ quantitates*, & *ſi incognitæ quantitates in reſultantibus æquationibus ſint rationales quantitates*, *ut exterminata etiam ſit rationalis quantitas*; *viz. per vulgarem methodum* Bartholini *ita reducantur duæ vel plures æquationes*, *ut tandem reſultet* $Px = Q$, *ubi* P & Q *ſunt rationales functiones reliquarum incognitarum quantitatum*, *ſubſtituatur* $\frac{Q}{P}$ *pro* x *in datis æquationibus* & *perficietur reductio*; *ſi vero nunquam pervenire liceat ad formulam* $Px = Q$ *ſed tandem perventum ſit ad duas vel plures reſultantes æquationes hujuſ-*

modi

modi $Ax^2 + Bx = C$ & $Ax^2 + Bx = D$, *&c.*, *vel* $Ax^3 + Bx^2 + Cx = D$ & $Ax^3 + Bx^2 + Cx = E$, *&c.*; *ubi* A, B, C, D, E, *&c.* *reſpective denotent quaſcunque rationales functiones incognitarum quantitatum* (y, v, z, *&c*) *in datâ æquatione contentarum*; *in quo caſu omnis valor e ſingulis incognitis quantitatibus* (y, z, v, *&c.*) *habet duos, &c. valores incognitæ quantitatis* (x) *ſibi ipſi reſpondentes, & per hanc methodum haud ita exterminari poteſt quantitas* x; *inveniatur una vel plures incognitæ quantitates rationalibus terminis reliquarum & novarum aſſumptarum, ſi modo fieri poſſit, & ſubſtituantur hi valores*; *& id, quod requiritur, perficietur.* Fermatius *invenit omnem integrum eſſe triangulum vel e duobus vel tribus triangulis compoſitum cum pluribus aliis ejuſmodi*; *e. g. denotante* (m) *integrum numerum aſſerit omnem primum numerum* $(4m + 1)$ *componi e duobus quadratis. In ſecundâ editione hujuſce operis aſſeritur omnem numerum vel eſſe cubum vel ex* 1, 2, 3, 4, ... 8, 9 *cubis compoſitum*; *etiamque omnem numerum, qui ſuperat numerum inveſtigandum, eſſe ſummam numerorum hujuſce generis* $pa^2 + qb^2 + rc^2 + sd^2$, *ubi* a, b, c & d *ſunt quicunque numeri, &* p, q, r & s *ſunt numeri inter ſe primi: dicitur ſummam duorum cuborum* $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ *non æquari tertio cubo* (c^3); *nec* $ae \times (a^2 - e^2)$, *nec* $x \times \frac{x+1}{2}$ *poſſe eſſe quadratum, ni* $x = 1$; *harum propoſitionum demonſtrationes dedit* Eulerus; *& conſimiles adjecit, viz.* $\alpha^3 \beta x^4 \pm \beta^3 \alpha y^4$, *&c. non quadratos eſſe demonſtravit ex eodem principio, viz. ſi modo ſit quadratus, deducit quantitates minores ejuſdem formulæ, quæ etiam erunt quadrati*; *& ſic deinceps in infinitum*; Le Grange *idem principium (minoribus datis) ad majores valores incognitarum datæ æquationis detegendos applicavit:* Fermatius *etiam detegit* $a^n - 1$ *ſemper diviſibilem eſſe per* n, *ſi modo* n *ſit primus numerus*; Eulerus *hanc propoſitionem demonſtravit, & quædam corollaria elegantia adjecit: aſſerit ille numeros* N^{2m+1} & N^{2m+2} *componi poſſe* $(m + 1)$ *diverſis modis e numeris hujuſce formulæ* $(p^2 + rq^2)$, *ubi* $N = a^2 + rb^2$; *& exinde quædam corollaria adjecit: in ſecunda editione hujuſce operis primum traditæ fuere* $(m + 1)$ *diverſæ compoſitiones prædictæ formulæ* $p^2 + rq^2$.

Albert Girard *invenit ſummam quadratorum, cuborum, biquadratorum, e ſingulis*

e ſingulis radicibus; regulam dedit Neutonus, *ex quâ inveniri poteſt ſumma datæ æquationis radicum, quadratorum e ſingulis radicibus, cuborum, &c.; ex ſummis inventis omnium inferiorum poteſtatum e ſingulis radicibus ſummam proxime ſuperioris detegit: in primâ editione hujuſce operis datur ſeriei lex, quæ invenit prædictam ſummam ex coefficientibus datæ æquationis: eadem lex ab* Eulero *poſtea tradita fuit: ſumma prædicta uſui inſervit in detegendâ approximatione ad maximam radicem datæ æquationis, cum omnes radices ſint poſſibiles; vel cum maxima radix datæ æquationis ſit poſſibilis; ſed vel ad transformationem, vel ad reductionem & ulteriorem reſolutionem, vel ad exterminationem incognitarum datarum æquationum quantitatum abſque methodo ſubſequentia contenta detegendi vix aut ne vix applicari poteſt:* Cramer *fingit radices æquationis* $y^n - p y^{n-1} + q y^{n-2} - r y^{n-3} + \&c. = 0$ *eſſe* $\alpha, \beta, \gamma, \delta$, *&c.; ſubſtituit has radices reſpective pro* y *in æquatione* $V - T y + S y^2 - \&c. = 0$, *& multiplicat quantitates reſultantes in ſeſe; ex ſummis ſingulorum valorum quantitatum inferiorum dimenſionum formulæ* $\alpha^a \beta^b \gamma^c$ *&c. + &c. deducit ſummas e ſingulis valoribus quantitatum ejuſdem formulæ dimenſionis proxime ſuperioris: in primâ editione hujuſce operis primùm ex datis ſummis omnium poteſtatum e ſingulis radicibus datur lex haud difficilis, quam obſervat ſeries pro inveniendâ prædictâ ſummâ* $\alpha^a \beta^b \gamma^c \delta^d$ *&c. + &c.; adjicitur etiam in ſecundâ editione methodus inveniendi eandem ſummam ex coefficientibus datæ æquationis: in primâ editione traditur methodus inveniendi aggregatum e ſingulis valoribus cujuſcunque algebraicæ functionis datæ vel datarum æquationum radicum, quæ fuſius in ſecundâ & tertiâ editione explicatur: his regulis a me inventis, in primâ hujuſce operis editione novam exinde deduxi transformationem æquationum in alias, quarum radices quamcunque datam algebraicam habeant relationem ad radices datarum æquationum; viz. primo e doctrinâ combinationum inveni numerum radicum, & conſequenter dimenſiones æquationis quæſitæ; 2do. ex aſſignabili relatione (aſſumptis* $\alpha, \beta, \gamma, \delta$, *&c.;* A, B, Γ, Δ, *&c. pro radicibus datarum æquationum reſpective) detegi radices æquationis quæſitæ & exinde earum ſummam, & aggregata e ſingulis duabus, tribus, quatuor, &c. in ſeſe ductis reſpective; quoniam in his aggregatis ſimiliter involvuntur radices* $\alpha, \beta,$

γ, δ, &c.; &c. ex datis æquationibus regularum prædictarum ope inveniri poſſunt: nemo priùs transformavit radices, niſi ex tali reductione æquationum, ut incognitæ quantitates per ſubſtitutionem exterminentur; & nimirum, nam hæc transformatio exigit methodum detegendi aggregatum e ſingulis valoribus cujuſcunque algebraicæ functionis nunquam prius traditam: nemo enim ante me dedit ſummam e ſingulis valoribus cujuſcunque irrationalis vel fractionalis datæ æquationis radicum functionis: nec aliquis dedit methodum detegendi aggregatum e ſingulis valoribus quantitatum $\alpha^a \beta^b \gamma^c$ *&c. + &c. ex ſummis poteſtatum ſingularum radicum (α, β, γ, &c.) datæ æquationis; vel ex ejus coefficientibus datis per legem; niſi* Cramer *progrediendo ex omnibus inferioribus aggregatis prædicti generis datis, proxime ſuperiora colligit, quæ perraro in hâc re uſui inſervire poteſt: in eâdem editione hujuſce operis applicatur prædicta methodus tranſformationis ad proprietates curvarum inveſtigandas, in quâ ſæpe perutilis eſt, nam ſæpe proprietates ſolummodo pendent e primis quæſitæ æquationis terminis; etiamque ad inveniendas impoſſibiles radices, ad delendas ſurdas e datâ æquatione quantitates, & ad reſolutiones æquationum & ex meâ & ex* Tſchirnhauſi *methodo petitas. In ſecundâ editione hujuſe operis datur nova methodus inveniendi prædicta aggregata ex aſſumptis datis æquationibus, quarum radices innoteſcunt, & exinde aggregata reſpondentia quærendo; etiamque ex æquationibus inferiorum dimenſionum ad æquationes ſuperiorum progrediendo; quæ methodi in multis caſibus majorem habent facilitatem.* Schootenius *in ſingulis locis datæ æquationis deficientibus ſcribit ſignum + vel —; & affirmat tot ſaltem radices eſſe impoſſibiles, quot poteſt eſſe maxima differentia inter numerum mutationum ſignorum de + in — & — in + ex quibuſcunque duabus ſubſtitutionibus hujuſcemodi.* Neutonus *dedit regulam, ex quâ ſæpe cognoſci poteſt, quot radices ſint poſſibiles, & quot impoſſibiles:* Campbell & Mac Laurinus *conſimiles regulas adjecere: in ſecundâ editione hujuſce operis continentur forſan maxime generalia principia omnium adhuc editorum inveniendi functiones, in quibus plures quantitates ſimiliter involvuntur, quæ ſemper evadent affirmativæ, cum quantitates ipſæ ſint poſſibiles, hæc autem multo magis generalia reddi poſſunt; exhinc erui poſſunt multæ regulæ numerum impoſſibilium datæ*

datæ æquationis radicum detegendi: in ſecundâ editione hujuſce operis redduntur prædictæ regulæ magis generales, ducendo datam æquationem in x — a, *& exinde per prædictas regulas inveniendi numerum impoſſibilium radicum progrediendo: in primâ editione anno* 1762 *editâ traditur regula, cujus ope probabilitas verum inveniendi impoſſibilium radicum numerum videtur eſſe ad probabilitatem eundum inveniendi numerum per* Neutoni *regulam prope* :: $3^{n-2}:2^{n-2}$; *ſi* 2 n *ſit numerus dimenſionum datæ æquationis: deinde in primâ editione transformatur data æquatio in alteram, cujus radices ſint quadrata datæ æquationis radicum; & in ſecundâ editione in quamcunque alteram, cujus radices ſint affirmativæ, cum radices datæ æquationis ſint poſſibiles; & exinde ſæpe ex progreſſu ſignorum de* + *in* + *&* — *in* — *innoteſcet numerus ejus impoſſibilium radicum: in ſecundâ editione anno* 1769 *impreſsâ obſervatum eſt ex augendo vel diminuendo, &c. datæ æquationis radices per aſſumptas quantitates facile ſe manifeſtare impoſſibiles radices, quæ priùs latuere: at omnes hæ regulæ prædictæ perraro invenient verum numerum impoſſibilium radicum in æquationibus multarum dimenſionum, & adhuc demonſtratione egent; vulgares enim demonſtrationes ſolummodo probant impoſſibiles radices in datâ æquatione contineri; non vero quod ſaltem tot ſunt, quot invenit regula: vera reſolutio problematis eſt perdifficilis & valde laborioſa; cognitum eſt radices ex poſſibilitate per æqualitatem tranſire ad impoſſibilitatem, ergo in generali reſolutione hujuſce problematis neceſſe eſt invenire caſum, in quo radices datæ æquationis evadunt æquales: reſolutio autem hujus caſus valde laborioſa eſt; & conſequenter reſolutio generalis prædicti problematis magis erit laborioſa: in primâ editione hujuſce operis anno* 1762 *editâ transformatur data æquatio in alteram, cujus radices ſint quadrata differentiarum datæ æquationis radicum, & ibi animadvertitur, "ſi mutentur reſultantis æquationis ſigna alternatim de* + *in* — *&* — *in* +, *nullam impoſſibilem radicem habere datam æquationem, ſin aliter impoſſibiles habere radices," in actis philoſophicis* Londin. *anno* 1764 *editis dantur exempla hujuſce transformationis in æquationibus quinque dimenſionum, & aſſeritur duas radices datæ æquationis eſſe impoſſibiles; ſi ultimus terminus reſultantis æquationis ſit negativus; ſin affirmativus, tum vel nullas vel quatuor fore impoſſibiles;*

poſſibiles; regula etiam traditur de eo caſu, in quo ultimus terminus nihilo evadat æqualis: qui has bene intelligit, non poteſt non applicare eaſdem obſervationes ad æquationes ſuperiorum dimenſionum: etiamque in eâdem editione adjungitur ſi data æquatio ſit $x^n - px^{n-1} + qx^{n-2} -$ *&c.* $= 0$, *& aſſumantur æquationes* $v^{n-1} - \frac{n-1}{n} pv^{n-2} +$ *&c.* $= 0$ *&* $v^n - pv^{n-1} +$ *&c.* $= z$, *& transformentur duæ æquationes in unam, ita ut exterminetur* v, *reſultantis æquationis ope, cujus radix eſt* z, *ſæpe detegi poſſe impoſſibiles radices; aſſeritur æquationem datam nullam habere poſſibilem radicem, ſi modo mutentur ſigna terminorum æquationis, cujus radix eſt* z, *continuo viciſſim de* + *in* — *&* — *in* +: *transformatur etiam data æquatio in alteram, cujus radices ſint* $\frac{1}{\alpha-\beta}, \frac{1}{\alpha-\gamma}, \frac{1}{\beta-\gamma}$, *&c.; ubi* α, β, γ, *&c. ſunt radices datæ æquationis; unde inveniri poteſt* $\frac{1}{A}$ *major quam maxima æquationis reſultantis radix; & exinde reſultat* A *minor quam minima differentia inter quaſcunque duas radices, & hinc quantitates* A, 2A, 3A, 4A, *&c.* 0, —A, *&c. præbent limites inter ſucceſſivas radices; quibus datis, facile conſtat numerus impoſſibilium datæ æquationis radicum. In ſecundâ editione ex dato numero impoſſibilium, &c. radicum datæ æquationis datur methodus inveniendi numerum impoſſibilium radicum in æquatione, cujus radices datam habeant relationem ad radices datæ æquationis; hujus regulæ dantur exempla, quorum unum eſt caſus, in quo radices reſultantis æquationis ſint quadrata differentiarum inter quaſcunque duas datæ æquationis radices, &c.: ex methodo detegendi numerum impoſſibilium radicum datæ æquationis, in quâ continetur una incognita quantitas datur methodus inveniendi limites, inter quos conſiſtant poſſibiles correſpondentes valores incognitarum quantitatum datæ æquationis duas vel plures incognitas quantitates habentis.* Clairant *in algebrâ ſuâ* 1760 *editâ methodum inveniendi numerum radicum impoſſibilium in biquadraticâ æquatione contentarum tradidit.*

Mac Laurinus *primus, quantum ſcio, animadvertit radices æquationis* $nx^{n-1} - \overline{n-1}\,px^{n-2} + \overline{n-2}\,qx^{n-3} -$ *&c.* $= 0$ *eſſe limites inter radices*

dices æquationis $x^n - px^{n-1} + qx^{n-2} -$ &c. $= 0$; Milner *invenit æquationem* $ax^n - \overline{a+b}\,px^{n-1} + \overline{a+2b}\,qx^{n-2} - \overline{a+3b}\,rx^{n-3} + {} = 0$ *non ſemper habere radicem inter minimam affirmativam & minimam negativam radicem æquationis* $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} +$ &c. $= 0$ *poſitam. Sit æquatio* (A) $x^n - px^{n-1} + qx^{n-2} -$ &c. $= 0$, *cujus radices ſint* α, β, γ, δ, &c., *quarum* α *major ſit quam* β, β *quam* γ, &c.; *ſint* π, ϱ, σ, &c. *radices æquationis* (B) $nx^{n-1} - \overline{n-1}\,px^{n-2} +$ &c. $= 0$ *inter* α & β, β & γ, &c. *reſpective poſitæ*; *tum, ſi* h & m *eadem habeant ſigna, in hoc opere probatur radices æquationis* $hA + mB = 0$ *inter* α & π, β & ϱ, &c. *reſpective poſitas eſſe*; *ſi vero* h & m *diverſa habeant ſigna, tum probatur unam radicem æquationis* $hA + mB = 0$ *majorem eſſe quam* α; *cæteras vero inter* π & β, ϱ & γ, &c. *reſpective poni*; *ſi vero* h & m *eadem habeant ſigna, tum duas radices æquationis* $hA + mBx = 0$ *inter minimam affirmativam* θ & *minimam negativam* $-i$ *datæ æquationis* $A = 0$ *poni, quarum una affirmativa, altera vero negativa erit: ſi modo detur mutatio ſignorum de* + *in* − *vel* − *in* + *a penultimo ad ultimum datæ æquationis terminum*; *tum affirmativam radicem inter* θ & 0, *negativam vero inter* $-u$ & $-i$ *poni*; *ſin aliter, tum affirmativam radicem inter* θ & u, & *negativam inter* 0 & $-i$ *poni, ubi* u *ſit radix æquationis* $nx^{n-1} - \overline{n-1}\,px^{n-2} +$ &c. $= 0$ *inter minimam affirmativam & minimam negativam datæ æquationis* $x^n - px^{n-1} +$ &c. $= 0$ *radicem poſita, quæ erit negativa vel affirmativa prout detur mutatio ſignorum de* + *in* − *vel* − *in* + *a penultimo ad ultimum datæ æquationis, necne*; *plura conſimilia de hâc re in hoc opere continentur. Radicem quadraticam quantitatis* $a + \sqrt{(b)} + \sqrt{(c)} + \sqrt{(d)} +$ &c., *ſi modo radix exprimi poſſit per quantitates ejuſdem generis, extraxit* Neutonus; *quaſdam æquationes* (2n) *dimenſionum ad æquationes* (n) *dimenſionum reduxit*; *reductiones earundem conſimiles ad* $\frac{n}{2}$ *dimenſiones, ſi modo* $\frac{n}{2}$ *ſit integer numerus, in primâ editione hujuſce operis traduntur*; *ſed in his & conſimilibus reductionibus primum inveſtigandi ſunt denominatores, quos recipere poſſunt quantitates in reductis æquationibus, ita ut præbeant coefficientes integros numeros in datis æquationibus*;

 etiam-

etiamque in eâdem editione adjicitur methodus detegendi, annon data æquatio reſolvi poteſt extractionis radicis quadraticæ, &c. ope: ſequentem quoque dedit Neutonus *regulam de electione terminorum ad calculum ineundum ſcil. ſi duæ incognitæ quantitates* (x & y) *ſimiliter involvantur in datis æquationibus, quòd tum neutrum adhibere convenit, ſed eorum vice tertium* (z) *eligere, qui ſimilem utrique relationem gerit: in primâ editione hujuſce operis primum prolata fuit demonſtratio hujuſce propoſitionis, nempe probando quantitatem* (z) *ſolummodo habere dimidiatum numerum valorum, quos habet quantitas* x *vel* y; *ratiocinatio hæc eſt, ſi valor quantitatis* x *ſit* α, *& ejus correſpondens valor quantitatis* y *ſit* β; *tum quoniam quantitates* (x & y) *ſimiliter involvuntur; ſi* x *ſit* β, y *erit* α; *deinde ſubſtituantur hi valores* α & β, *vel* β & α *reſpective pro* x & y *in quantitate rationali* (z), *in quâ* x & y *ſimiliter involvuntur, & ex utraque ſubſtitutione idem reſultabit valor quantitatis* (z), *& conſequenter* (z) *duos habet valores æquales, unde dimidiatum habet numerum valorum quantitatis* (y); *&c.; ſi vero numerus valorum quantitatum* x *vel* y *ſit impar; tum unus valor quantitatis* (x) *æqualis erit ejus correſpondenti valori quantitatis* (y); *& hinc ſi numerus valorum æquationis, cujus radix eſt* x *vel* y, *ſit m, tum numerus radicum* (z) *æquationis, quæ ſint rationalis functio quantitatum* x & y, *in quâ* x & y *ſimiliter involvuntur, erit* $\frac{m}{2}$ *vel* $\frac{m+1}{2}$: *& ſic de pluribus, &c. In ſecundâ editione magis generale redditur hoc problema; eadem ratiocinatio applicatur ad* (n) *æquationes totidem* (n) *incognitas quantitates habentes, in quibus* (m, r, s, *&c.*) *quantitates ſimiliter involvuntur. Eadem ratiocinatio ſic applicari poteſt ad priorem* Neutoni *regulæ partem, "ſcilicet cum duorum terminorum talis obvenit affinitas, ſive ſimilitudo relationis, ad cæteros terminos quæſtionis, ut oporteret æquationes per omnia ſimiles, ex utrovis adhibito produci tum neutrum adhibere convenit, &c: ſi vero æquationes per omnia ſimiles ex utrovis* (x *vel* y) *producantur; ſit* α *valor quantitatis* x, *&* π *ejus correſpondens valor quantitatis* y, *& in prædictis æquationibus pro* x *ſcribatur* y, *& pro* y *ſcribatur* x; *tum, ſi* α *ſit valor quantitatis* y, *ſequitur* π *eſſe ejus correſpondentem valorem quantitatis* x; *unde patet duos diverſos valores* (α & π) *quantitatum* (x & y) *unum ſolummodo habere valorem* (α + π

vel

vel $x+\alpha$) *quantitatis* ($z=x+y$) *ſibi ipſis reſpondentem; & conſequenter numerum valorum quantitatis* (x *vel* y) *duplum eſſe numerum valorum (exceptis excipiendis) quantitatis* z. *In hâc editione hujuſce operis probatur, ſi modo dentur æquationes, in quibus quantitates* x *&* y *ſimiliter involvuntur, & aſſumatur* $P=z$ *quantitas rationalis, in cujus quadrato* (P^2) *ſimiliter involvuntur* x *&* y, *tum æquationem, cujus radix eſt* z, *habere ejus negativas & affirmativas radices inter ſe æquales, & conſequenter quoad formulam, dimidias habere dimenſiones æquationis, cujus radix eſt* x *vel* y, *&c.* Neutonus *non videtur cognoviſſe dimenſiones æquationis, cujus radix eſt* z, *dimidias eſſe æquationis, cujus radix eſt* x *vel* y; *primus ſubſequentem propoſitionem aſſerui & demonſtravi, ſi modo duæ dentur æquationes, in quibus (*x *&* y, *&c.) ſimiliter involvuntur, æquationem, cujus radix eſt rationalis functio quantitatum* (x *&* y), *in quâ x &* y *ſimiliter involvuntur, dimidias (exceptis excipiendis) ſolummodo habere dimenſiones æquationis, cujus radix eſt* x *vel* y, *&c.: adjicitur etiam in hoc opere ſubſequens propoſitio, ſit* $x=\phi:z$, *ita ut* z *ſit eadem functio quantitatis* (x), *viz.* $z=\phi:x$; *& in quacunque aſſumptâ æquatione* $A=0$ *quæ ſit functio quantitatum* x *&* y *pro* (x) *ſcribatur ejus valor* $\phi:z$, *& in æquatione reſultante pro* z *ſcribatur* (x), *& ſit æquatio reſultans* $B=0$; *tum valores quantitatis* x *in aſſumptâ æquatione* $x=\phi:x$ *erunt etiam valores quantitatis* (x) *in aſſumptis æquationibus* $A=0$ & $B=0$; *etiamque ſit* v *rationalis functio quantitatum* x *&* $\phi:x$, *in quâ* x *&* $\phi:x$ *ſimiliter involvuntur; tum æquatio, (exceptis excipiendis) cujus radix eſt* v, *haud plures habet quam dimidias dimenſiones æquationis, cujus radix eſt* x *vel* $\phi:x$: *plura de his rebus adjiciuntur.*

Fractionem $\frac{1}{x^n - p x^{n-1} + q x^{n-2} - \&c.}$ *in fractiones* $\frac{p}{x-\alpha} + \frac{q}{x-\beta}$ + *&c. primum diſtinxit* Leibnitzius *vel* Bernoullius; Eulerus *invenit* $\frac{\alpha^r}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)\ \&c.} + \frac{\beta^r}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)\ \&c.} + \frac{\gamma^r}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)\ \&c.} + \&c. = 0$, *ſi modo* r *ſit integer numerus minor quam numerus* (n) *literarum* $\alpha, \beta, \gamma, \delta$, *&c.: hæc propoſitio generalior redditur a me in actis philoſ.* Londin. *viz. idem probatur, cum* α^r,

β^r, γ^r, *&c.*

β^r, γ^r, *&c. evadant diverſa rectangula vel contenta earundem dimenſionum e diverſis radicibus ex novis principiis; in hâc editione hujuſce operis ex prædicto theoremate* Bernoullii *vel* Leibnitzii *deducitur valor ſummæ prædictæ; cum* r *ſit quicunque numerus affirmativus vel negativus; unde deduci poteſt ſeries, quæ exprimit prædictam ſummam, cum* r *ſit fractio: reſolutionem æquationis hujuſce formulæ* $x^n - npx^{n-2} + n.\frac{n-3}{2}p^2x^{n-4} - \&c. = A$, *& reductionem æquationis* $x^{2n} + px^{2n-1} \ldots + px + 1$ *ad æquationem* $x^n - ax^{n-1} + \&c. = 0$, *reperiit* Moivræus: *in primâ editione hujuſce operis dantur quatuor radices æquationis* $x^4 + 1 = 0$, *viz.* $\pm\surd(\frac{1}{2}) \pm \surd(-\frac{1}{2})$; *quinque radices æquationis* $x^5 + 1 = 0$, *viz.* -1, $\pm\frac{\surd(1\frac{1}{4}) + \frac{1}{2}}{2} \pm \frac{\surd(-2\frac{1}{2} \pm \surd(1\frac{1}{4}))}{2}$; *ſex radices æquationis* $x^6 + 1 = 0$ *viz.* $\pm\surd(-1)$, $\frac{\pm\surd(3) \pm \surd(-1)}{2}$; *æquatio* $x^{2n} \pm x^{2n-1} + x^{2n-2} \pm x^{2n-3} + x^{2n-4}\ \&c. = 0$ *in quadraticam reducitur æquationis ope* $v^n \mp v^{n-1} - (n-1)v^{n-2} \pm \overline{n-2}\,v^{n-3} + \overline{n-2}.\frac{n-3}{2}v^{n-4} \pm \&c. = 0$; *datur æquatio* $p^n - np^{n-2} + n.\frac{n-3}{2}p^{n-4} - n.\frac{n-4}{2}.\frac{n-5}{3}p^{n-6} + \&c. = (\surd(\frac{1}{4}p^2 - 1) + \frac{1}{2}p)^n + (-\surd(\frac{1}{4}p^2 - 1) + \frac{1}{2}p)^n = \alpha^n + \frac{1}{\alpha^n}\ \&c.$; Bezout *aſſumit* $y = \frac{x+a}{x+b}$, *ſubſtituit hanc quantitatem pro* y *in æquatione* $y^n + h = 0$, *reſultat* $x^n + n\frac{a+hb}{1+h}x^{n-1} + n.\frac{n-1}{2}\frac{a^2+hb^2}{1+h}x^{n-2} + \&c. = 0$; *invenit ſeriem* $x^n = a^{n-1}b \pm a^{n-2}b^2 + na^{n-2}bx + n.\frac{n-3}{2}a^{n-3}bx^2 + n.\frac{n-4}{2}.\frac{n-5}{3}a^{n-4}bx^3 + \&c.$, *cujus reſolutio eſt* $x = \sqrt[n]{(a^{n-1})b} + \sqrt[n]{(a^{n-2})b^2}$; *in actis philoſ.* Londin.: *& hâc editione hujuſce operis dantur æquationes* $x^{2n} - 2b^npx^n - \frac{n}{1.2}.$

2 n $b^{n-1}a^2$

$2n\,b^{n-2}a^2 p\,x^{n-2} - \frac{n.(n^2-1)}{1.2.3.4} \times 2n\,b^{n-4}a^4 p\,x^{n-4} - \&c. = a^{2n}p - b^{2n}p^2$; *etiamque* $x^n - p\,(n\,a^{n-3}b\,x^2 + n.\frac{n-5}{2}a^{n-6}b^2x^4 + \&c.) \pm p^2(n\,a^{\frac{n-3}{2}}b^{\frac{n+1}{2}}x - \&c.) = a^n p + b^n p^3$, *cujus reſolutio eſt* $x = a\sqrt[n]{(p)} + b\sqrt[n]{(p^3)}$; *cum quibuſdam ſimilibus; hæ reſolutiones facile magis generales reddi poſſunt; viz. detegi poſſunt æquationes, quarum radices ſunt* $x = a\sqrt[n]{(p^m)} + b\sqrt[n]{(p^{m+2})}$. *In ſecundâ editione plura de reductionibus recurrentium æquationum adjiciuntur. Æquationem* $x^n - p\,x^{n-1} + \&c. = 0$ *ad æquationem* $x^m - a\,x^{m-1} + \&c. = 0$ *ope æquationis* $n.\frac{n-1}{2}..\frac{n-m+1}{m}$ *dimenſionum deprimi poſſe oſtendit* Le Seur; *(eadem principia ad biquadraticas æquationes prius applicavit* Sanderſon) *atque inde demonſtravit æquationem* $x^{4n+2} + \&c. = 0$ *unum quadraticum diviſorem poſſibilem ad minimum habere; & ex iiſdem ſane principiis demonſtravit* Eulerus *quantitatem* $x^n - p\,x^{n-1} + \&c.$ *conſtare e ſimplicibus vel quadraticis diviſoribus in ſeſe ductis; non niſi poſt edita mea miſcell. i. e. primam hujuſce operis editionem librum domini* Le Seur *videram, nec* Euleri *methodum a meâ tamen multo diſcrepantem; nempe diſtinxi biquadraticam æquationem* $x^2 - 2(a + b\sqrt{(-1)})x - c - d\sqrt{(-1)} = 0$ *in duas rationales quadraticas. In ſecundâ editione hujuſce operis animadvertitur; ſi modo reducatur data æquatio* $x^n - p\,x^{n-1} + q\,x^{n-2} - \&c. = 0$ *dimenſionum* (n) *in æquationem* $x^m - a\,x^{m-1} + b\,x^{m-2} - \&c. = 0$ *dimenſionum* (m); *æquationes, quarum radices ſint vel* a *vel* b *vel* c, *&c., eaſdem habere dimenſiones; & correſpondentes valores incognitarum quantitatum* (a, b, c, d, &c.) *eandem continere irrationalitatem, ni duo vel plures valores unius incognitæ quantitatis ſint inter ſe æquales; &c. In eâdem editione aſſeritur, ſi modo duæ æquationes duas incognitas quantitates* (x & y) *involvant, correſpondentes valores quantitatum* (x & y) *ſemper eandem habere irrationalitatem, ni duo vel plures* (m) *valores unius (e. g.* x*) ſint inter ſe æquales, tum autem eandem irrationalitatem habere quadraticam, &c. æquationem,*

tionem, cujus radix eſt x, *quam habet ejus correſpondens valor quantitatis* (y): *Methodum reducendi duas æquationes* $y^n + ay^{n-1} + by^{n-2} + \&c. = 0$ & $y^m + Ay^{m-1} + By^{m-2} + \&c. = 0$ *in unam, ita ut exterminetur incognita quantitas* y, *docuit* Eulerus *ducendo prædictas duas æquationes in duas quantitates* $Py^{m-1} + a'y^{m-2} + b'y^{m-3} + \&c.$, & $P'y^{n-1} + A'y^{n-2} + B'y^{n-3} + \&c.$, & *rectangula addendo & ita aſſumendo coefficientes* $A, a', b', \&c.$; $P, A', B', \&c.$; *ut evaneſcant ſinguli termini reſultantis æquationis, in quibus invenitur quantitas* (y): *conſimilia etiam applicari poſſunt ad tres vel plures æquationes tres vel plures incognitas quantitates habentes.* Bezout *reduxit duas vel plures æquationes in unam, ita ut incognitæ quantitates exterminentur, ducendo datas æquationes* $Ay^n + (a + bx + cz + \&c.)\,y^{n-1} + (d + ex + fz + gx^2 + hxz + \&c.)\,y^{n-2} + \&c.$, $A'y^m + (a' + b'x + c'z + \&c.)\,y^{m-1} + \&c. = 0$, $A''y^r + (a'' + b''x + c''z + \&c.)\,y^{r-1} + \&c. = 0$, &c. *in quantitates ſubſequentium formularum* $Py^{nmr\&c-n} + (p + qx + rz + \&c.)\,y^{nmr\&c.-n-1} + \&c.$, $P'y^{nmr\&c.-m} + (p' + q'x + r'z + \&c.)\,y^{nmr\&c.-m-1} + \&c.$, $P''y^{nmr\&c.-r} + \&c.$; &c. *reſpective; & rectangula exinde reſultantia addendo; deinde ita aſſumendo coefficientes harum quantitatum, ut nihilo evadant æquales omnes termini, in quibus inveniuntur quantitates exterminandæ: in multis caſibus dedit etiam magis ſimplices formulas multiplicatorum, quæ exterminationem ſæpe magis facilem reddent: & ſubſequentes elegantes propoſitiones detexit; ex principiis* Taylori *incrementorum; etiamque propoſitione facile deducendâ, attamen à me in primâ editione hujuſce operis datâ, viz. quod numerus coefficientium in æquatione* (n) *dimenſionum* $(y^n + (a + bx + cz + \&c.)\,y^{n-1} + \&c.) = 0$, (m) *incognitas quantitates habente, erit* $(n+1) \times \frac{n+2}{2} \times \frac{n+3}{3} \ldots \frac{n+m}{m} = S$; & *ex propoſitione eleganti a ſe inventâ, viz. quod* $N(h)^n$ *denotante numerum terminorum in æquatione* (n) *dimenſionum, quæ continet* (h) *incognitas quantitates* $(x, y, z, v, \&c.)$; *numerus terminorum in æquatione* (n) *dimenſionum diviſibilium vel per* x^p *vel* y^q *vel* z^r *vel* v^s *&c. erit* $= (N(h)^{n-p} + N(h)^{n-q} + N(h)^{n-r} + N(h)^{n-s} + \&c.) - (N(h)^{n-p-q} + N(h)^{n-p-r} + N(h)^{n-p-s} + N(h)^{n-q-s} + \&c.) + (N(h)^{n-p-q-r} +$

N(h)

N $(h)^{n-p-q-s}$ + &c.) — (N $(h)^{n-p-q-r-s}$ + &c.) + &c.: & ſi ſint (h) æquationes (n, m, l, k, &c.) dimenſionum reſpective totidem incognitas quantitates (x, y, z, v, &c.) involventes; & ſint p, q, r, s, &c.; p′, q′, r′, &c.; p″, q″, &c.; maximæ dimenſiones, ad quas aſcendunt incognitæ quantitates x, y, z, v, &c., in reſpectivis æquationibus (n, m, l, k, &c.) dimenſionum; tum æquationem, cujus radix eſt x vel y vel z, &c. haud aſcendere ad plures quam $n \times m \times l \times k \times$ &c. — $(n-p) \times (m-p') \times (l-p'') \times$ &c. — $(n-q) \times (m-q') \times (l-q'') \times$ &c. — $(n-r) \times (m-r') \times$ &c. — &c. = P dimenſiones: ſi vero dimenſiones quantitatum (x & y) ſimul ſumptarum haud ſuperent dimenſiones a, a′, a″, &c. in prædictis æquationibus; tum æquationem, cujus radix eſt x, &c. haud plures habere quam $P + (n-a) \times (m-a') \times$ &c. — $(p+q-a) \times (p'-a') \times$ &c.; hæ propoſitiones forſan facilius e principiis in prob. 41, a me traditis, quæ prius edita fuere in ſecundâ editione hujuſce operis, erui poſſunt. In primâ editione hujuſce operis reducuntur duæ æquationes in unam ita ut incognitæ quantitates exterminentur infinitarum ſerierum ope, quæ methodus in hâc editione ad plures æquationes plures incognitas quantitates habentes applicatur; etiamque datur modus hoc problema reſolvendi, quæ conjungit methodos Crameri & Bezout. Le Grange ope fluxionum & logarithmorum idem problema reſolvit. In primâ editione hujuſce operis primum, quod ſcio, animadvertitur, ſi termini duarum datarum æquationum (n & m) dimenſionum reſpective duas incognitas quantitates (x & y) involventium, in quibus maximæ inveniuntur dimenſiones prædictarum quantitatum, communem habeant diviſorem, æquationem haud aſſurgere ad $n \times m$ dimenſiones; & in eâdem editione etiam adjicitur in proprietatibus curvarum, ſi modo tres ſint æquationes (n, m, r) dimenſionum tres incognitas quantitates (x, y, z) habentes, æquationem, cujus radix eſt x vel y vel z aſcendere poſſe ad $n \times m \times r$ dimenſiones. In ſecundâ editione hujuſce operis, ſi modo dentur (h) æquationes, quarum dimenſiones ſunt reſpective n, m, r, s, &c.; totidem (h) incognitas quantitates (x, y, z, &c.) habentes, docetur methodus e terminis maximarum datarum æquationum dimenſionum inveniendi, annon æquatio, cujus radix eſt x vel y vel z, &c., habet $n \times m \times r \times s \times$ &c. dimenſiones; quæ methodus in hâc editione fuſius explicatur: eâdem editione animadvertitur, ſi modo plures ſint

ſint æquationes quam incognitæ quantitates, & inveniatur communis diviſor unam incognitam quantitatem (x) *ſolummodo involvens, tot eſſe datarum æquationum radices, quot diverſos valores habet incognita quantitas* (x)*; ni duas vel plures radices æquales habeat prædicta incognita quantitas, &c.; etiamque ſi numerus æquationum major ſit quam numerus incognitarum quantitatum, & exterminentur omnes incognitæ quantitates, reſultare æquationes cognitas quantitates ſolummodo involventes, quæ detegunt quibus caſibus poſſunt eſſe æquationes bujuſmodi; &c.: dantur etiam exempla, in quibus continentur tres æquationes totidem incognitas quantitates habentes; attamen una ex iis ad libitum aſſumi poteſt; in his exemplis datur æquatio relationem inter cognitas quantitates exprimens: & in hâc editione aſſeritur, ſi modo dentur* n *æquationes* (m) *incognitas quantitates habentes, & in prædictis contineantur* (r) *independentes æquationes relationem ſolummodo inter cognitas quantitates exhibentes, tum aſſumi poſſe* $n - m + r$ *incognitas quantitates ad libitum; haud poſſum dicere, annon quædam de his in æquationibus dimenſionum unam ſuperantium ab aliis prius tradita fuere:* Bezout *dedit novam methodum inveniendi valorem e ſingulis incognitis quantitatibus in ſimplicibus æquationibus; nova methodus inveniendi prædictos valores etiam in hoc opere traditur.* Eulerus *dedit methodum inveniendi nonnullas æquationes, quarum reſolutiones innoteſcunt, viz. aſſumit* $x = \sqrt[n]{(\alpha)} + \sqrt[n]{(\beta)} + \sqrt[n]{(\gamma)} + \sqrt[n]{(\delta)} + \&c.$ *pro reſolutione, ubi* $\alpha, \beta, \gamma, \delta$, *&c. denotant reſpectivas* (m) *radices æquationis* (m) *dimenſionum detegendæ. In primâ editione bujus operis datur generalis methodus inveniendi æquationem ex ejus reſolutione datâ, quæ ſolummodo exigit exterminationem irrationalium quantitatum ex datâ reſolutione; in quâ duæ dantur novæ generales methodi irrationales quantitates exterminandi, viz. datæ reſolutionis inveniantur ſinguli diverſi valores, contentum ex his omnibus valoribus erit rationalis quantitas quæſita; i. e. ſi irrationalis quantitas ſit* $\sqrt[n]{(A)}$, *tum* $\pi\sqrt[n]{(A)}$, $\varrho\sqrt[n]{(A)}$, $\sigma\sqrt[n]{(A)}$, *&c. erunt diverſi valores irrationalis quantitatis* $\sqrt[n]{(A)}$, *ſi modo* $\pi, \varrho, \sigma, \tau$, *&c. ſint reſpectivæ radices æquationis* $x^n - 1 = 0$, *&c. ut docetur exemplis: ex hâc methodo exterminandi irrationales quantitates a me primum inventâ*

inventâ, & in primâ editione hujuſce operis traditâ, immediate innoteſcet numerus & conſequenter dimenſiones reſultantis æquationis; & exinde judicium de generalitate reſolutionis aſſumptæ formari poteſt; utpote magis generalis videtur eſſe reſolutio, in cujus æquatione datarum dimenſionum plures dantur incognitæ independentes quantitates ad libitum aſſumendæ; & in generali reſolutione æquationis (h) *dimenſionum requiruntur ſaltem* (h) *independentes quantitates hujuſcemodi; unde reſolutio* Euleri *minime eſt generalis, nam in ejus reſolutione ab irrationalibus quantitatibus liberatâ aſcendet incognita quantitas* (x) *ad* n^m *dimenſiones, & ſolummodo continet* (m) *incognitas quantitates: facile etiam conſtat novam irrationalem quantitatem* $\sqrt[k]{\ }$ *in reſolutione contentam augere numerum dimenſionum æquationis, cujus radix eſt* x *in ratione* k : 1; *exinde ex introductione plurium irrationalium apparet multum augeri dimenſiones reſultantis æquationis ſine introductione multarum independentium incognitarum quantitatum; unde excogitavi, ſi in reſolutione aſſumantur diverſæ poteſtates* 1, 2, 3, 4, ... n — 2, n — 1; *tum* (n — 1) *incognitas independentes quantitates in eam introduci poſſe ſine ullâ novâ irrationali quantitate introductâ, & exinde inveni reſolutionem,* "$x = a\sqrt[n]{(p)} + b\sqrt[n]{(p^2)} + c\sqrt[n]{(p^3)} \ldots . + B\sqrt[n]{(p^{n-3})} + C\sqrt[n]{(p^{n-2})} + D\sqrt[n]{(p^{n-1})}$, *in quâ ſi exterminentur irrationales quantitates, reſultat æquatio* $x^n - n\,(aD + bC + cB + \&c.)\,x^{n-2} + \&c. = 0$." *Hæc methodus continetur in primâ editione hujuſce operis; & in initio anni* 1763, *ad clariſſimum* Euler *exemplar mittebatur; &* Menſe Maio *anno* 1770 *exemplaria ſecundæ editionis hujuſce operis ad ſummos mathematicos* D'Alembert, Bezout & Monteclu; Euler, Le Grange, Friſi, *&c. mittebantur; utrum ea receperint, neſcio; recepiſſe autem nunquam literis ad me miſſis agnoverunt, celeberrimo* Friſio *excepto, cujus eximium opus me recepiſſe agnoſco, & cujus amicitia mihi dum vivam cordi erit: poſteaquam hæc a me edita ſunt,* Eulerus & Bezout *eandem prorſus reſolutionem protulere* $x = a\sqrt[n]{(p)} + b\sqrt[n]{(p^2)} + c\sqrt[n]{(p^3)} + \&c.$ *exemplis illuſtratam: domini* Bezout *methodus exterminandi irrationales quantitates a meâ diverſa eſt; ipſe primus applicavit idem principium ad*

reductionem æquationum, quæ vix differt a meâ poſtea in philoſoph. tranſact. editâ: in ſecundâ editione hujuſce operis annis 1768 *&* 1769 *impreſsâ, & in initio anni* 1770 *editâ ex datis* (n) *valoribus* $\alpha, \beta, \gamma, \delta,$ *&c. invenitur reſolutio prædicti generis, viz.* $x = a\sqrt[n]{(p)} + b\sqrt[n]{(p^2)} + c\sqrt[n]{(p^3)} +$ *&c. cujus radices ſunt prædictæ quantitates* $\alpha, \beta, \gamma,$ *&c.; (ibi majoris facilitatis gratiâ ſupponitur* $p = 1$; *quod prius ſuppoſuit* Bezout) *&* $\alpha = \pi a + \pi^2 b + \pi^3 c +$ &c., $\beta = \varrho a + \varrho^2 b + \varrho^3 c +$ &c., $\gamma = \sigma a + \sigma^2 b + \sigma^3 c +$ &c., &c.; *ubi* $\pi, \varrho, \sigma,$ &c. *ſunt radices æquationis* $z^n - 1 = 0$; *hinc facile conſtat* $n \times a = \pi^{n-1} \times \alpha + \varrho^{n-1}\beta + \sigma^{n-1} \times \gamma + \tau^{n-1}\delta +$ &c. $= \pi^{n-1}(\pi a + \pi^2 b + \pi^3 c +$ &c.$) + \varrho^{n-1}(\varrho a + \varrho^2 b + \varrho^3 c +$ &c.$) + \sigma^{n-1}(\sigma a + \sigma^2 b + \sigma^3 c +$ &c.$) +$ &c.; $n \times b$ $\pi^{n-2}\alpha + \varrho^{n-2}\beta + \sigma^{n-2}\gamma +$ &c. $= \pi^{n-2}(\pi a + \pi^2 b + \pi^3 c +$ &c.$) + \varrho^{n-2}(\varrho a + \varrho^2 b + \varrho^3 c +$ &c.$) +$ &c.; $n \times c = \pi^{n-3}\alpha + \varrho^{n-3}\beta + \sigma^{n-3}\gamma +$ &c. $= \pi^{n-3}(\pi a + \pi^2 b + \pi^3 c +$ &c.$) + \varrho^{n-3}(\varrho a + \varrho^2 b + \varrho^3 c +$ &c.$) +$ &c.; &c.; *& exinde* $\alpha = \pi a + \varrho b + \sigma c +$ &c. $= \frac{1}{n}(\pi(\pi^{n-1}\alpha + \varrho^{n-1}\beta +$ &c.$) + \varrho(\pi^{n-2}\alpha + \varrho^{n-2}\beta +$ &c.$) + \sigma(\pi^{n-3}\alpha + \varrho^{n-3}\beta +$ &c.$) +$ &c.$)$: *hinc ex hâc reſolutione a me primum excogitatâ acquiri poteſt reſolutio æquationis, cujus generalis reſolutio ex quâcunque aliâ methodo innoteſcit: plures conſimiles dari poſſunt reſolutiones; e. g.* 1. *Sit reſolutio*

$$x = \frac{a\sqrt[n]{(p)} + b\sqrt[n]{(p^2)} + c\sqrt[n]{(p^3)} \ldots + H\sqrt[n]{(p^{n-1})}}{\alpha\sqrt[n]{(p)} + \beta\sqrt[n]{(p^2)} + \gamma\sqrt[n]{(p^3)} \ldots + \theta\sqrt[n]{(p^{n-1})}} + \frac{a'\sqrt[n]{(p)} + b'\sqrt[n]{(p^2)} + c'\sqrt[n]{(p^3)} \ldots + H'\sqrt[n]{(p^{n-1})}}{\alpha'\sqrt[n]{(p)} + \beta'\sqrt[n]{(p^2)} + \gamma'\sqrt[n]{(p^3)} \ldots \theta'\sqrt[n]{(p^{n-1})}} + \frac{a''\sqrt[n]{(p)} + b''\sqrt[n]{(p^2)} + \text{\&c.}}{\alpha''\sqrt[n]{(p)} + \beta''\sqrt[n]{(p^2)} + \text{\&c.}} + \text{\&c.};$$

æquatio, cujus radix eſt x, (n) *ſolummodo habet dimenſiones, cum exterminentur irrationalis quantitas* $\sqrt[n]{(p)}$ *& ejus poteſtates.* 2. *Sit* $n = m \times r \times$ &c., *tum aſſumatur* $x = A + (a\sqrt[m]{(p)} + b\sqrt[m]{(p^2)} + c\sqrt[m]{(p^3)} +$ &c.$) + \sqrt[r]{(q)}\,(a'\sqrt[m]{(p)} + b'\sqrt{(p^2)}$

$b'\sqrt[m]{(p^2)} \ldots + H'\sqrt[m]{(p^{m-1})} + \sqrt[r]{(q^2)}\,(a''\sqrt[m]{(p)} + b''\sqrt[m]{(p^2)} + \&c.)$ $\ldots \sqrt[r]{(q^{r-1})}\,(a'^{r-1}\sqrt[m]{(p)} + b'^{r-1}\sqrt[m]{(p^2)} + c'^{r-1}\sqrt[m]{(p^3)} + \&c.) +$ $(a'^{r}\sqrt[r]{(q)} + b'^{r}\sqrt[r]{(q^2)} + c'^{r}\sqrt[r]{(q^3)} \ldots + H'^{r}\sqrt[r]{(q^{r-1})})$; *ita reducatur hæc æquatio, ut exterminentur irrationales quantitates* $\sqrt[m]{(p)}$ *&* $\sqrt[n]{(q)}$, *& reſultat æquatio* $(n \times m)$ *dimenſionum libera ab irrationalibus, habens* $(n \times m)$ *incognitas quantitates* A, a, b, c, &c.; a′, b′, c′, &c.; &c.; *ad libitum aſſumendas, cum duabus* p *&* q, *quæ cum reliquis habent connectionem.*

3. *Sit* $x = \dfrac{a\sqrt[m]{(p)} + b\sqrt[m]{(p^2)} + c\sqrt[m]{(p)} \times \sqrt[r]{(q)} + d\sqrt[r]{(q)} + \&c.}{\alpha\sqrt[m]{(p)} + \beta\sqrt[m]{(p^2)} + \gamma\sqrt[m]{(p)}\,\sqrt[r]{(q)} + \delta\sqrt[r]{(q)} + \&c.}$ + &c.; *i. e. generaliter ſit* $n = m \times r \times s \times t \times \&c.$; *&* $(x) =$ *cuicunque rationali algebraicæ functioni quantitatum* $\sqrt[m]{(p)}$, $\sqrt[r]{(q)}$, $\sqrt[s]{(k)}$, $\sqrt[t]{(l)}$, &c.; *in quâ dimenſiones prædictarum incognitarum quantitatum* $\sqrt[m]{(p)}$, $\sqrt[r]{(q)}$, $\sqrt[s]{(k)}$, $\sqrt[t]{(l)}$, *&c. haud ſuperent* $m-1$, $r-1$, $s-1$, $t-1$, &c. *reſpective; tum æquatio, cujus radix eſt* x, *habet ſolummodo* $m \times r \times s \times \&c. = n$ *dimenſiones: omnia hæc etiam applicari poſſunt ad reductiones æquationum, vel ad æquationes plures incognitas quantitates habentes. Ex iiſdem principiis* Euleri *reſolutio prædicta* $x = \sqrt[n]{(\alpha)} + \sqrt[n]{(\beta)} + \sqrt[n]{(\gamma)}$ + &c.; *ubi* $\alpha, \beta, \gamma,$ &c. *ſunt radices æquationis quæſitæ, ſic generalior reddi poteſt: aſſumatur* $x = a\,(\sqrt[n]{(\alpha)} + \sqrt[n]{(\beta)} + \sqrt[n]{(\gamma)} + \&c.) + b\,(\sqrt[n]{(\alpha^2)} + \sqrt[n]{(\beta^2)} + \sqrt[n]{(\gamma^2)} + \&c.) + c\,(\sqrt[n]{(\alpha\beta)} + \sqrt[n]{(\alpha\gamma)} + \sqrt[n]{(\beta\gamma)} + \&c.) + d\,(\sqrt[n]{(\alpha^3)} + \sqrt[n]{(\beta^3)} + \sqrt[n]{(\gamma^3)} + \&c.) + e\,(\sqrt[n]{(\alpha^2\beta)} + \sqrt[n]{(\beta^2\alpha)} + \sqrt[n]{(\alpha^2\gamma)} + \sqrt[n]{(\gamma^2\alpha)} + \sqrt[n]{(\beta^2\gamma)} + \&c.) + f\,(\sqrt[n]{(\alpha\beta\gamma)} + \&c.) + \&c.$, *in cujus reſolutione ſimiliter involvuntur incognitæ quantitates* $\alpha, \beta, \gamma, \delta,$ &c.; *vel magis generaliter aſſumatur* $x =$ *cuicunque rationali algebraicæ functioni incognitarum quantitatum* $\alpha, \beta, \gamma, \delta,$ &c.; *in quâ ſimiliter involvuntur prædictæ quantitates* $(\alpha, \beta, \gamma, \&c.)$; *reducatur*

reducatur hæc reſolutio, ita ut exterminentur irrationales quantitates; reſultat æquatio, in quâ ſimiliter involvuntur incognitæ quantitates (α, β, γ, δ, &c.); *unde inveniri poteſt æquatio, cujus radices ſunt* α, β, γ, δ, &c.; *& æquatio reſultans haud aſcendet ad majores dimenſiones, quam ſi reſolutio fuiſſet* $x = \sqrt[n]{(\alpha)} + \sqrt[n]{(\beta)} + \sqrt[n]{(\gamma)} + \&c.$; *attamen continet plures incognitas quantitates ad libitum aſſumendas, & conſequenter magis generalis erit reſolutio: quod ſi haud ſimiliter in prædictâ functione involvantur quantitates* (α, β, γ, δ, &c.), *idem etiam affirmari poteſt: ſin autem reſolutio quamcunque aliam habeat generalem formulam, tum ex iiſdem principiis deduci poteſt æquatio, cujus fuit reſolutio.*

Dominus Vaudremonde *aſſumit pro radicibus quæſitæ æquationis quantitates eaſdem ac prius traditas; i. e. ſint* $1, r', r''$, &c.; *vel* π, ρ, σ, &c. *radices æquationis* $z^n - 1 = 0$; *cognitum eſt* π, π^2, π^3, &c.; ρ, ρ^2, ρ^3, &c.; &c.; *eſſe radices æquationis* $z^n - 1 = 0$; *unde, ſi modo pro* π, ρ, σ, &c.; π, π^2, π^3, &c.; ρ, ρ^2, ρ^3, &c.; &c.; *ſubſtituantur* $1, r', r''$, &c.; *& pro* α, β, γ, δ, &c. *ſcribantur* a, b, c, d, &c. *reſpective, reſultabit* $a = \frac{1}{n}((a + b + c + d + \&c.) + (a + r'b + r''c + \&c.) + (a + r'^2 b + r''^2 c + \&c.) + \&c.)$, $b = \frac{1}{n}((a + b + c + d + \&c.) + r'^{n-1}(a + r'b + r''c + \&c.) + r'^{n-2}(a + r''^2 b + r''^2 c + \&c.) + \&c.)$, $c = \frac{1}{n}((a + b + c + d + \&c.) + r''^{n-1}(a + r'b + r''c + \&c.) + \&c.)$, &c.; *deinde cum magno acumine pro prædictis quantitatibus* $a + r'b + r''c + \&c.$, $a + r'^2 b + r''^2 c + \&c.$ *in prædictâ reſolutione ſubſtituit* $\sqrt[n]{(a + r'b + r'c + \&c.)^n}$, $\sqrt[n]{(a + r'^2 + r''^2 + \&c.)^n}$, &c.; *tum quantitates reſultantes ex expanſione poteſtatum,* $(a + r'b + r''c + \&c.)^n = a^n + n a^{n-1} r'b + n a^{n-1} r''c + \&c.$, $(a + r'^2 b + r''^2 c + \&c.)^n = a^n + \&c.$; &c.: *hæ quantitates in alias dividantur, in quibus radices* a, b, c, d, &c. *ſimiliter involvantur, (ſi modo poſſibile ſit); cum aliis quarum quædam poteſtates ſimiliter involvunt prædictas radices; cum aliis, quarum poteſtates continent prædictas quantitates ſimiliter involutas, etiamque habent quantitates, quarum poteſtates continent prædictas quantitates ſimiliter involutas; & ſic deinceps in infinitum; tum facile e noſtro theoremate pro inveneniendis aggregatis e ſingulis contentis* $\alpha^a \beta^b \gamma^c \&c. + \alpha^b \beta^a \gamma^c \&c. + \&c.$

ex

ex ſummis poteſtatum datæ æquationis radicum datis anno 1762 *edito, vel ex conſimili regulâ a ſe poſtea editâ (mutatâ vero notatione, &c.) erui poteſt ſumma cujuſcunque algebraicæ functionis radicum* (a, b, c, d, &c.), *in quibus ſimiliter involvuntur. Si modo exprimi poſſit reſolutio per radices datarum quantitatum, tum ſemper per hanc regulam exprimi poteſt, i. e. datur reſolutio.*

Tſchirnhaus *propoſuit æquationem* $y = ax^{n-1} + bx^{n-2} + cx^{n-3} + \&c.$ *pro tollendis* (n) *terminis datæ æquationis* $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0$; *æquationis reſultantis omnes terminos intermedios evaneſcere fingit; i. e. æquationem reſultantem eſſe* $y^n + H = 0$; *& hâc methodo reſolvit cubicam æquationem* $x^3 + px^2 + qx + r = 0$ *tollendo ſecundum & tertium* px^2 *&* qx *datæ æquationis terminos: hâc methodo in ſecundâ editione hujuſce operis primus exterminavi ſecundum & quartum datæ æquationis terminos, & exinde reſolutionem biquadraticæ æquationis detexi. In ſecundâ editione hujuſce operis animadvertitur* 1mo. *reſolutionem*

$$\sqrt[3]{\left(-\tfrac{1}{2}b + \sqrt{\left(\tfrac{1}{4}b^2 + \frac{a^3}{27}\right)}\right)}\,(A) + \sqrt[3]{\left(-\tfrac{1}{2}b - \sqrt{\left(\tfrac{1}{4}b^2 + \frac{a^3}{27}\right)}\right)}\,(B)$$

habere novem valores, & eſſe reſolutionem trium cubicarum $x^3 + ax + b = 0$, $x^3 - \frac{1 + \sqrt{(-3)}}{2}ax + b = 0$ *&* $x^3 - \frac{1 - \sqrt{(-3)}}{2}ax + b = 0$; 2do. *tres radices datæ æquationis* $x^3 + ax + b = 0$ *eſſe* $A + B$, $\alpha A + \frac{1}{\alpha}B$, $\beta A + \frac{1}{\beta}B$; *ubi* 1, α, β *ſunt tres radices æquationis* $x^3 - 1 = 0$: 3tio. *ſi ſit data æquatio* $x^3 + qx - r = 0$, *cujus radices ſint* α, β, $-\alpha - \beta$; *& transformetur hæc æquatio in aliam, cujus radix ſit* z, *ubi* $z^2 - zx = q = -\frac{\alpha^2 + \alpha\beta + \beta^2}{3}$, *ſex valores quantitatis* z *fore* $z = \frac{\alpha}{2} + \frac{(\alpha + 2\beta)\sqrt{(-3)}}{6}$, $\frac{\beta}{2} + \frac{(\beta + 2\alpha)\sqrt{(-3)}}{6}$, $\frac{-\alpha - \beta}{2} + \frac{(\alpha - \beta)\sqrt{(-3)}}{6}$: *conſimilem methodum ſubſecutus eſt* Le Grange, *at pro* $\frac{\alpha}{2} + \frac{\alpha + 2\beta}{6}\sqrt{(-3)} = \frac{3\alpha + (\beta - \gamma)\sqrt{(-3)}}{6}$, *&c.; ſi modo* α,

β & $\gamma = -\alpha - \beta$ *ſint tres radices datæ æquationis, ſcribit* $\frac{1}{3}\left(\alpha - \frac{1+\sqrt{(-3)}}{2}\beta - \frac{1-\sqrt{(-3)}}{2}\gamma\right)$, &c. *ex his valoribus collegi radices æquationis* $z^6 - rz^3 = \frac{q^3}{27}$ *eſſe impoſſibiles, cum radices datæ æquationis ſint poſſibiles; hìc forſan haud indignum eſt obſervatu cubum* $6^3 z^3 = (3\beta + (\beta + 2\alpha)\sqrt{(-3)})^3$ *vel* $(3\alpha + (\alpha + 2\beta)\sqrt{(-3)})^3$ *vel* $(-3\alpha - 3\beta + (\alpha - \beta)\sqrt{(-3)})^3$ *tantummodo habere duos diverſos valores, viz.* $-27(4\beta^2\alpha + 4\alpha^2\beta) + (24\beta^3 + 36\beta^2\alpha - 36\alpha^2\beta - 24\alpha^3)\sqrt{(-3)} = A + B\sqrt{(-3)}$, & $A - B\sqrt{(-3)}$; *rectangulum ſub his duobus valoribus* $(A + B\sqrt{(-3)})(A - B\sqrt{(-3)}) = A^2 + 3B^2$ *erit quantitas, in quâ* α & β, α & $-\alpha-\beta$, β & $-\alpha-\beta$, *etiamque* α, β & $\gamma = -\alpha - \beta$, *ſimiliter involvuntur: in ſecundâ editione hujuſce operis aſſeritur reſolutionem* $x = \sqrt[3]{(a+\sqrt{(c)})} + \frac{h}{\sqrt[3]{(a+\sqrt{(c)})}}$ *eſſe reſolutionem æquationis* $3 \times 2 = 6$ *dimenſionum; etiamque reſolutionem generalem æquationis* (n) *dimenſionum eſſe reſolutionem æquationis* $1.2.3.4..(n-1).n = N$ *dimenſionum; cui adjici poteſt, ſi haud aſſurgat ad majores quam* N *dimenſiones prædicta æquatio, tum quantitates in eâ contentas neceſſario habere denominatores, qui ſunt functiones datæ æquationis coefficientium, &c.: in primâ editione hujuſce operis datur ſubſequens reſolutio biquadraticæ æquationis* $x^4 + 2px^3 = qx^2 + rx + s$, *in quâ haud tollitur ſecundus terminus; addatur ad utramque æquationis partem* $(p^2 + 2n)x^2 + 2pnx + n^2$, & *exinde reſultat æquatio* $(x^2 + px + n)^2 = (p^2 + 2n + q)x^2 + (2pn + r)x + s + n^2$: *fiat* $4(p^2 + 2n + q)(s + n^2) = (2pn + r)^2$; *unde* $8n^3 + 4qn^2 + (8s - 4rp)n + 4qs + 4p^2s - r^2 (A) = 0$ & $x^2 + px + n = \sqrt{(p^2 + 2n + q)}x + \sqrt{(s + n^2)}$: *in ſecundâ editione probatur* $\frac{\alpha\beta + \gamma\delta}{2}$, $\frac{\alpha\gamma + \beta\delta}{2}$, $\frac{\alpha\delta + \beta\gamma}{2}$ *eſſe tres valores quantitatis* n, *ſi modo* α, β, γ & δ *ſint radices datæ æquationis;* & $\frac{\alpha + \beta - \gamma - \delta}{2}$,

$\frac{\gamma+\delta-\alpha-\beta}{2}$; $\frac{\alpha+\gamma-\beta-\delta}{2}$, $\frac{\beta+\delta-\alpha-\gamma}{2}$; $\frac{\alpha+\delta-\beta-\gamma}{2}$, $\frac{\beta+\gamma-\alpha-\gamma}{2}$ *eſſe ſex valores quantitatis* $\sqrt{(p^2+2n+q)}$; & $\frac{\alpha\beta-\gamma\delta}{2}$, $\frac{\gamma\delta-\alpha\beta}{2}$; $\frac{\alpha\gamma-\beta\delta}{2}$, *&c. eſſe ſex radices quantitatis* $\sqrt{(s+n^2)}$; *hæc poſtea edidit* Le Grange: *ſin autem veræ radices cubicæ* ($A=0$) *tantummodo aſſumantur, tum in reſolutione continebuntur (24) radices; quarum duodecim, ut in eâdem editione oſtenditur, ſunt falſæ; duodecim autem veræ; i. e. radices datæ biquadraticæ; unaquæque enim ejus radix ter occurret.*

Le Grange *ſubſtituit radices* π, ρ, σ, &c. *æquationis* $y^n+h=0$ *reſpective pro* y *in aſſumptâ* Tſchirnhauſi *æquatione* $a+bx+cx^2+dx^3+$ &c. $=y$, *ubi* x *ſit radix datæ æquationis* $x^n-px^{n-1}+qx^{n-2}-$ &c. $=0$, *cujus radices ſint* α, β, γ, δ, &c.; & *quoniam valor* (π) *quantitatis* y *æqualiter refert ad omnes valores* (α, β, γ, δ, &c.) *datæ æquationis; ergo vel* $a+b\alpha+c\alpha^2+$ &c. $=\pi$ *vel* $a+b\beta+c\beta^2+$ &c. $=\pi$, &c.; & *ſic de cæteris valoribus* ρ, σ, τ, &c.; *unde facile deducit numerum diverſorum valorum, quos habere poſſunt coefficientes* a, b, c, &c.: *in ſecundâ editione hujuſce operis aſſeritur dimenſiones æquationis, cujus radix eſt* a, *&c. eſſe* $1.2.3.4\ldots(n-1)$; *ſi reſolutio data ſit* $x=a\sqrt[n]{(1)}+b\sqrt[n]{(1^2)}+c\sqrt[n]{(1^3)}+$ &c.; & α, β, γ, δ, &c. *ſint radices datæ æquationis*; & π, ρ, σ, τ, &c. *radices æquationis* $x^n-1=0$, *tum aſſumuntur* $\alpha=\pi a+\pi^2 b+\pi^3 c+$ &c.; $\beta=\rho a+\rho^2 b+\rho^3 c+$ &c.; &c.: *ſed quoniam* π, ρ, σ, &c. *nec ad unam* (α) *nec ad alteram* (β) *datæ æquationis radicem reſtinguantur, ſed æque ad omnes referant; hinc* Le Grange *detegit numerum valorum, quos habere poſſunt ſingulæ coefficientes* a, b, c, &c. *Ille primus aſſerit* α, α^2, α^3, α^4, $\ldots$ $\alpha^n=1$ *eſſe* n *radices diverſas æquationis* $x^n-1=0$, *ſi modo* n *ſit primus numerus* & α *haud ſit unitas: plura conſimilia de prædictis radicibus, &c. in hâc editione hujuſce operis adjunguntur.*

De Moivre *dedit methodum inveniendi ſummam ſeriei æqualis vel omnibus*

nibus alternis, vel omnibus terminis binis, ternis, &c. intervallis a ſe invicem diſtantibus ſeriei $a + bx + cx^2 + \&c.$ *ex diviſione unitatis per multinomium rationale* $p + qx + rx^2 + \&c.$ *Anno* 1757 *remittebam ad regiam ſocietatem primam editionem hujuſce operis, quam legebat* Simpſon; *deinde anno* 1758 *in actis philoſoph.* Lond. *inſeruit chartulam, quæ continet regulam a me in prædictis chartulis datam, viz. ſit* S *data functio quantitatis* x, *quæ expanſa in ſeriem ſecundum dimenſiones quantitatis* x *progredientem, ſit* $a + bx + cx^2 + dx^3 + \&c.$; *in quantitate* S *pro* x *ſcribantur reſpective* $\alpha x, \beta x, \gamma x, \delta x$, &c.; *ubi* $\alpha, \beta, \gamma, \delta$, &c. *ſunt radices æquationis* $x^n - 1 = 0$, *& reſultent* (n) *quantitates* A, B, C, D, &c.; *tum erit* $\frac{A + B + C + D + \&c.}{n}$ = *ſummæ primi, & terminorum, quorum diſtantiæ a primo ſunt reſpective* n, 2n, 3n, &c.; *ſeriei* $a + bx + cx^2 + \&c.$ *attamen ad finem hujuſce chartulæ dicit ſe dediſſe ſolutionem hujuſce problematis exempli methodo paululum diverſâ in alio loco, ubi principia generalis methodi indicantur: ſed affirmo me hanc generalem methodum nunquam prius ut credo excogitatam ad regiam ſocietatem anno* 1757 *miſiſſe*; *&* Simpſon *illas chartulas legiſſe, in quâ contenta fuit prædicta methodus: — In hâc editione hujuſce operis ulterius extenditur hæc regula; ſed hic animadvertendum eſt,* 1mo. *ſi ſeries ſecundum dimenſiones quantitatis* x *progrediens hanc habeat formulam* $a + bx^{\frac{n}{m}} + cx^{\frac{2n}{m}} + dx^{\frac{3n}{m}} + \&c.$, *& requiratur ſumma prædictorum terminorum, & pro* $x^{\frac{n}{m}}$ *ſcribantur reſpective* $\alpha x^{\frac{n}{m}}, \beta x^{\frac{n}{m}}, \gamma x^{\frac{n}{m}}, \delta x^{\frac{n}{m}}$, &c. *& reſultent quantitates* $a + b\alpha x^{\frac{n}{m}} + c\alpha^2 x^{\frac{2n}{m}} + \&c. = A, B, C, \&c.$; *tum fore* $\frac{A + B + C + \&c.}{n}$ *ſummam quæſitam.* 2do. *Si irrationales in datâ quantitate contineantur; tum eadem radix* (π) *quantitatis irrationalis in ſingulis quantitatibus* A, B, C, D, &c. *aſſumenda eſt; e. g. ſit irrationalis quantitas* $\sqrt[r]{(e + fx + gx^2 + \&c.)}$, *& radices æquationis* $x^r - 1 = 0$ *reſpective* π, ρ, σ, &c.; *tum, ſi in quantitate* $A = \pi\sqrt[r]{(e + f\alpha x + g\alpha^2 x^2 + \&c.)}$ *ſcribatur rectangulum* $\pi \times \sqrt[r]{(e + \&c.)}$, *in reliquis quantitatibus* $B = \pi\sqrt[r]{(e + f\beta x + g\beta^2 x^2 + \&c.)}$,

$+$ &c.), $C = \pi\sqrt[r]{(e + f\gamma x + g\gamma^2 x^2 + \&c.)}$, D, E, &c. *ſcribenda etiam ſunt rectangula ex eâdem radice* π *in diverſos valores prædictos* $\sqrt[r]{(e + \&c.)}$. 3^{tio}. *Si quantitas ſub radice* $\sqrt[s]{(a x^h + \&c.)}$ *contenta incipiat a quantitate, in quâ continetur poteſtas vel radix quantitatis* (x); *tum ita reducenda eſt, ut incipiat a quantitate, in quâ haud continetur* (x). *e.g. ſit quantitas* $\sqrt[n]{(a x^h + b x^{h+m} + \&c.)}$, *tum ita ſcribenda eſt* $x^{\frac{h}{n}} \times \sqrt[n]{(a + b x^m + \&c.)}$; *e. 2. ſit quantitas* $\sqrt[n]{(a x^r + b x^{r+s} + \&c. + \sqrt[m]{(a' x^t + b' x^{t+v} + \&c.)} + \&c.)}$, *tum primum reducenda eſt in quantitatem* $x^{\frac{r}{n}} \times \sqrt[n]{(a + b x^s + \&c. + x^{\frac{t}{m} - r} \times \sqrt{(a' + b x^v + \&c.)}\ \&c.)}$; *ſi vero* $\frac{t}{m} - r$ *minor ſit quam* 0; *deinde ita reducenda eſt* $x^{\frac{r}{n}} \times \sqrt[n]{(a + b x^s + \&c. + x^{\frac{t}{m} - r} \times \sqrt[m]{(a' + b' x^v + \&c.)}\ \&c.)} = x^{\frac{t}{nm}} \times \sqrt[n]{(\sqrt[m]{(a' + b' x^v + \&c.)} + x^{r - \frac{t}{m}}(a + b x^v + \&c.)\ \&c.)}$; *& ſic de reducendis quibuſcunque aliis radicibus, ita ut ſemper e quantitate, in quâ haud continetur* (x) *vel ejus poteſtas vel radix, incipiat ſingula quantitas ſub radice contenta: omnia hæc etiam ad quantitates, quæ haud incipiunt a primo ſeriei termino; vel quæ plures quantitates* (x, y, z, &c.) *continent; vel ad æquationes applicari poſſunt.*

In detegendis integris vel rationalibus valoribus incognitarum quantitatum datarum æquationum dantur reſolutiones ſæpe generales, ſæpe autem tantummodo particulares; cum vero ſint particulares, tum particulares valores quarundam incognitarum quantitatum aſſumendi ſunt, qui ex nullâ generali ratiocinatione colligi poſſunt (generalis enim ratiocinatio ad generalem ſolutionem ducit) niſi ex rebus prius cognitis. Plerique caſus in æquationibus omnium dimenſionum (ſimplicibus exceptis) nullam generalem reſolutionem recipient, niſi per continuas approximationes vel tentamina: ſi generalem reſolutionem recipiant, & numerus incognitarum quantitatum major ſit quam numerus æquationum per unitatem; tum generales valores incognitarum quantitatum continentur in formulis primâ editione hujuſce operis pro iis datis, viz. $A z^n + B z^{n-1} + C z^{n-2} + \&c.$, $a z^m + b z^{m-1}$

 $+ c z^{m-2}$

$+ c z^{m-2} + \&c., \&c.;$ *vel* $\frac{A z^{n} + B z^{n-1} + C z^{n-2} + \&c.}{P z^{m} + Q z^{m-1} + R z^{m-2} + \&c.}$, $\frac{a z^{r} + b z^{r-1} + c z^{r-2} + \&c.}{P z^{m} + Q z^{m-1} + \&c.}$, &c.; *vel quæ in præcedentibus continentur* $\frac{A z^{n} + B z^{n-1} + \&c.}{P z^{m} + Q z^{m-1} + \&c.}$, $\frac{a z^{r} + b z^{r-2} + \&c.}{p z^{s} + q z^{s-1} + \&c.}$, &c.; &c., *in quibus* n, m; r, s, &c. *denotant integros numeros, & A, B,* &c.; P, Q, &c.; a, b, &c.; p, q, &c.; *ſunt rationales quantitates; &* z *quantitas ad libitum aſſumenda; ſubſtituantur hæ quantitates pro ſuis valoribus in datis æquationibus, & ex æquatis inter ſe vel nihilo coefficientibus correſpondentium terminorum reſultantium æquationum erui poſſunt prædicti integri numeri & rationales quantitates: ſi vero numerus incognitarum quantitatum major ſit quam numerus æquationum per* m, *tum aſſumendæ ſunt generales rationales functiones rationalium quantitatum* (z, w, u, &c.) *ad libitum aſſumendarum, quæ continent* m *prædictas quantitates* (z, w, u, &c.), *& pro ſingulis incognitis quantitatibus diverſis ſubſtituendæ ſunt quantitates formulæ* $\frac{a z^{n} + (b + c w + d u + \&c.) z^{n-1} + \&c.}{P z^{m} + (Q + R w + S u + \&c.) z^{m-1} + \&c.}$; &c.; *& ſingulæ coefficientes correſpondentium terminorum reſultantium æquationum nihilo æquales reddendæ ſunt; & exinde rationales valores coefficientium* a, b, &c.; P, Q, &c.; &c.; *& integri valores numerorum* n *&* m *acquirendi. Si autem dentur quædam correſpondentes integrales vel rationales valores incognitarum quantitatum datarum, & ex iis requiratur methodus plures correſpondentes valores detegendi; ſubſtituantur in datis æquationibus pro quantitatibus quæſitis generales rationales functiones datarum quantitatum, vel datarum & aliarum ad libitum aſſumendarum; & ex æquationibus reſultantibus per methodos ſupra traditas inveniantur rationales correſpondentes valores quæſiti: ſi vero irrationales valores dati generis incognitarum quantitatum datarum æquationum requirantur, tum pro iis aſſumantur generales functiones irrationalium quantitatum prædicti generis, &c. ſed plerumque hæ generales ſubſtitutiones calculum nimis laborioſum exigunt; ergo ad peculiares ſubſtitutiones faciliores confugiendum eſt. Non poſſum enumerare quam plurimas elegantes ſolutiones problematum hujuſce generis, a præclariſſimis viris*

ris inventas; in hâc præfatione pauca adjicere inventa de diviſoribus & reſiduis numerorum, &c. viſum eſt. Eulerus *dedit ſubſequentem elegantem numerorum proprietatem; viz. ſi* S a *denotet ſummam diviſorum numeri* (a), *tum erit* $Sn = S(n-1) + S(n-2) - S(n-5) - S(n-7) + S(n-12) + S(n-15) -$ &c. *invenit etiam, ſi modo* α, β, γ, δ, &c. *ſint reſidua e diviſione quadratorum* 1, 4, 9, 16, &c. *per primum numerum, &* π *non reſiduum; tum erunt* $\pi\alpha$, $\pi\beta$, $\pi\gamma$, &c. *etiam non reſidua: unde ſubſequentem propoſitionem deduxit, viz. ſit* p *primus numerus; &* a^m *minima poteſtas quantitatis* (a), *quæ per* p *diviſa, relinquat unitatem; & ſit* m *minor quam* $p-1$, *tum haud major erit quam* $\frac{p-1}{2}$, &c. *Conſimilis propoſitio ad quoſdam alios caſus extendi poteſt: in ſecundâ editione hujuſce operis datur methodus inveniendi, annon data ſeries exprimi poteſt per rationalem fractionem, necne;* 2. *adjicitur non poſſe* (n) *primos numeros in arithmeticâ progreſſione, ni eorum communis differentia diviſibilis ſit per* 1 . 2 . 3 . 4 . . n, &c. *Nonnulla præſtantia de diviſoribus numerorum detegendis adjecere* Euler, Le Grange, Beguelin, *&c. In hâc editione aſſeritur numerum reſiduorum e diviſione numerorum* 1^m, 2^m, 3^m, 4^m, &c. *per primum numerum* (p), *eſſe* $\frac{p-1}{r}$; *ſi modo* r *ſit maximus communis diviſor quantitatum* m & $p-1$: *non niſi poſt hoc opus impreſſum fuit, vidi propoſitionem* Euleri, *viz. ſit* $m \times s + 1 = p$, *tum erit numerus reſiduorum prædictorum* $= s$, *quæ quidem propoſitio eſt particularis pars prædictæ a me inventæ: dicitur numerum* 1 . 2 . 3 . 4 . . n *habere ad ſummam ejus diviſorum minorem rationem quam quicunque minor numerus ad ſummam ejus diviſorum: demonſtratur ſummam* $1^r + 2^r + 3^r + 4^r \ldots x^r$ *ſemper diviſibilem eſſe per* x *vel* $x+1$, *ſi* x *vel* $x+1$ *ſit primus numerus major quam* $r+1$; *& in genere aggregatum e ſingulis contentis hujuſce* $\alpha^a \beta^b \gamma^c \delta^d$ &c. + &c.; *ubi* α, β, γ, δ, &c. *reſpective denotant numeros* 1, 2, 3, 4, . . x; *& ſumma datorum indicum* $a+b+c+d+$ &c. *minor eſt quam* $x-1$ *vel* x; *ſemper diviſibile eſſe per prædictum primum* x *vel* $x+1$. *Magis generaliter aſſeri poteſt* $1 . 2 . 3 . 4 \ldots (r+1) \times S$ *ſemper diviſibilem eſſe per* $(2x+1) \times (x+1) \times x$; *ſi* r *ſit par; & per* $x^2 \times (x+1)^2$, *ſi* r

ſit

ſit impar numerus, ex quibus facile deduci poteſt demonſtratio elegantis proprietatis parabolarum, proprietati Archimedis *conicæ parabolæ conſimilis, a me in meis proprietatibus curvarum editæ. Datur etiam demonſtratio elegantis proprietatis primorum numerorum poſtea traditæ, viz. quod numerus* $1 . 2 . 3 . 4 . . (n - 1) + 1$ *ſemper erit diviſibilis per* n, *ſi* n *ſit primus numerus: hujuſce theorematis demonſtrationem elegantem prius invenit* Le Grange.

Poſt tot atque tantorum virorum in re algebraicâ labores, eſt quod miretur forſan aliquis, me alium de eodem argumento edere tractatum, nec chartæ jam parcere perituræ: delictum tamen meum, ſi quod fuerit, hæc erunt, quæ minuant: 1mo. *librum, qui omnia recentiorum ſcriptorum inventa complectatur, extare nullum; deinde, me quoque quædam, quæ luce forſan non indigna erunt, inveniſſe; ſed de iis judicet eruditus lector: officio certe meo fideliter inſervire, ſcientiæque fines, quâ potui, promovere, in votis erat. Quid in hoc libro, de rebus potius quam de verbis laborans, præſtiti, paucis accipe.*

In primo capite 1. *traditur lex, quam obſervat ſeries pro inveniendâ ſummâ* $\alpha^m + \beta^m + \gamma^m + \delta^m +$ &c., *ubi* $\alpha, \beta, \gamma, \delta$, &c. *ſunt radices datæ æquationis* $x^n - p x^{n-1} + q x^{n-2} - r x^{n-3} +$ &c. $= 0$ *in terminis ejus coefficientium* (p, q, r, s, &c.). 2. *Adjicitur ſeries, quæ exprimit aggregatum e pluribus prædictis ſummis.* 3. *Conſectatur ſeries pro aggregato detegendo omnium quantitatum hujuſce generis* $\alpha^a \beta^b \gamma^c$ &c. $+ \alpha^b \gamma^a \beta^c$ &c. + &c. *ex ſummis prædicti generis* $\alpha^m + \beta^m + \gamma^m +$ &c. *datis, cujus ſeriei lex mirâ gaudet ſimplicitate.* 4. *Adjicitur methodus idem aggregatum detegendi ex coefficientibus* (p, q, r, s, &c.) *datis.* 5. *Conſectatur regula pro inveniendo aggregato contentorum e quibuſque* (m) *radicibus (datis earum ſummâ, etiamque ſummâ quadratorum, cuborum, &c. e ſingulis radicibus).* 6. *Docetur methodus inveniendi ſummam quarumcunque functionum algebraicarum rationalium vel irrationalium e radicibus datæ vel datarum æquationum compoſitarum; in ſecundâ & tertia editione hæc methodus fuſius explicatur: unde detegi poteſt aggregatum e ſingulis valoribus infinitæ poteſtatum ſeriei datarum æquationum radicum, ſi modo illa infinita poteſtatum ſeries per algebraicam radicum functionem exprimi poſſit.*

poſſit. 7. *Datâ unâ, duabus vel pluribus æquationibus docetur methodus inveniendi æquationem, cujus radices ſunt quæcunque algebraica radicum datarum æquationum functio ex inveniendâ ſummâ ſingulorum valorum prædictæ functionis; deinde ex inveniendis aggregatis rectangulorum ſub quibuſque duobus, contentorum ſub quibuſque tribus, &c. valoribus prædictæ functionis; quæ erunt reſpective coefficientes ſecundi, tertii, &c. terminorum æquationis, cujus radix eſt functio prædicta. Hæc transformatio, cujus principia bene nota ſunt, nunquam in uſum priuſquam a me recepta fuit.* 8. *Invenitur æquatio, cujus radices ſunt* (m) *poteſtates datæ æquationis radicum, & exinde datur methodus delendi ſurdas quantitates e datâ æquatione; etiamque dantur æquationes, quarum radices ſunt vel ſumma vel differentia duarum datæ æquationis radicum, vel quadrata prædictæ differentiæ; cum quibuſdam aliis.* 9. *Docetur utilis methodus ex aſſumptis æquationibus, quarum radices cognitæ ſunt, eruendi coefficientes quæſitæ æquationis; etiamque ex terminis inferioris methodus detegendi quoſdam terminos ſuperioris æquationis.*

In capite ſecundo 1. *probatur omnem algebraicam æquationem, cujus radices ſunt omnes impoſſibiles, fore aggregatum e diverſis quadratis.* 2. *Demonſtratur omnem æquationem eſſe ſummam vel differentiam duorum poſſibilium quadratorum.* 3. *Sit æquatio* (A) $x^n - px^{n-1} + qx^{n-2} -$ &c. $= 0$, *cujus radices ſunt* α, β, γ, &c., *quarum* α *major ſit quam* β, β *quam* γ, &c.; *ſint* π, ϱ, σ, &c. *radices æquationis* (B) $nx^{n-1} - (n-1)px^{n-2} +$ &c. $= 0$ *inter* α & β, β & γ, &c. *reſpective poſitæ; tum, ſi* h & m *eadem habeant ſigna, erunt radices æquationis* $hA + mB = 0$ *inter* α & π, β & ϱ, &c. *reſpective poſitæ; ſi diverſa habeant ſigna, tum una radix major erit quam* α, *cæteræ vero inter* π & β, ϱ & γ, &c. *reſpective ponentur.* 4. *Duæ radices æquationis* $hA + mB = 0$ *inter minimam affirmativam & minimam negativam datæ æquationis* $A = 0$ *ponentur; quarum una erit affirmativa, altera vero negativa, ſi* h & m *eadem habeant ſigna; &c.* 5. *Datis duabus æquationibus, quarum quædam radices ſunt inter ſe limites; datur methodus alias æquationes deducendi, quarum radices erunt inter ſe limites; &c.* 6. *Ex æquatione, cujus radices ſunt* $\frac{1}{\alpha - \beta}$,

$\frac{1}{\alpha-\gamma}$, $\frac{1}{\beta-\gamma}$, &c., *ubi* α, β, γ, &c. *ſunt radices datæ æquationis, traditur methodus inveniendi limites inter radices datæ æquationis.* 7. *Sint* $\alpha, \beta, \gamma, \delta$, &c. *quæcunque poſſibiles quantitates, tum ſumma e ſingulis valoribus contentorum* $\alpha^{2a}\beta^{2b}\gamma^{2c}$ &c. $+$ $\alpha^{2b}\beta^{2a}\gamma^{2c}$ &c. $+$ &c. *habet ad ſummam e ſingulis valoribus contentorum* $\alpha^{p}\beta^{q}\gamma^{r}$ &c. $+$ $\alpha^{q}\beta^{p}\gamma^{r}$ &c. $+$ &c. *eandem vel majorem rationem quam numerus contentorum in priori ſummâ ad numerum contentorum in poſteriori; ſi modo* 2 a *major ſit quam* p, *&c.* 8. *E ſerie datarum quantitatum, quæ ſemper evadent affirmativæ, cum radices datæ æquationis ſint poſſibiles, dantur principia; e quibus ſæpe dici poteſt, quot radices impoſſibiles habeat data æquatio.* 9. *Adjicitur regula pro inveniendis impoſſibilibus radicibus, quæ ſemper eas deducit, cùm* Neutoni *regula detegat, & ſæpe radices impoſſibiles deteget, cùm prædicta fallat.* 10. *Traditur regula pro inveniendis impoſſibilibus radicibus ex detegendo æquationem, cujus radices ſunt quadrata, &c. datæ æquationis radicum.* 11. *Transformatur data æquatio in alteram, cujus radices ſunt quadrata differentiarum datæ æquationis radicum; & exinde detegi poteſt, annon omnes radices datæ æquationis ſint poſſibiles, necne; &c.* 12. *Sit data æquatio* $x^n - px^{n-1} + qx^{n-2} -$ &c. $= 0$, *aſſumantur* $nx^{n-1} - (n-1)px^{n-2} +$ &c. $= 0$ & $x^n - px^{n-1} +$ &c. $= w$, *deinde inveniatur æquatio cujus radix eſt* w; *ex eâ ſæpe deduci poſſunt impoſſibiles radices.* 13. *Sint duæ æquationes* $Hx^n - ax^{n-1} +$ &c. $= 0$ & $Lx^m - px^{m-1} +$ &c. $= 0$, *quarum radices ſint reſpective* $\alpha, \beta, \gamma, \delta$, &c.; π, ρ, σ, τ, &c.; *aſſumantur* $Hx^n - ax^{n-1} +$ &c. $= 0$ & $Lx^m - px^{m-1} +$ &c. $= w$; *deinde* $Lx^m - px^{m-1} +$ &c. $= 0$ & $Hx^n - ax^{n-1} +$ &c. $= u$, *aſſeritur contentum e ſingulis valoribus quantitatis* w *ad contentum ſub ſingulis valoribus quantitatis* u : : $\mp L^n : H^m$. 14. *Traduntur regulæ, e quibus detegi poteſt numerus poſſibilium, affirmativarum & negativarum radicum, quas habet data æquatio* $x^n + Ax^m + B = 0$. 15. *Adjicitur generalis methodus inveniendi numerum affirmativarum & negativarum radicum in biquadraticâ æquatione contentarum.* 16. *Ex conceſsâ methodo detegendi numerum impoſſibilium cujuſcunque datæ æquationis radicum, in quâ continetur una ſolummodo incognita quantitas, &c. & datâ æquatione plu-*

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res incognitas quantitates habente; datur methodus inveniendi, annon hæ incognitæ quantitates admittant poſſibiles correſpondentes valores. 17. *Ex dato numero poſſibilium, affirmativarum & negativarum radicum in datâ æquatione; conſequitur methodus detegendi numerum impoſſibilium, &c. radicum in æquatione, cujus radices datam habeant aſſignabilem relationem ad radices datæ æquationis: e. g. quadrata differentiarum datæ æquationis radicum, &c.* 18. *Sit data æquatio* $x^3+qx-r=0$, *aſſumatur* $z^2-zx=\frac{q}{3}$; *dantur ſex valores* $\frac{\alpha}{2}+\frac{(\alpha+2\beta)\sqrt{(-3)}}{6}$, *&c. quantitatis* z, *ſi modo* α, β, &c. *ſint radices datæ æquationis; unde colligitur, ſi modo radices datæ æquationis ſint omnes poſſibiles, tum radices æquationis, cujus radix eſt* z, *fore omnes impoſſibiles.* 19. *Sit æquatio* $x^4+qx^2-rx+s=0$; *& inveniatur æquatio, cujus radix* (z) *eſt* $\alpha+\beta$, $\alpha+\gamma$, $\beta+\gamma$, &c.; *ſi modo* α, β, γ, δ *ſint radices datæ æquationis; deinde inveniuntur caſus in quibus poſſibiles, &c. radices reſultantis æquationis, cujus radix eſt* z, *reſpondent poſſibilibus, &c. radicibus datæ æquationis.*

In capite tertio 1. *datur methodus inveniendi illas datæ æquationis radices, quæ datam inter ſe habent relationem.* 2. *Traduntur reſidua e diviſione quantitatis* $x^n-px^{n-1}+qx^{n-2}-$ &c. *per* $(x-a)^m$. 3. *Datur infinita æquatio* $1-\frac{a}{1-r}x+\frac{ra^2}{(1-r)(1-r^2)}-$ &c. $=0$ *quæ habet infinitas poſſibiles radices.* 4. *Sint plures æquationes quam incognitæ quantitates; animadvertitur tot & non plures eſſe valores ſingularum incognitarum quantitatum, quot ſunt valores unius incognitæ quantitatis* (x), *ni duo valores quantitatis* (x) *ſint inter ſe æquales, &c.* 5. *Docetur methodus inveniendi, annon datæ irrationales quantitates poſſibiles habeant valores.* 5. *Æquatio* $x^{2n}\pm x^{2n-1}+$ &c. $=0$ *reducitur in æquationem* $x^2+vx+1=0$ *ope æquationis* $v^n\mp v^{n-1}-(n-1)v^{n-2}+$ &c. $=0$, *unde primum detegitur quatuor radices æquationis* $x^4+1=0$ *eſſe* $\mp\sqrt{(\frac{1}{2})}\mp\sqrt{(-\frac{1}{2})}$, *quinque radices æquationis* $x^5+1=0$, *& ſex radices æquationis* $x^6+1=0$. 7. *Quædam etiam nova de recurrentibus ſeriebus adjiciuntur.* 8. *Transformantur irrationales quantitates in alias, quæ datis irrationalibus æquales ſunt, & tamen ſub diverſâ irrationalitatis formulâ latent.*

latent. 9. *Adjiciuntur quædam de radicibus æquationis* $x^n \pm 1 = 0$, *etiamque de correſpondentibus radicibus* $a + \surd(-b^2) + a - \surd(-b^2)$. 10. *Inveniuntur correſpondentes impoſſibiles radices datæ reſolutionis.* 11. *Si modo data æquatio* $x^n - px^{n-1} + \&c. = 0$ *deprimatur in æquationem* $x^m - ax^{m-1} + bx^{m-2} - \&c. = 0$; *docetur æquationem, cujus radix eſt* a *vel* b *vel* c, *&c., eaſdem habere dimenſiones;* & *datâ unâ coefficiente, tum ex eâ per methodum communes diviſores detegendi, deduci poſſe correſpondentem valorem e ſingulis reliquis, ni duo vel plures valores datæ coefficientis ſint inter ſe æquales; etiamque omnes coefficientes eandem habere irrationalitatem.* 12. *Ex radicibus æquationis* $P^6 + 2qP^4 + (q^2 - 4s)P^2 - r^2 = 0$ *datis, deteguntur radices æquationis* $x^4 + qx^2 - rx + s = 0$. 13. *Biquadratica æquatio hujuſce formulæ* $x^2 - 2(a + b\surd(-1))x - c - d\surd(-1) = 0$ *diſtinguitur in duas æquationes poſſibiles quantitates ſolummodo habentes.* 14. *Traditur reſolutio biquadraticæ æquationis* $x^4 + 2px^3 = qx^2 + rx + s$ *ex addendo* $(p^2 + 2n)x^2 + 2pnx + n^2$ *utrique æquationis parti,* & *ſupponendo* $4(s + n^2) \times (p^2 + 2n + q) = (2pn + r)^2$; *etiamque inveniuntur radices quantitatis* (n) *reſpective* $\frac{\alpha\beta + \gamma\delta}{2}$, $\frac{\alpha\gamma + \beta\delta}{2}$, &c.; *quantitatis* $\surd(q + p^2 + 2n)$ *reſpective* $\frac{\alpha + \beta - \gamma - \delta}{2}$, $\frac{\alpha + \gamma - \beta - \delta}{2}$, &c.; *quantitatis* $\surd(s + n^2)$ *reſpective* $\frac{\alpha\beta - \gamma\delta}{2}$, $\frac{\alpha\gamma - \beta\delta}{2}$, &c.; &c. 15. *Oſtenditur etiam methodus, per quam irrepſerunt radices in reſolutionem, quæ in datâ æquatione haud inveniuntur.* 16. *Per conſimilem methodum datur reſolutio quarundam æquationum quinque* & *ſex dimenſionum; in hâc reſolutione occurrunt tres independentes æquationes tres incognitas quantitates habentes; attamen, quod primum a me obſervatum fuit, una incognita quantitas ad libitum aſſumi poteſt, quoniam datur una æquatio relationem inter cognitas quantitates ſolummodo exprimens.* 17. *Aſſumitur æquatio pro reſolutione datæ æquationis; ſed in aſſumptâ æquatione cavendum eſt, ne ex æquatis datæ* & *reſultantis æquationis terminis ullæ oriantur æquationes ex aliis pendentes, quarum hîc dantur exempla.* 18. *Datur reſolutio biquadraticæ*

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æquationis delendo ſecundum & quartum terminum: hâc methodo reducatur data æquatio $x^n - p x^{n-1} + q x^{n-2} - \&c. = 0$ *ad æquationem* $v^n + Q = 0$, *tum a me primum docetur reducentem æquationem exigere* $1.2.3..n-1$ *dimenſiones.* 19. *Traditur methodus inveniendi æquationes, quas deprimere liceat.* 20. *Datâ quantitate utcunque irrationali, invenitur altera, quæ in datam multiplicata, rationalem facit quantitatem; exhinc datâ æquatione irrationales quantitates involvente, inveniuntur dimenſiones æquationis prædictæ a ſurdis liberatæ.* 21. *Obſervatum eſt, quod in multiplicatione irrationalium quantitatum facile eſt in errores induci, ni reſpectus habeatur ad diverſas radices; e. g. ducatur* $\sqrt{(-a^2)}$ *in* $\sqrt{(-a^2)}$, *productum vel erit* $= -a^2$ *vel* $= \sqrt{(+a^4)} = +a^2$ *vel* $-a^2$; *ſed* $+a^2$ *haud eſt vera radix: ſi neceſſe eſt ut reſpectus habeatur ad diverſa ſigna & radices in rebus mathematicis, multo magis in phyſicis; in rebus mathematicis natura vix alium ſaltum agnoſcit, niſi ex poſſibilitate per æqualitatem ad impoſſibilitatem; etiamque in paribus poteſtatibus revertit; in imparibus progreditur: in phyſicis in paribus poteſtatibus natura admittit ſaltum, reſpectus enim phyſice habendus eſt ad directionem, in quâ vires agunt; ſi vero par ſit poteſtas, mathematice eadem erit poteſtas quantitatis datæ, utrum ſit negativa vel affirmativa, i. e. phyſice utrum agat in hâc vel ejus oppoſitâ directione; ſi vero impar ſit poteſtas, tum diverſa erunt ſigna, i. e. mathematicè apparent oppoſitæ vires, prout vires oppoſitis directionibus agunt.* 22. *Quantitate datâ, quæ ſimplicium terminorum juxta dimenſiones cujuſlibet literæ* x *progredientium ſeriem exprimit; quantitas æqualis terminis, quorum diſtantiæ a ſe invicem ſint reſpective* n, 2 n, 3 n, &c. *acquiritur.* 23. *Dantur formulæ quantitatum, ex quibus in ſeſe ductis orientur quæcunque datæ quantitates.* 24. *Oſtenditur* $x = \sqrt[3]{(a + \sqrt{(b)})} + \sqrt[3]{(a - \sqrt{(b)})}$ *eſſe reſolutionem æquationis novem dimenſionum. i. e. trium diverſarum cubicarum, &c.* 25. *Invenitur aggregatum e ſingulis quantitatibus hujuſce generis* $\alpha^a \beta^b \gamma^c$ &c., *ubi* α, β, γ, &c. *ſunt radices æquationis* $x^n - 1 = 0$. 26. *Adjiciuntur novæ methodi reſolvendi cubicas & biquadraticas æquationes; etiamque reſolutiones æquationum* $x^{2n} - 2 b^n p x^n - \frac{p}{1.2} \times 2 n b^{n-1} a^2 x^{n-1} -$

k &c.

&c. $= a^{2n} p - b^{2n} p^2$, &c. 27. *Traditur reſolutio æquationum, viz.* $x = H + a\sqrt[n]{(p)} + b\sqrt[n]{(p^2)} \ldots \sqrt[n]{(p^{n-1})}$; *& ex datis radicibus datur reſolutio hujuſce generis, quæ continet prædictas radices.* 28. *Sint* $\alpha, \beta, \gamma, \delta$, &c. $(n-1)$ *radices æquationis* $v^{n-1} - pv^{n-2} +$ &c. $= 0$; *inveniuntur dimenſiones, ad quas aſſurgit æquatio, cujus radix eſt* $x = \sqrt[n]{(\alpha)} + \sqrt[n]{(\beta)} + \sqrt[n]{(\gamma)} +$ &c. 29. *Ex omni generali reſolutione æquationum ſuperiorum dimenſionum deducuntur reſolutiones omnium æquationum inferiorum dimenſionum.* 30. *Demonſtratur generalem reſolutionem æquationis* (n) *dimenſionum ſaltem eſſe reſolutionem æquationis* $1.2.3\ldots(n-1) \times n$ *dimenſionum.* 31. *In inveſtigandis approximationibus ad radices datæ æquationis, probatur approximationem inventam non pendere e ratione, quam quantitas aſſumpta pro valore habet ad verum valorem, ſed ex approximatione ad unum valorem propiori quam ad reliquos.*

In capite quarto 1. *datur methodus (quantum ſcio) nova reducendi* n *ſimplices æquationes ad* $n-1$, &c. 2. *Deteguntur radices incognitarum quantitatum, quæ irrepunt in æquationes reſultantes ex vulgari methodo reducendi duas vel plures æquationes in unam, ita ut omnes præter unam incognitæ quantitates exterminentur.* 3. *Sint duæ æquationes (*n *&* m *dimenſionum) duas incognitas quantitates (*x *&* y*) habentes*; *oſtenditur æquationem, cujus radix eſt* x *vel* y, *non aſcendere ad* $n \times m$ *dimenſiones*; *ſi termini duarum æquationum, quorum dimenſiones ſunt maximæ, communem habeant diviſorem.* 4. *Sint tres, &c. æquationes, quarum dimenſiones ſunt reſpective* n, m, o, &c.; *tres, &c. incognitas quantitates (*x, y, z*) involventes, oſtenduntur caſus, in quibus prædicta æquatio haud aſcendet ad* $n \times m \times o \times l \times$ &c. *dimenſiones.* 5. *Demonſtrantur termini datarum æquationum*; *e quibus pendet ſumma radicum, quadratorum e ſingulis radicibus, &c.* 6. *Datur methodus transformandi duas vel plures* (h) *æquationes in unam, ita ut (*$h-1$*) incognitæ quantitates (*x, y, z, &c.*) exterminentur vel per ſubſtitutionem finitarum quantitatum vel infinitarum quantitatum ſerierum pro ſuis valoribus in datâ æquatione.* 7. *Sint* (h) *æquationes, quarum dimenſiones ſunt* n, m, l, k, &c. *habentes* (h) *incognitas quantitates (*x, y, z, &c.*), & ſint* a, b, c, d, &c. *maximæ dimenſiones,*

ad

ad quas aſſurgunt termini, in quibus continentur $(h-1)$ incognitæ quantitates (x, z, v, &c.) probatur dimenſiones, ad quas aſcendere poteſt æquatio, cujus radix eſt y, non plures eſſe quam maximum ſubſequens contentum $n \times a \times b \times c \times$ &c., $m \times a \times c \times d \times$ &c., &c. 8. Datur nova methodus tranſformandi duas vel plures æquationes duas vel plures incognitas quantitates habentes in unam, ita ut incognitæ quantitates exterminentur, ex ſubſtitutione & multiplicatione in aſſumptas quantitates conjunctim. 9. Ex datis æquationibus deducuntur aliæ æquationes, quarum incognitarum quantitatum radices eædem erunt ac radices incognitarum quantitatum datarum æquationum. 10. Datur etiam methodus detegendi, annon omnes radices (m) æquationum ſint etiam omnes radices (m) aliarum datarum æquationum. 11. Dantur principia, e quibus ſæpe dici poteſt numerus radicum incognitæ quantitatis in prioribus æquationibus contentarum, quas non habent poſteriores æquationes. 12. Ex datâ algebraicâ quantitate in nonnullis caſibus invenitur ejus valor. 13. Datâ æquatione (n) dimenſionum unam vel plures (h) incognitas quantitates (x, y, z, &c.) habente, invenitur conſtitutio ejus coefficientium; & ex datis $\left((n+1) \cdot \frac{n+2}{2} \cdots \frac{n+h}{h} - 1\right)$ correſpondentibus valoribus incognitarum quantitatum (x, y, z, &c.) ſequitur conſtitutio quæſita. 14. Æquatio n dimenſionum m incognitas quantitates involvens $(n+1) \cdot \frac{n+2}{2} \cdots \frac{n+m}{m} - 1$ incognitas coefficientes ad libitum aſſumendas habere poteſt. 15. Sint n æquationes m incognitas quantitates habentes, & ſi ita reduci poſſint æquationes, ut reſultent (r) æquationes independentes cognitas quantitates ſolummodo involventes; tum aſſeritur in prædictis æquationibus aſſumi poſſe $(m-n+r)$ incognitas quantitates ad libitum. 16. Sint duæ vel plures æquationes duas vel plures (x, y, z, &c.) incognitas quantitates habentes, quarum radices ſint α, β, γ, &c.; π, ϱ, σ, &c.; $\varkappa, \lambda, \mu$, &c.; invenitur aggregatum e ſingulis valoribus quantitatum hujuſce generis $\alpha^m \pi^2 \varkappa^l$ &c. 19. Traditur regula, ex quâ nonnunquam deduci poſſunt limites, inter quos conſiſtant poſſibiles datarum æquationum radices. 18. Datâ quantitate duas vel plures (x, y, z, &c.) incognitas quantitates involvente, in ſeriem ſecundum

dum dimenſiones quantitatum (x, y, z, &c.) *reducatur, methodus hìc docetur inveniendi ſummam e ſingulis terminis hujuſce generis* $A y^{pm+d} z^{p'm+d'}$ &c. 19. *Iiſdem datis traditur methodus inveniendi illas datæ æquationis radices, quæ datam inter ſe habeant relationem.* 20. *Conſequitur methodus inveniendi numerum correſpondentium radicum incognitarum quantitatum, quæ nihilo evadunt æquales.* 21. *Sint duæ vel plures æquationes totidem incognitas quantitates* (x, y, z, &c.) *habentes, aſſeritur tot habere impoſſibiles radices quantitatem* y, *quot habet quantitas* (z) *vel* (x), *&c. ni duo vel plures valores harum quantitatum inter ſe æquales inveniantur.* 22. *Datis duabus vel pluribus æquationibus totidem incognitas quantitates habentibus, quæ diviſores recipiunt, datæ æquationes in alias minores dimenſiones habentes deprimuntur; exhinc facile erui poſſunt dimenſiones æquationis, cujus radix eſt incognita quantitas in duabus vel pluribus æquationibus duas vel plures incognitas quantitates involventibus contenta.* 25. *Inveniuntur duæ vel plures æquationes duas vel plures incognitas quantitates habentes, quæ reſolvi poſſunt:* 2. *datur maxime generalis reſolutio adhuc cognita duarum vel plurium æquationum duas vel plures incognitas quantitates habentium; & conſimili methodo ex principiis in capite tertio traditis deduci poteſt maxime generalis reductio prædictarum æquationum adhuc cognita.* 24. *Dantur reſolutiones quarundam æquationum.* 25. *Sint tres vel plures æquationes, in quibus dimenſiones terminorum ſunt reſpective* n, n — m, n — 2 m, &c.; n′, n′ — m, n′ — 2 m, &c.; &c.; *aſſeritur æquationem, cujus radix eſt incognita quantitas* (x) *in prædictis æquationibus contenta, habere formulam* $x^{\alpha} + P x^{\alpha - m} + Q x^{\alpha - 2m} +$ &c. $= 0$: &c. 26. *Sint duæ vel plures æquationes incognitas quantitates* (x & y) *ſimiliter involventes; probatur æquationem, cujus radix eſt rationalis functio, in quâ* x & y, *vel in cujus quadrato quantitates* x & y *ſimiliter involvuntur, dimidias ſolummodo habere dimenſiones, vel formulam dimidiarum dimenſionum æquationis, cujus radix eſt* x *vel* y; *quorum caſuum multa adjiciuntur exempla.* 27. *Dantur conſimiles propoſitiones de pluribus quantitatibus ſimiliter inter ſe involutis.* 28. *Sit æquatio* $A = 0$, *quæ continet duas cognitas quantitates* x & y, *in eâ pro* x *ſcribatur* $\frac{1}{x}$, *&*

reſultet:

reſultet æquatio B = o, *tum ex duabus æquationibus* A = o & B = o *deducatur æquatio, cujus radix eſt* x; *aſſeritur æquationem ſic deductam habere duas radices quantitatis* (x) + 1 & — 1; *omnes vero reliquas radices contineri in recurrente æquatione: multæ conſimiles propoſitiones etiam adjiciuntur.* 29. *Sint quantitates* A & B *functiones literarum* x & y; *datur methodus inveniendi, utrum* A *ſit functio quantitatis* B, *necne.* 30. *Sit æquatio* $x^n - p\,x^{n-1} + q\,x^{n-2} -$ &c. = o, *etiamque* $p' = P$, $pP - q' = Q$, $pQ - qP + r' = R$, &c. *ſint* a, b, c, d, &c. *ſumma datæ æquationis radicum, quadratorum e ſingulis radicibus, &c. tum aſſeritur* $a^{\cdot} = P$, $\frac{1}{2}b^{\cdot} = Q$, $\frac{1}{3}c^{\cdot} = R$, &c.; *ſi modo in iis pro* $p^{\cdot}$, $q^{\cdot}$, &c. *ſcribantur* p', q', &c. 31. *Probatur rationem quantitatum inventarum per methodum vulgarem approximationis in duabus vel pluribus æquationibus duas vel plures quantitates incognitas habentibus ad veras radices pendere ex hoc, nempe quo magis ad unam radicem quam ad reliquas ſit approximatio aſſumpta: idem principium etiam applicari poteſt, cum magis appropinquet quantitas aſſumpta ad duas vel tres, &c. radices quam ad reliquas.* 32. *Docetur methodus inveniendi approximationes ad impoſſibiles radices prædictarum æquationum.* 33. *Traditur etiam methodus ex incrementis coefficientium detegendi incrementa diverſarum radicum; & exhinc probatur omnem æquationem vel poſſibilem vel impoſſibilem hujuſce formulæ* $\alpha + \beta\sqrt{(-1)}$ *radicem habere.*

In capite quinto, 1. *continetur methodus inveniendi quadraticos, &c. datæ æquationis diviſores e* Carteſii *principiis, & quædam dantur methodi probandi nullas rationales radices in datâ æquatione contineri.* 2. *Datis duabus vel pluribus æquationibus duas vel plures incognitas quantitates* (x, y, z, &c.) *habentibus; demonſtratur correſpondentes valores omnium harum quantitatum ſemper eandem habere irrationalitatem, ni duo vel plures valores incognitæ quantitatis ſint inter ſe æquales.* 3. *Traditur methodus inveniendi integrales correſpondentes valores incognitarum quantitatum in duabus vel pluribus æquationibus contentarum.* 4. *Datâ æquatione, cujus radix eſt* x, *datur methodus detegendi, annon ea reſolvi poteſt extractionum radicum quadraticarum, cubicarum, &c. ope.* 5. *Datis quibuſcunque rationalibus & integralibus functionibus quantitatum* x, y, z, &c.; *invenitur quantitas in quâ univerſi communes diviſores datarum quantitatum continentur.*

 nentur.

nentur. 6. Probatur numeros $(a+b)^n$ & $(a-b)^n$, &c. ubi a & b ſunt inter ſe primi, nullum habere communem diviſorem, niſi 2^m. 7. Traditur methodus detegendi, annon æquatio (4n), &c. dimenſionum reduci poteſt in æquationem (n) dimenſionum radicum quadraticarum extractionum ope. 8. Probatur radicem $(\surd(e)+\surd(f))^{\frac{1}{2n+1}}$ non recipere quantitatem hujuſce formulæ $\sqrt[2n+1]{(g)}$, ſi $(e-f)^{\frac{1}{2n+1}}$ ſit rationalis quantitas; etiamque adjiciuntur novæ regulæ inveniendi ejus poſſibiles & impoſſibiles radices; explicantur regulæ Schooteni & Neutoni; & oſtenditur, ut in nonnullis caſibus neceſſe ſit datam binomialem ducere in 2^n. 9. Conſimiles regulæ adjiciuntur de extrahendis radicibus n poteſtatis duarum vel plurium irrationalium quantitatum hujuſce formulæ $\sqrt[m]{(e)}+\sqrt[m]{(f)}+$ &c. quæ facile etiam reddi poſſunt magis generales. 10. Datur methodus inveniendi, annon data rationalis quantitas ex infinitis partibus conflata exprimi poteſt per rationalem functionem aſſumptæ quantitatis. 11. Datur lex, quam obſervat ſeries in terminis quotientium expreſſa, quæ proxime datam fractionem ita exhibeant; ut nulla fractio, cujus termini ipſis haud ſunt majores, propius accedat; &c. 12. Datâ ſimplici æquatione duas vel tres vel plures incognitas quantitates habente, inveniuntur earum integri correſpondentes valores in terminis quantitatis ad libitum aſſumendæ, &c. 13. Reducuntur æquationes duas vel plures incognitas quantitates habentes in unam, ita ut exterminentur incognitæ quantitates; &, ſi incognitæ quantitates reſultantium æquationum ſint rationales, exterminatæ etiam ſint rationales. 14. Docetur methodus inveniendi æquationem incognitas quantitates (x, y, &c.) involventem, quarum correſpondentes valores ſunt $x = Az^n + Bz^{n-1} +$ &c., $y = az^m + bz^{m-1} +$ &c. &c. 15. Datâ æquatione duas vel plures incognitas quantitates involvente, traditur methodus detegendi, annon hæ quantitates exprimi poſſunt per rationalem functionem quantitatis (z). 16. Pauca adjiciuntur de generalibus integralibus valoribus incognitarum quantitatum in datis æquationibus contentarum; & datur methodus ex dato correſpondente valore quarundam quantitatum detegendi infinitos valores incognitarum quantitatum in datâ vel datis æquationibus contentarum. 17. Conſequitur methodus transformandi numeros per vulgarem notationem deſignatos

ſignatos in diverſam notationem. 18. *Sit* $a^x b^z c^y d^v$ &c. $=$ A, *adjungitur contenta* $(x+1)(z+1)(y+1)$ &c., & $\frac{a^{x+1}-1}{a-1} \times \frac{b^{z+1}-1}{b-1} \times$ &c. *eſſe minima, cum* $a^{x+1} = b^{z+1} =$ &c. 19. *Affirmatur ſummam diviſorum numeri* $1.2.3.4..x$ *habere ad numerum ipſum majorem rationem quam ſummam diviſorum cujuſcunque minoris numeri ad numerum ipſum.* 20. *Pauca adjiciuntur de integris numeris, e.g. omnem numerum componi e novem cubis*; &c. *ſed non demonſtrantur.* 21. *Sit* $a^2 + r b^2 = N$, & N^{2m+1} *vel* N^{2m+2} *componi poſſe* $(m+1)$ *diverſis modis e quantitatibus ejuſdem formulæ ab aliis affirmatum eſt*; *hìc dantur & demonſtrantur prædictæ compoſitiones.* 22. *Conſimiles propoſitiones hìc affirmantur & demonſtrantur de* $(a^2 + ab + b^2)^n$; *e.g. omnes numeros, qui continentur in formulâ* $a^2 + ab + b^2$, *etiam in formulâ* $a^2 - ab + b^2$ *contentos eſſe.* 23. *Docetur numerum non reſiduorum e diviſione numerorum hujuſce formulæ* $A^h B^k C^l$ &c. *per primum numerum vel eſſe* o *vel* n *vel* 2 n, &c.; *ubi* n *denotat numerum reſiduorum.* 24. *Sit* r *maximus communis diviſor numerorum* m & $n-1$, *adjicitur* $\frac{n-1}{r}$ *numerum eſſe reſiduorum e diviſione numerorum* $1^m, 2^m, 3^m$, &c. *per primum numerum* n. 25. *Ex non reſiduis demonſtratur nullos correſpondentes integros valores incognitarum quantitatum in nonnullis æquationibus contineri.* 26. *Si* 0, 1, 2, 3, &c. *ſint ultimæ figuræ radicis*; *tum* 0, 1, 2, 3, &c. *etiam erunt ultimæ figuræ poteſtatum* 1, 5, 9, 13, &c., &c., &c. *unde facile probari poteſt quatuor quadratos non dari in arithmeticâ progreſſione.* 27. *Probatur non dari tres, quinque, ſeptem, &c. primos numeros in arithmeticâ progreſſione, quorum communis differentia non diviſibilis eſt per* $1.2.3$, $1.2.3.4.5$, &c. 28. *Sit* $1^r + 2^r + 3^r + 4^r +$ &c. $=$ S, *tum aſſeritur* $1.2.3..r(r+1)$ *diviſibilem eſſe per* $x^2 \times (x+1)^2$, *ſi* r *ſit impar*; *per* $x(x+1)(2x+1)$ *ſi* r *ſit par numerus*; *etiamque ſi* $\alpha, \beta, \gamma, \delta$, &c. *ſint primi numeri &* $a+b+c+d+$&c. *minor ſit quam* x; *tum* $\alpha^a \beta^b \gamma^c \delta^d$ &c. $+$ &c. *eſſe diviſibilem per* $x+1$, *ſi* $x+1$ *ſit primus numerus: exhinc etiam datur demonſtratio theorematis maxime elegantis primorum numerorum ab amiciſſimo & in omni matheſeos parte verſatiſſimo viro* Joanne Wilſon *armigero*

armigero inventi atque mibi communicati. 29. *Traditur theor. de æqualitate numerorum ex Noſtris Proprietatibus curvarum aſſumptum.* 30. *Datis* m *correſpondentibus duarum vel trium, &c. incognitarum quantitatum valoribus reſpective, docetur methodus inveniendi functionem quantitatum* (x, y, z, &c.), *quæ habet prædictos correſpondentes valores.* 31. *Oſtenduntur caſus, in quibus exponentialis æquatio* $a^x = a$ *vel* $x^x = a$ *habet unam affirmativam vel negativam, vel duas poſſibiles radices. In addendis quædam adjiciuntur propoſitiones e præcedentibus facile deducendæ. Hæc ſunt inventa, quæ in hoc libro continentur.*

Multa ſparſim inſuper adjeci, & ut labores hoſce meos benigne excipiant, & defectus in calculis adeo intricatis rerum algebraicarum cultores ſuppleant, ex animo oro & exopto.

Jul. 12, 1782.

CORRIGENDA: Pag. 2. lin. antepen. pro incognitis, lege cognitis; p. 5. l. 10. pro prædictæ Schooteni & Neutoni regulæ, lege prædicta Neutoni regula; l. 17. dele vel per Schootenii vel; p. 7. l. 13. pro P, lege P'; p. 14. l. 17. pro dicitur, lege dicit; l. 25. pro 1, lege a; p. 16. l. 17. pro ; lege , ubi α & $\frac{1}{\alpha}$ ſunt radices æquationis $x^2 - px + 1 = 0$; l. penult. pro b & b^2, lege b) & b^2); p. 18. l. 7. pro A & P, lege P & P'; p. 24. l. 23. pro r'^2 & r''^2, lege $r'^2 b$ & $r''^2 c$.

MEDITA-

MEDITATIONES ALGEBRAICÆ.

C A P U T I.

DE *methodo inveniendi æquationem, cujus radices ſint quæcunque algebraica radicum datæ æquationis vel datarum æquationum functio.*

P R O B. I.

Sint $\alpha, \beta, \gamma, \delta, \varepsilon$, &c. *Radices datæ æquationis*
$x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - tx^{n-5} + vx^{n-6} - wx^{n-7} + zx^{n-8}$ &c. $= o$.
inveuire ſummam $\alpha^m + \beta^m + \gamma^m + \delta^m + \varepsilon^m +$ &c.

Summa quæſita erit

$$p^m - mqp^{m-2} + mrp^{m-3} \left.\begin{array}{l} -m \times s \\ +m\times\frac{m-3}{2}\times q^2 \end{array}\right) p^{m-4} \left.\begin{array}{l} +m \times t \\ -m\times\overline{m-4}\times qr \end{array}\right) p^{m-5} \left.\begin{array}{l} -m \times v \\ +m\times\overline{m-5}\times qs \\ +m\times\frac{m-5}{2}\times r^2 \\ -m\times\frac{m-5}{2}\times\frac{m-4}{3}\times q^3 \end{array}\right\} p^{m-6}$$

$$\left.\begin{array}{l} +m \times w \\ -m\times\overline{m-6}\times qt \\ -m\times\overline{m-6}\times rs \\ +m\times\overline{m-6}\times\frac{m-5}{2}\times q^2r \end{array}\right) p^{m-7} \left.\begin{array}{l} -m \times z \\ +m\times\overline{m-7}\times qv \\ +m\times\overline{m-7}\times rt \\ +m\times\frac{m-7}{2}\times s^2 \\ -m\times\overline{m-7}\times\frac{m-6}{2}\times q^2s \\ -m\times\overline{m-7}\times\frac{m-6}{2}\times qr^2 \\ +m\times\frac{m-7}{2}\times\frac{m-6}{3}\times\frac{m-5}{4}\times q^4 \end{array}\right) p^{m-8} + \&c.$$

Duplex eſt lex, quam obſervat hæc ſeries; altera, quæ ad contenta literalia; altera, quæ ad coefficientes ſeu uncias ſpectat.

Contentorum literalium lex eſt; ut datæ æquationis coefficientium exponentes in ſingulis contentis eidem poteſtati quantitatis *(p)* annectendis eandem conficiant ſummam, quæ quidem ſumma eſt u, ſi modo p^{m-n} ſit poteſtas quantitatis p, cui annectuntur contenta hæc literalia.

Per datæ æquationis coefficientium exponentes intelligo numeros deſignantes earum diſtantias a primo æquationis termino; ſc. exponens coefficientis q, eſt 2; coefficientis r, 3; coefficientis s, 4; coeffi-

cientis t, 5γ & ſic deinceps; quo fit, ut exponens cujuſlibet poteſtatis coefficientis q, qualis q^{μ}, ſit 2μ; exponens cujuſlibet poteſtatis coefficientis r, qualis r^{ν}, ſit 3ν; exponens cujuſlibet poteſtatis coefficientis s, qualis s^{π}, ſit 4π; exponens cujuſlibet poteſtatis coefficientis t, qualis t^{ϱ}, ſit 5ϱ; &c.

Cuncta inſuper contenta literalia, quæ ſummam u poſſunt conficere, poteſtati p^{m-u} ſunt ſemper adjungenda.

Coefficientium ſeu unciarum hæc eſt lex. Cujuſcunque literalis contenti $p^{m-u}\ q^{\mu}\ r^{\nu}\ s^{\pi}\ t^{\varrho}\ v^{\sigma}\ w^{\tau}$ &c. Coefficiens erit fractio, cujus numerator eſt

$$m\times\overline{m-u+1}\times\overline{m-u+2}\times\overline{m-u+3}\times\overline{m-u+4}\times\overline{m-u+5}\times\overline{m-u+6}\times\&c.$$

in quo tot factores erunt, quot unitates in ſummâ indicum μ, ν, π, ϱ, σ, τ, &c. denominator vero erit contentum ortum e multiplicatione ſerierum

$1\times2\times3\times4\times$ &c. $1\times2\times3\times4\times$ &c. $1\times2\times3\times4\times$ &c. $1\times2\times3\times4\times$ &c.

quarum prima continet tot terminos, quot ſunt unitates in μ; ſecunda, quot ſunt in ν; tertia, quot ſunt in π; quarta, quot ſunt in ϱ; &c. i. e. coefficiens literalis contenti $p^{m-u}\ q^{\mu}\ r^{\nu}\ s^{\pi}\ t^{\varrho}$ &c. erit

$$m\times\frac{\overline{m-u+1}\times\overline{m-u+2}\times\overline{m-u+3}\times\overline{m-u+4}\ldots\overline{m-u+\mu+\nu+\pi+\varrho+\&c.-1}}{1.2.3..\mu\times1.2.3..\nu\times1.2.3..\pi\times1.2.3..\varrho\times\&c.}$$

Coefficientis literalis contenti $p^{m-u}\ q^{\mu}\ r^{\nu}\ s^{\pi}\ t^{\varrho}$ &c. ſignum erit affirmativum, vel negativum, (modo u ſit par numerus) prout ſumma $\mu+\nu+\pi+\varrho+$ &c. ſit par vel impar: quod ſi u ſit impar numerus, ſignum erit affirmativum, vel negativum, prout impar vel par fuerit ſumma $\mu+\nu+\pi+\varrho+$&c. i. e. erit affirmativum vel negativum prout ſumma $u+\mu+\nu+\pi+\varrho+$ &c. ſit par vel impar numerus.

DEMONSTRATIO.

* " Sit $p=a$, $p^2-2q=b$, $pb-qa+3r=c$, $pc-qb+ra-4s=d$, " $pd-qc+rb-sa+5t=e$, $pe-qd+rc-sb+ta-6v=f$, & ſic " deinceps. Et erit a ſumma radicum, b ſumma quadratorum ex " ſingulis radicibus, c ſumma cuborum, d ſumma quadrato-quadra- " torum, e ſumma quadrato-cuborum, f ſumma cubo-cuborum, et " ſic in reliquis."

Ex

* Newt. Arith. Univ. pag. 251.

Ex hâc regulâ ſatis facile colligitur coefficientem literalis contenti $p^{m-u} q^{\mu} r^{\nu} s^{\pi} t^{\rho}$ &c. in ſumma $\alpha^m + \beta^m + \gamma^m + \delta^m +$ &c. inveniendâ, æqualem eſſe aggregato coefficientium literalium contentorum $p^{m-u-1} q^{\mu} r^{\nu} s^{\pi} t^{\rho}$ &c., $p^{m-u} q^{\mu-1} r^{\nu} s^{\pi} t^{\rho}$ &c., $p^{m-u} q^{\mu} r^{\nu-1} s^{\pi} t^{\rho}$ &c., $p^{m-u} q^{\mu} r^{\nu} s^{\pi-1} t^{\rho}$ &c., $p^{m-u} q^{\mu} r^{\nu} s^{\pi} t^{\rho-1}$ &c., in ſummis reſpectivis $\alpha^{m-1}+\beta^{m-1}+\gamma^{m-1}+\delta^{m-1}+$ &c., $\alpha^{m-2}+\beta^{m-2}+\gamma^{m-2}+\delta^{m-2}+$ &c., $\alpha^{m-3}+\beta^{m-3}+\gamma^{m-3}+\delta^{m-3}+$ &c., $\alpha^{m-4}+\beta^{m-4}+\gamma^{m-4}+\delta^{m-4}+$ &c., $\alpha^{m-5}+\beta^{m-5}+\gamma^{m-5}+\delta^{m-5}+$ &c., &c.

Hæ autem coefficientes reſpective erunt

$$\overline{m-1}\times\overline{m-u}\times\frac{\overline{m-u+1}\times\overline{m-u+2}\times\overline{m-u+3}\ldots\ldots\times\overline{m-u+\mu+\nu+\pi+\rho+\&c.-2}}{1.2.3\ldots\mu\times1.2.3\ldots\nu\times1.2.3\ldots\pi\times1.2.3\ldots\rho\times\&c.}$$

$$\mu\times\overline{m-2}\times\frac{\overline{m-u+1}\times\overline{m-u+2}\times\overline{m-u+3}\ldots\ldots\times\overline{m-u+\mu+\nu+\pi+\rho+\&c.-2}}{1\times2.3\ldots\mu\times1.2.3\ldots\nu\times1.2.3\ldots\pi\times1.2.3\ldots\rho\times\&c.}$$

$$\nu\times\overline{m-3}\times\frac{\overline{m-u+1}\times\overline{m-u+2}\times\overline{m-u+3}\ldots\ldots\times\overline{m-u+\mu+\nu+\pi+\rho+\&c.-2}}{1.2.3\ldots\mu\times1.2.3\ldots\nu\times1.2.3\ldots\pi\times1.2.3\ldots\rho\times\&c.}$$

$$\pi\times\overline{m-4}\times\frac{\overline{m-u+1}\times\overline{m-u+2}\times\overline{m-u+3}\times\ldots\ldots\overline{m-u+\mu+\nu+\pi+\rho+\&c.-2}}{1.2.3\ldots\mu\times1.2.3\ldots\nu\times1.2.3\ldots\pi\times1.2.3\ldots\rho\times\&c.}$$

$$\rho\times\overline{m-5}\times\frac{\overline{m-u+1}\times\overline{m-u+2}\times\overline{m-u+3}\times\ldots\ldots\overline{m-u+\mu+\nu+\pi+\rho+\&c.-2}}{1.2.3\ldots\mu\times1.2.3\ldots\nu\times1.2.3\ldots\pi\times1.2.3\ldots\rho\times\&c.}$$

&c. &c.

Et aggregatum harum unciarum erit

$$\left.\begin{array}{lll} m^2 & -u\,m & +u \\ & -1 & -2\mu \\ & +\mu & -3\nu \\ & +\nu & -4\pi \\ & +\pi & -5\rho \\ & +\rho & \&c. \\ & +\&c. & \end{array}\right\}\times\frac{\overline{m-u+1}\times\overline{m-u+2}\times\overline{m-u+3}\times\ldots\ldots\overline{m-u+\mu+\nu+\pi+\rho+\&c.-2}}{1.2.3\ldots\mu\times1.2.3\ldots\nu\times1.2.3\ldots\pi\times1.2.3\ldots\rho\times\&c.}$$

In primo hujus aggregati factore pro u ſubſtituatur ejus valor $(2\mu + 3\nu + 4\pi + 5\rho +$ &c.); & fiet aggregatum

$$m\times\overline{m-u+\mu+\nu+\pi+\rho+\&c-1}\times\frac{\overline{m-u+1}\times\overline{m-u+2}\times\overline{m-u+3}\times\ldots\overline{m-u+\mu+\nu+\pi+\rho+\&c.-2}}{1.2.3\ldots\mu\times1.2.3\ldots\nu\times1.2.3\ldots\pi\times1.2.3\ldots\rho\times\&c.}$$

Hoc vero aggregatum eſt coefficiens ſeu uncia, quæ in datâ ſerie pro unciâ literalis contenti $p^{m-1} q^{\mu} r^{\nu} s^{\pi} t^{\rho}$ &c. in ſummâ $\alpha^m+\beta^m+\gamma^m+\delta^m$ + &c. ſubſtituitur; conſequenter, cum ſeries prædicta inferiores ſummas $\alpha^{m-1}+\beta^{m-1}+\gamma^{m-1}+\delta^{m-1}+$&c., $\alpha^{m-2}+\beta^{m-2}+\gamma^{m-2}+\delta^{m-2}+$&c., $\alpha^{m-3}+\beta^{m-3}+\gamma^{m-3}+\delta^{m-3}+$&c., $\alpha^{m-4}+\beta^{m-4}+\gamma^{m-4}+\delta^{m-4}+$&c.,&c., recte oſtendat, ſummam quoque proxime ſuperiorem $\alpha^m+\beta^m+\gamma^m+\delta^m+$ &c. vere exprimet.

Ex. Sit æquatio $x^4 - 2x^3 - 7x^2 + 20x - 12 = 0$. In hoc exemplo $p, q, r, s, t,$ ſunt reſpective $2, -7, -20, -12, 0$. Quare ſumma quadratorum e ſingulis radicibus $(p^2 - 2q)$ eſt $2^2 + 2 \times 7 = 18$; cuborum ſumma $(p^3 - 3pq + 3r)$ $2^3 + 3 \times 2 \times 7 - 3 \times 20 = -10$; quadrato-quadratorum ſumma $\left(\begin{array}{l} p^4 - 4pq^2 + 4rp - 4s \\ \qquad\qquad\qquad\quad + 2q^2 \end{array}\right)$ $2^4 + 4 \times 7 \times 2^2 - 4 \times 20 \times 2 + 4 \times 12 + 2 \times 7^2 = 114$. Quadrato-cuborum ſumma $\left(p^5 - 5qp^3 + 5rp^2 \begin{array}{l} -5s \\ +5q^2 \end{array}\right\} p \begin{array}{l} +5t \\ -5qr \end{array}\Big)$ $2^5 + 5 \times 7 \times 2^3 - 5 \times 20 \times 2^2 + 5 \times 12 \times 2 + 5 \times 7^2 \times 2 - 5 \times 7 \times 20 = -178$.

Non abs re forſan erit obſervare ſeriem

$$\left(p^m - mqp^{m-2} + mrp^{m-3} \begin{array}{l} -m\,s \\ +m \times \frac{m-3}{2} q^2 \end{array}\right\} p^{m-4} \begin{array}{l} +m \times t \\ -m \times \overline{m-4}\; qr \end{array}\Big\} p^{m-5} - \&c.\Big)$$

in infinitum excurrentem, in alteram ſeriem ſui ſimilem

$$\left(p^n - nqp^{n-2} + nrp^{n-3} \begin{array}{l} -n\,s \\ +n \times \frac{n-3}{2} q^2 \end{array}\right\} p^{n-4} \begin{array}{l} +n\,t \\ -n \times \overline{n-4}\,qr \end{array}\Big\} p^{n-5} - \&c.$$

in infinitum) ductam, ſeriem infinitam

$$\left(p^{n+m} - \overline{n+m}\,q\,p^{n+m-2} + \overline{n+m}\,r\,p^{n+m-3} \begin{array}{l} -\overline{n+m} \times s \\ +\overline{n+m} \times \frac{n+m-3}{2}\, q^2 \end{array}\right\}$$

$$p^{n+m-4} \begin{array}{l} +\overline{n+m} \times t \\ -\overline{n+m} \times \overline{n+m-4}\; qr \end{array}\Big\} p^{n+m-5} + \&c.\Big)$$

eandem prorſus legem quoad terminos ſervantem efficere.

PROB. II.

Invenire aggregatum e pluribus hujuſcemodi ſeriebus confectum.

Addantur ſimul ſerierum termini; tum conficietur problema.

Ex. Sit a ſumma radicum, b ſumma quadratorum e ſingulis radicibus, c ſumma cuborum, d ſumma quadrato-quadratorum, e ſumma quadrato-cuborum, f ſumma cubo-cuborum, &c.

Invenire ſummam (m) terminorum ab initio hujuſce ſeriei

$$a + b + c + d + e + \&c.\ M.$$

Per problema præcedens

$a =$

$a = p$

$b = p^2 - 2q$

$c = p^3 - 3qp + 3r$

$d = p^4 - 4qp^2 + 4rp - 4s + 2q^2$

$e = p^5 - 5qp^3 + 5rp^2 - 5sp + 5q^2p + 5t - 5qr$

&c. &c. &c. &c. &c. &c.

$M = p^m - mqp^{m-2} + mrp^{m-3} - msp^{m-4} + m \times \frac{m-3}{2} q^2 p^{m-4} + mtp^{m-5} - m \times \overline{m-4}\, qrp^{m-5} - \&c.$

ſumma autem primæ columnæ

$$p + p^2 + p^3 + p^4 + p^5 \ldots\ldots p^m = \frac{p^{m+1} - p}{p - 1}$$

ſecundæ vero

$$2q + 3qp + 4qp^2 + 5qp^3 \ldots mqp^{m-2} = \frac{2 - p - \overline{m+1}\, p^{m-1} + mp^m}{\overline{1-p}^2} \times q$$

tertiæ

$$3r + 4rp + 5rp^2 \ldots\ldots mrp^{m-3} = \frac{3 - 2p - \overline{m+1}\, p^{m-2} + mp^{m-1}}{\overline{1-p}^2} \times r$$

quartæ

$$4s + 5sp + 6sp^2 \ldots\ldots msp^{m-4} = \frac{4 - 3p - \overline{m+1}\, p^{m-3} + mp^{m-2}}{\overline{1-p}^2} \times s$$

quintæ

$$2q^2 + 5q^2p + 9q^2p^2 \ldots m \times \frac{m-3}{2} q^2 p^{m-4} = \frac{4 - 2p - \overline{m+1} \times \overline{m-2}\, p^{m-3} + \overline{2m^2 - 4m - 4}\, p^{m-2} - m \times \overline{m-3}\, p^{m-1}}{2 \times \overline{1-p}^3} \times q^2$$

&c. = &c.

& conſequenter ſumma quæſita erit

$$\frac{p^{m+1} - p}{p - 1} - \frac{2 - p - \overline{m+1}\, p^{m-1} + mp^m}{\overline{1-p}^2} \times q + \frac{3 - 2p - \overline{m+1}\, p^{m-2} + mp^{m-1}}{\overline{1-p}^2} \times r$$

$$- \frac{4 - 3p - \overline{m+1}\, p^{m-3} + mp^{m-2}}{\overline{1-p}^2} \times s + \frac{5 - 4p - \overline{m+1}\, p^{m-4} + mp^{m-3}}{\overline{1-p}^2} \times t$$

$$+ \frac{4 - 2p - \overline{m+1} \times \overline{m-2} \times p^{m-3} + \overline{2m^2 - 4m - 4}\, p^{m-2} - m \times \overline{m-3}\, p^{m-1}}{2 \times \overline{1-p}^3} \times q^2$$

$$- \frac{5 - 3p - \overline{m+1} \times \overline{m-3}\, p^{m-4} + \overline{2m^2 - 6m - 5}\, p^{m-3} - m \times \overline{m-4}\, p^{m-2}}{\overline{1-p}^3} \times qr$$

&c.

Eadem

Eadem erunt literalia hujuſce ac præcedentis ſeriei contenta, abjectâ quantitate p, ejuſque poteſtatibus.

Regula, quam obſervant coefficientes, hæc eſt; ſcribatur $u = 2\mu + 3\nu + 4\pi + 5\varrho + 6\sigma +$ &c. & $l = 1 + \mu + \nu + \pi + \varrho + \sigma +$ &c. coefficiens literalis contenti $q^{\mu}\, r^{\nu}\, s^{\pi}\, t^{\varrho}\, v^{\sigma}$ &c. erit æqualis contento $1 \times 2 \times 3 \times 4 \times 5 \ldots \overline{l-2}$ in fractionem multiplicato, cujus denominator eſt

$$\overline{1-p}^{\,l} \times 1 \times 2 \times 3 \times \ldots \mu \times 1.2.3 \ldots \nu \times 1 \times 2.3 \ldots \pi \times 1.2.3 \ldots \varrho \times 1.2.3 \ldots \sigma \times \text{\&c.}$$

Numerator vero

$$u - \overline{u-l+1} \times p$$

$$-\overline{m+1} \times \frac{m-u+2}{1} \times \frac{m-u+3}{2} \times \frac{m-u+4}{3} \times \frac{m-u+5}{4} \times \frac{m-u+6}{5} \ldots\ldots \frac{m-u+l-1}{l-2}\, p^{m-n+1}$$

$$\left.\begin{array}{l} +u \times \frac{m-u+1}{1} \times \qquad \frac{m-u+3}{1} \times \frac{m-u+4}{2} \times \frac{m-u+5}{3} \times \frac{m-u+6}{4} \times \frac{m-u+7}{5} \times \frac{m-u+8}{6} \ldots \frac{m-u+l}{l-2} \\ -\overline{u-l+1} \times \frac{m-u}{1} \times \qquad \frac{m-u+2}{1} \times \frac{m-u+3}{2} \times \frac{m-u+4}{3} \times \frac{m-u+5}{4} \times \frac{m-u+6}{5} \times \frac{m-u+7}{6} \ldots \frac{m-u+l-1}{l-2} \end{array}\right\} p^{m-n+2}$$

$$\left.\begin{array}{l} -u \times \frac{m-u+1}{1} \times \frac{m-u+2}{2} \times \qquad \frac{m-u+4}{1} \times \frac{m-u+5}{2} \times \frac{m-u+6}{3} \times \frac{m-u+7}{4} \times \frac{m-u+8}{5} \ldots \frac{m-u+l}{l-3} \\ +\overline{u-l+1} \times \frac{m-u}{1} \times \frac{m-u+1}{2} \times \qquad \frac{m-u+3}{1} \times \frac{m-u+4}{2} \times \frac{m-u+5}{3} \times \frac{m-u+6}{4} \times \frac{m-u+7}{5} \ldots \frac{m-u+l-1}{l-3} \end{array}\right\} p^{m-n+3}$$

$$\left.\begin{array}{l} +u \times \frac{m-u+1}{1} \times \frac{m-u+2}{2} \times \frac{m-u+3}{3} \times \qquad \frac{m-u+5}{1} \times \frac{m-u+6}{2} \times \frac{m-u+7}{3} \times \frac{m-u+8}{4} \ldots \frac{m-u+l}{l-4} \\ -\overline{u-l+1} \times \frac{m-u}{1} \times \frac{m-u+1}{2} \times \frac{m-u+2}{3} \times \qquad \frac{m-u+4}{1} \times \frac{m-u+5}{2} \times \frac{m-u+6}{3} \times \frac{m-u+7}{4} \ldots \frac{m-u+l-1}{l-4} \end{array}\right\} p^{m-n+4}$$

$$\left.\begin{array}{l} -u \times \frac{m-u+1}{1} \times \frac{m-u+2}{2} \times \frac{m-u+3}{3} \times \frac{m-u+4}{4} \times \qquad \frac{m-u+6}{1} \times \frac{m-u+7}{2} \times \frac{m-u+8}{3} \ldots \frac{m-u+l}{l-5} \\ +\overline{u-l+1} \times \frac{m-u}{1} \times \frac{m-u+1}{2} \times \frac{m-u+2}{3} \times \frac{m-u+3}{4} \times \qquad \frac{m-u+5}{1} \times \frac{m-u+6}{2} \times \frac{m-u+7}{3} \ldots \frac{m-u+l-1}{l-5} \end{array}\right\} p^{m-n+5}$$

$$\left.\begin{array}{l} +u \times \frac{m-u+1}{1} \times \frac{m-u+2}{2} \times \frac{m-u+3}{3} \times \frac{m-u+4}{4} \times \frac{m-u+5}{5} \times \qquad \frac{m-u+7}{1} \times \frac{m-u+8}{2} \ldots \frac{m-u+l}{l-6} \\ -\overline{u-l+1} \times \frac{m-u}{1} \times \frac{m-u+1}{2} \times \frac{m-u+2}{3} \times \frac{m-u+3}{4} \times \frac{m-u+4}{5} \times \qquad \frac{m-u+6}{1} \times \frac{m-u+7}{2} \ldots \frac{m-u+l-1}{l-6} \end{array}\right\} p^{m-n+6}$$

$$\left.\begin{array}{l} -u \times \frac{m-u+1}{1} \times \frac{m-u+2}{2} \times \frac{m-u+3}{3} \times \frac{m-u+4}{4} \times \frac{m-u+5}{5} \times \frac{m-u+6}{6} \times \qquad \frac{m-u+8}{1} \ldots \frac{m-u+l}{l-7} \\ +\overline{u-l+1} \times \frac{m-u}{1} \times \frac{m-u+1}{2} \times \frac{m-u+2}{3} \times \frac{m-u+3}{4} \times \frac{m-u+4}{5} \times \frac{m-u+5}{6} \times \qquad \frac{m-u+7}{1} \ldots \frac{m-u+l-1}{l-7} \end{array}\right\} p^{m-n+7}$$

& ſic deinceps ad penultimum terminum $p^{n-u+l-1}$; ultimus autem terminus erit

$\pm m$

$$\pm m\times\frac{m-u+1}{1}\times\frac{m-u+2}{2}\times\frac{m-u+3}{3}\times\frac{m-u+4}{4}\ldots\frac{m-u+l-2}{l-2}\times p^{m-u+l}.$$

E methodo inveniendi ſummam $(m-u+1)$ terminorum ſeriei emergentis per diviſionem quantitatis $u-\overline{u-l+1}\times p$ per $\overline{1-p}^{l}$, facile erui & demonſtrari poteſt hoc problema.

Ex. Datâ æquatione $x^4-3x^3+x^2+3x-2=0$. invenire ſummam $a+b+c+d$.

Pro p, q, r, s, m, ſcribantur reſpective $3, 1, -3, -2, 4$. Erit ſumma

$$a+b+c+d=\frac{p-p^5}{1-p}(120)-q\times\frac{2-p-5p^3+4p^4}{\overline{1-p}^2}(-47)+r\times\frac{3-2p-5p^2+4p^3}{\overline{1-p}^2}(-45)-s\times\frac{4-3p-5p+4p^2}{\overline{1-p}^2}(8)=38.$$

$$+q^2\times\frac{4-2p-10p+12p^2-4p^3}{2\times\overline{1-p}^3}(2).$$

Cor. Si $p=1$, tum $\frac{p-p^{m+1}}{1-p}$ fit (m), & $q\times\frac{2-p-\overline{m+1}\,p^{m-1}+mp^m}{\overline{1-p}^2}$ fit $\left(q\times\overline{m+2}\times\frac{m-1}{2}\right)$, &c.

Summa quæſita eſt

$$m-\overline{m+2}\times\frac{m-1}{2}q+\overline{m+3}\times\frac{m-2}{2}r-\overline{m+4}\times\frac{m-3}{2}\times s+\&c.$$

$$+\tfrac{1}{2}\times\overline{2m+4}\times\frac{m-3}{2}\times\frac{m-2}{3}q^2-\&c.$$

coefficiens cujuſlibet literalis contenti $q^{\mu}\,r^{\nu}\,s^{\pi}\,t^{\varrho}$ &c. hujuſce ſeriei æqualis eſt fractioni

$$\overline{\overline{l-1}\times m+u}\times\frac{\overline{m-u+1}\times\overline{m-u+2}\times\overline{m-u+3}\ldots\ldots\overline{m-u+l-1}}{l\times\overline{l-1}\times1.2.3\ldots\mu\times1.2.3\ldots\nu\times1.2.3\ldots\pi\times1.2.3\ldots\varrho\times\&c.}$$

DEMONSTRATIO.

Sit S ſumma coefficientium terminorum $q^{\mu}\,r^{\nu}\,s^{\pi}\,t^{\varrho}$ &c., $pq^{\mu}\,r^{\nu}\,s^{\pi}\,t^{\varrho}$ &c., $p^2q^{\mu}\,r^{\nu}\,s^{\pi}\,t^{\varrho}$ &c., ... $p^{m-u}q^{\mu}\,r^{\nu}\,s^{\pi}\,t^{\varrho}$ &c., &c., in ſummis reſpectivis poteſtatum

teſtatum $u, u+1, u+2, \ldots\ldots m$ ex ſingulis radicibus; tum erit

$$S + \overline{m+1} \times \frac{\overline{m-u+2} \times \overline{m-u+3} \times \overline{m-u+4} \times \ldots \times \overline{m-u+l-1}}{1.2\ldots\mu \times 1.2\ldots\nu \times 1.2\ldots\pi \times 1.2\ldots\varrho \times \&c.}$$

(quæ eſt per prob. 1. coefficiens (τ) termini $p^{m-u+1}\, q^{\mu}\, r^{\nu}\, s^{\pi}\, t^{\varrho}$ &c. in ſummâ poteſtatis $(m+1)$ ex ſingulis radicibus) $= S'$ proximæ ſummæ, in quâ poteſtas (m) unitate augetur: ſed per hoc coroll. S & S' erunt reſpective

$$\frac{\overline{l-1} \times \overline{m+u} \times \overline{m-u+1} \times \overline{m-u+2} \times \overline{m-u+3} \ldots\ldots \overline{m-u+l-1}}{l \times \overline{l-1} \times 1.2.3\ldots\mu \times 1.2.3\ldots\nu \times 1.2.3\ldots\pi \times 1.2.3\ldots\varrho \times \&c.}$$

$$\& \frac{\overline{\overline{l-1} \times \overline{m+1} + u}}{l \times \overline{l-1}} \times \frac{\overline{m+1-u+1} \times \overline{m+1-u+2} \times \overline{m+1-u+3} \ldots \overline{m+1-u+l-1}}{1.2.3\ldots\mu \times 1.2.3\ldots\nu \times 1.2.3\ldots\pi \times 1.2.3\ldots\varrho \times \&c.};$$

unde differentia $S' - S = (\overline{\overline{l-1} \times \overline{m+1} + u} \times \overline{m-u+l} - \overline{\overline{l-1} \times \overline{m+u} \times \overline{m-u+1}})$

$$\frac{\overline{m-u+2} \times \overline{m-u+3} \ldots\ldots \overline{m-u+l-1}}{l \times \overline{l-1} \times 1.2\ldots\mu \times 1.2\ldots\nu \times 1.2\ldots\pi \times 1.2\ldots\varrho \times \&c.} =$$

$$l \times \overline{l-1} \times \overline{m+1} \times \frac{\overline{m-u+2} \times \overline{m-u+3} \ldots\ldots \overline{m-u+l-1}}{l \times \overline{l-1} \times 1.2\ldots\mu \times 1.2\ldots\nu \times 1.2\ldots\pi \times 1.2\ldots\varrho \times \&c.} =$$

$$\overline{m+1} \times \frac{\overline{m-u+2} \times \overline{m-u+3} \ldots \overline{m-u+l-1}}{1.2\ldots\mu \times 1.2\ldots\nu \times 1.2\ldots\pi \times 1.2\ldots\varrho \times \&c.},$$

quæ eſt coefficiens prædicta τ termini $p^{m-u+1}\, q^{\mu}\, r^{\nu}\, s^{\pi}\, t^{\varrho}$ &c.; conſequenter ſi in coroll. S recte deſignet ſummam prædictam ad poteſtatem (m) ex ſingulis radicibus, S' etiam ſummam proxime ſuperiorem ad poteſtatem $(m+1)$ vere oſtendet.

PROB. III.

Sint $\alpha, \beta, \gamma, \delta, \varepsilon$, &c. *radices datæ æquationis*

$$x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0,$$

& ſint a, b, c, d, e, f, &c. *dati indices, quorum numerus ſit* m; m *aut æqualis aut minor erit quam* n.

Invenire ſummam omnium hujuſmodi quantitatum.

$\alpha^a \beta^b \gamma^c \delta^d \varepsilon^e$ &c. $+ \alpha^b \beta^a \gamma^c \delta^d \varepsilon^e$ &c. $+ \alpha^c \beta^a \gamma^b \delta^d \varepsilon^e$ &c. $+$
$\alpha^d \beta^a \gamma^b \delta^c \varepsilon^e$ &c. $+ \alpha^e \beta^a \gamma^b \delta^c \varepsilon^d$ &c. $+ \alpha^a \beta^c \gamma^b \delta^d \varepsilon^e$ &c. $+$ &c.

in quibus numerus factorum eſt (m).

Scribantur

Scribantur Sa, Sb, Sc, &c. pro ſummis reſpectivis a, b, c, &c. poteſtatum e ſingulis radicibus, *i. e.*

$Sa = \alpha^a + \beta^a + \gamma^a + \delta^a + \varepsilon^a +$ &c.
$Sb = \alpha^b + \beta^b + \gamma^b + \delta^b + \varepsilon^b +$ &c.
$Sc = \alpha^c + \beta^c + \gamma^c + \delta^c + \varepsilon^c +$ &c.
$Sd = \alpha^d + \beta^d + \gamma^d + \delta^d + \varepsilon^d +$ &c.
$Se = \alpha^e + \beta^e + \gamma^e + \delta^e + \varepsilon^e +$ &c.
&c.= &c. &c.
$S\overline{a+b} = \alpha^{a+b} + \beta^{a+b} + \gamma^{a+b} + \delta^{a+b} + \varepsilon^{a+b} +$ &c.
$S\overline{a+c} = \alpha^{a+c} + \beta^{a+c} + \gamma^{a+c} + \delta^{a+c} + \varepsilon^{a+c} +$ &c.
$S\overline{b+c} = \alpha^{b+c} + \beta^{b+c} + \gamma^{b+c} + \delta^{b+c} + \varepsilon^{b+c} +$ &c.
$S\overline{a+d} = \alpha^{a+d} + \beta^{a+d} + \gamma^{a+d} + \delta^{a+d} + \varepsilon^{a+d} +$ &c.
&c. = &c. &c.
$S\overline{a+b+c} = \alpha^{a+b+c} + \beta^{a+b+c} + \gamma^{a+b+c} + \delta^{a+b+c} + \varepsilon^{a+b+c} +$ &c.
$S\overline{a+b+d} = \alpha^{a+b+d} + \beta^{a+b+d} + \gamma^{a+b+d} + \delta^{a+b+d} + \varepsilon^{a+b+d} +$ &c.
$S\overline{a+b+e} = \alpha^{a+b+e} + \beta^{a+b+e} + \gamma^{a+b+e} + \delta^{a+b+e} + \varepsilon^{a+b+e} +$ &c.
$S\overline{a+c+d} = \alpha^{a+c+d} + \beta^{a+c+d} + \gamma^{a+c+d} + \delta^{a+c+d} + \varepsilon^{a+c+d} +$ &c.
$S\overline{b+c+d} = \alpha^{b+c+d} + \beta^{b+c+d} + \gamma^{b+c+d} + \delta^{b+c+d} + \varepsilon^{b+c+d} +$ &c.
&c. = &c. &c. &c.
$S\overline{a+b+c+d} = \alpha^{a+b+c+d} + \beta^{a+b+c+d} + \gamma^{a+b+c+d} + \delta^{a+b+c+d} +$ &c.
$S\overline{a+b+c+e} = \alpha^{a+b+c+e} + \beta^{a+b+c+e} + \gamma^{a+b+c+e} + \delta^{a+b+c+e} +$ &c.
$S\overline{a+b+d+e} = \alpha^{a+b+d+e} + \beta^{a+b+d+e} + \gamma^{a+b+d+e} + \delta^{a+b+d+e} +$ &c.
$S\overline{b+c+d+e} = \alpha^{b+c+d+e} + \beta^{b+c+d+e} + \gamma^{b+c+d+e} + \delta^{b+c+d+e} +$ &c.
&c. = &c.
$S\overline{a+b+c+d+e} = \alpha^{a+b+c+d+e} + \beta^{a+b+c+d+e} + \gamma^{a+b+c+d+e} +$ &c.
&c. = &c.

& ſic deinceps.

Scribatur etiam

$$A = Sa\times Sb\times Sc\times Sd\times Se\times \&c.$$

$$B = \frac{S\overline{a+b}\times A}{Sa\times Sb} + \frac{S\overline{a+c}\times A}{Sa\times Sc} + \frac{S\overline{b+c}\times A}{Sb\times Sc} + \frac{S\overline{a+d}\times A}{Sa\times Sd} + \&c.$$

$$C = \frac{S\overline{a+b+c}\times A}{Sa\times Sb\times Sc} + \frac{S\overline{a+b+d}\times A}{Sa\times Sb\times Sd} + \frac{S\overline{a+c+d}\times A}{Sa\times Sc\times Sd} + \frac{S\overline{b+c+d}\times A}{Sb\times Sc\times Sd} + \frac{S\overline{a+b+e}\times A}{Sa\times Sb\times Se} + \&c.$$

$$\begin{cases} D = \dfrac{S\overline{a+b+c+d}\times A}{Sa\times Sb\times Sc\times Sd} + \dfrac{S\overline{a+b+c+e}\times A}{Sa\times Sb\times Sc\times Se} + \dfrac{S\overline{a+c+d+e}\times A}{Sa\times Sc\times Sd\times Se} + \dfrac{S\overline{b+c+d+e}\times A}{Sb\times Sc\times Sd\times Se} + \&c. \\ BB = \dfrac{S\overline{a+b}\times S\overline{c+d}\times A}{Sa\times Sb\times Sc\times Sd} + \dfrac{S\overline{a+c}\times S\overline{b+d}\times A}{Sa\times Sc\times Sb\times Sd} + \dfrac{S\overline{a+e}\times S\overline{b+c}\times A}{Sa\times Se\times Sb\times Sc} + \&c. \end{cases}$$

$$\begin{cases} E = \dfrac{S\overline{a+b+c+d+e}\times A}{Sa\times Sb\times Sc\times Sd\times Se} + \dfrac{S\overline{a+b+c+d+f}\times A}{Sa\times Sb\times Sc\times Sd\times Sf} + \dfrac{S\overline{a+b+c+e+f}\times A}{Sa\times Sb\times Sc\times Se\times Sf} + \&c. \\ BC = \dfrac{S\overline{a+b}\times S\overline{c+d+e}\times A}{Sa\times Sb\times Sc\times Sd\times Se} + \dfrac{S\overline{a+c}\times S\overline{b+d+e}\times A}{Sa\times Sc\times Sb\times Sd\times Ce} + \dfrac{S\overline{a+d}\times S\overline{b+c+e}\times A}{Sa\times Sd\times Sb\times Sc\times Se} + \&c. \end{cases}$$

$$\begin{cases} F = \dfrac{S\overline{a+b+c+d+e+f}\times A}{Sa\times Sb\times Sc\times Sd\times Se\times Sf} + \&c. \\ BD = \dfrac{S\overline{a+b}\times S\overline{c+d+e+f}\times A}{Sa\times Sb\times Sc\times Sd\times Se\times Sf} + \dfrac{S\overline{a+c}\times S\overline{b+d+e+f}\times A}{Sa\times Sc\times Sb\times Sd\times Se\times Sf} + \&c. \\ CC = \dfrac{S\overline{a+b+c}\times S\overline{d+e+f}\times A}{Sa\times Sb\times Sc\times Sd\times Se\times Sf} + \dfrac{S\overline{a+b+d}\times S\overline{c+e+f}\times A}{Sa\times Sb\times Sd\times Sc\times Se\times Sf} + \&c. \\ BBB = \dfrac{S\overline{a+b}\times S\overline{c+d}\times S\overline{e+f}\times A}{Sa\times Sb\times Sc\times Sd\times Se\times Sf} + \dfrac{S\overline{a+c}\times S\times\overline{b+d}\times S\overline{e+f}\times A}{Sa\times Sc\times Sb\times Sd\times Se\times Sf} + \&c. \end{cases}$$

& ſic deinceps.

Obſervandum eſt terminos prædictos Sa, Sb, Sc, &c. $S\overline{a+b}$, $S\overline{a+c}$, $S\overline{b+c}$, &c. $S\overline{a+b+c}$, $S\overline{a+b+d}$, &c. deſignare quantitates, quæ ſibimetipſis per notam æqualitatis adjungantur.

Idem intelligendum eſt de quantitatibus BB, BC, BD, CC, BBB, &c.

Summa quæſita eſt

$$A - B + 1.2C - 1.2.3D + 1.2.3.4E - 1.2.3.4.5F$$
$$+ 1.1BB - 1.1.2BC + 1.1.2.3BD$$
$$+ 1.2.1.2CC$$
$$- 1.1.1BBB$$
$$+ 1.2.3.4.5.6G - \&c.$$
$$- 1.1.2.3.4BE + \&c.$$
$$- 1.2.1.2.3CD + \&c.$$
$$+ 1.1.1.2BBC \quad \&c.$$

Literalium quantitatum hæc eſt lex. Exponentes literalium quantitatum terminorum infra ſe poſitorum eandem conficiunt ſummam, quæ quidem ſumma erit ſucceſſive 1, 2, 3, 4, 5, 6, &c.

Per exponentes literarum intelligo numeros deſignantes ſedes illas, quas in ſerie naturali literarum ſingulæ literæ occupant; ſic exponens literæ *A* eſt 1; literæ *B*, 2; literæ *C*, 3; & ſic deinceps: quo fit, ut exponens literalis quantitatis *BBB* ... *CCC* ... *DDD* ... *EEE* ... *FFF* ... *GGG* ... &c.; in quâ tot ſunt literæ *B*, *C*, *D*, *E*, *F*, *G*, &c. contentæ, quot unitates in μ, ν, π, ϱ, σ, τ, &c. reſpective; fiat $2\mu + 3\nu + 4\pi + 5\varrho + 6\sigma + 7\tau +$ &c.

Cunctæ inſuper literales quantitates, quæ ſummas ſucceſſivas (1, 2, 3, 4, 5, &c.) poſſunt efficere, ſemper ſunt adjungendæ.

Coefficiens literalis quantitatis *BBB* ... *CCC* ... *DDD* ... *EEE* ... *FFF* ... *GGG* ... &c. in quâ tot ſunt literæ *B*, quot unitates in μ; tot ſunt literæ *C*, quot unitates in ν; tot ſunt literæ *D*, *E*, *F*, *G*, &c. quot unitates in π, ϱ, σ, τ, &c. reſpective; æqualis eſt contento $1^{\mu} \times 1^{\nu} \times 2^{\nu} \times 1^{\pi} \times 2^{\pi} \times 3^{\pi} \times 1^{\varrho} \times 2^{\varrho} \times 3^{\varrho} \times 4^{\varrho} \times 1^{\sigma} \times 2^{\sigma} \times 3^{\sigma} \times 4^{\sigma} \times 5^{\sigma} \times 1^{\tau} \times 2^{\tau} \times 3^{\tau} \times 4^{\tau} \times 5^{\tau} \times 6^{\tau} \times$ &c.

Quod ſi ſint (k) indices quantitates æquales per (a) deſignatæ, indices vero (l) æquales per b deſignatæ, r' & s' indices æquales per c & d reſpective deſignatæ, &c. tum ſumma prædicta per contentum $1 . 2 . 3 \ldots k \times 1 . 2 . 3 \ldots l \times 1 . 2 . 3 \ldots r' \times 1 \times 2 \times 3 \ldots s' \times$ &c. dividenda eſt.

Signa affixa e prob. 1. facile deduci poſſunt, i. e. erunt affirmativa vel negativa, prout $\mu + 2\nu + 3\pi + 4\varrho +$ &c. ſit par vel impar numerus.

Ex. Sit æquatio $x^7 - 6x^6 + 9x^5 + 10x^4 - 45x^3 + 54x^2 - 29x + 16 = 0$, cujus radices ſint α, β, γ, δ, ε, &c.; invenire ſummam

$\alpha^2\beta^3 + \beta^2\alpha^3 + \alpha^2\gamma^3 + \gamma^2\alpha^3 + \beta^2\gamma^3 + \gamma^2\beta^3 + \alpha^2\delta^3 + \delta^2\alpha^3 + \beta^2\delta^3 + \delta^2\beta^3 + \gamma^2\delta^3 + \delta^2\gamma^3 +$ &c.

per problema primum inveniatur

$$S\,2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \varepsilon^2 + \&c. = p^2 - 2q = 18$$

$$S\,\underline{3} = \alpha^3 + \beta^3 + \gamma^3 + \delta^3 + \varepsilon^3 + \&c. = p^3 - 3pq + 3r = 24$$

$$S\,\overline{2+3} = \alpha^5 + \beta^5 + \gamma^5 + \delta^5 + \varepsilon^5 + \&c. = p^5 - 5p^3q + 5p^2r - 5ps + 5t + 5pq^2 - 5qr = 216$$

 in

in hoc caſu erit

$A = S2 \times S3 = 18 \times 24 = 432.$ $B = S\overline{2+3} = 216$

& conſequenter per problema ſumma quæſita erit $A - B = 432 - 216 = 216.$

Ex. 2. Sit æquatio $x^6 - 5x^5 + 4x^4 + 10x^3 - 11x^2 - 5x + 6 = 0$, cujus radices ſint $\alpha, \beta, \gamma, \delta, \epsilon$, &c. invenire ſummam quantitatum.

$\alpha^2\beta^3\gamma^4 + \alpha^2\gamma^3\beta^4 + \beta^2\alpha^3\gamma^4 + \beta^2\gamma^3\alpha^4 + \gamma^2\alpha^3\beta^4 + \gamma^2\beta^3\alpha^4 +$
$\alpha^2\beta^3\delta^4 + \delta^2\alpha^3\beta^4 + \delta^2\beta^3\alpha^4 +$ &c.

per problema primum inveniantur

$S2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \epsilon^2 +$ &c. $= 17,$
$S3 = \alpha^3 + \beta^3 + \gamma^3 + \delta^3 + \epsilon^3 +$ &c. $= 35,$
$S4 = \alpha^4 + \beta^4 + \gamma^4 + \delta^4 + \epsilon^4 +$ &c. $= 101;$

&

$S\overline{2+3} = \alpha^5 + \beta^5 + \gamma^5 + \delta^5 + \epsilon^5 +$ &c. $= 275,$
$S\overline{2+4} = \alpha^6 + \beta^6 + \gamma^6 + \delta^6 + \epsilon^6 +$ &c. $= 797,$
$S\overline{3+4} = \alpha^7 + \beta^7 + \gamma^7 + \delta^7 + \epsilon^7 +$ &c. $= 2315;$

&

$S\overline{2+3+4} = \alpha^9 + \beta^9 + \gamma^9 + \delta^9 + \epsilon^9 +$ &c. $= 20195;$

conſequenter fient

$A = S2 \times S3 \times S4 = 17 \times 35 \times 101 = 60095,$
$B = S\overline{2+3} \times S4 + S\overline{2+4} \times S^3 + S\overline{3+4} \times S2 = 275 \times 101 + 797 \times 35 + 2315 \times 17 = 95025,$
$C = S\overline{2+3+4} = 20195;$

& per problema ſumma quæſita erit $A - B + 2C = 5460.$

2. Iiſdem literis eaſdem quantitates denotantibus, ſit $S^a = P$, $P \times S^b - S^{a+b} = Q$, $Q \times S^c - (S^{c+a} \times S^b + S^{c+b} \times S^a) + 1.2\, S^{a+b+c} = R$, $R \times S^d - (S^{d+a} \times S^b \times S^c + S^{d+b} \times S^a \times S^c + S^{d+c} \times S^a \times S^b) + 1.2\,(S^{d+a+b} \times S^c + S^{d+a+c} \times S^b + S^{d+b+c} \times S^a) \begin{matrix} -1.2.3 \times S^{d+a+b+c} \\ +1.1 \times (S^{d+a} \times S^{b+c} + S^{d+b} + S^{a+c} + S^{d+c} \times S^{a+b}) \end{matrix} = T$,
$T \times S^e - (S^{e+a} \times S^b \times S^c \times S^d + S^{e+b} \times S^a \times S^c \times S^d + S^{e+c} \times S^a \times S^b \times S^d \times S^{e+d} \times S^a \times S^b \times S^c)$
$+ 1.2\,(S^{e+a+b} \times S^c \times S^d + S^{e+a+c} \times S^b \times S^d + S^{e+b+c} \times S^a \times S^d + S^{e+a+d} \times S^b \times S^c +$
$S^{e+b+d} \times S^a \times S^c + S^{e+c+d} \times S^a \times S^b) - 1.2.3 \times (S^{e+a+b+c} \times S^d + S^{e+a+b+d} \times S^c +$
$S^{e+a+c+d} \times S^b + S^{e+b+c+d} \times S^a) + 1.1 \times (S^{e+a} \times S^{b+c} \times S^d + S^{e+a} \times S^{b+d} \times S^c + S^{e+a} \times$
$S^{c+d} \times S^b + S^{e+b} \times S^{a+c} \times S^d + S^{e+b} \times S^{a+d} \times S^c + S^{e+b} \times S^{c+d} \times S^a + S^{e+c} \times S^{a+b} \times S^d +$
$S^{e+c} \times S^{a+d} \times S^b + S^{e+c} \times S^{b+d} \times S^a + S^{e+d} \times S^{a+b} \times S^c + S^{e+d} \times S^{a+c} \times S^b + S^{e+d} \times S^{b+c} \times S^a)$

$+ 1$

$+ 1 . 2 . 3 . 4\, S^{e+a+b+c+d} - 1 . 1 . 2 \times (S^{e+a} \times S^{b+c+d} + S^{e+b} \times S^{a+c+d} + S^{e+c} \times S^{a+b+d} + S^{e+d} \times S^{a+b+c} + S^{e+a+b} \times S^{c+d} + S^{e+a+c} \times S^{b+d} + S^{e+b+c} \times S^{a+d} + S^{e+a+d} \times S^{b+c} + S^{e+b+d} \times S^{a+c} + S^{e+c+d} \times S^{a+b}) = V$, & ſic deinceps; erunt

$P = \alpha^a + \beta^a + \gamma^a + \delta^a +$ &c.

$Q = \alpha^a\beta^b + \beta^a\alpha^b + \alpha^a\gamma^b + \gamma^a\alpha^b + \beta^a\gamma^b +$ &c.

$R = \alpha^a\beta^b\gamma^c + \alpha^a\beta^c\gamma^b + \alpha^b\beta^a\gamma^c +$ &c.

$S = \alpha^a\beta^b\gamma^c\delta^d + \alpha^b\beta^a\gamma^c\delta^d + \alpha^a\beta^b\gamma^d\delta^c +$ &c.

$T = \alpha^a\beta^b\gamma^c\delta^d\varepsilon^e +$ &c.

&c.= &c.

Lex hujuſce ſeriei eadem eſt ac præcedentis; & ex eâ, animadvertendo primos terminos $P \times S^b$, $Q \times S^c$, $R \times S^d$, $T \times S^e$, &c. hujuſce ſeriei æquales eſſe aggregatis ſingulorum terminorum præcedentis ſeriei, in ſummas S^b, S^c, S^d, S^e, &c. reſpective ductis; facile deduci poteſt.

3. Haud raro multo facilius e coefficientibus datæ æquationis ad quæſitum aggregatum progredi licet.

Hujuſce generis ſolutionem ſubſequens ſemper præbebit methodus.

Sit index a minor quam b, b quam c, c quam d, d quam e, &c. & ſit numerus indicum a, b, c, d, e, &c. $= v$, & data æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} \ldots\ldots \pm Px^{n-v+3} \mp Qx^{n-v+2} \pm Rx^{n-v+1} \mp Sx^{n-v} \pm \&c. = o$.

Primus terminus ſolutionis erit

$S^a \times R^{b-a} \times Q^{c-b} \times P^{d-c} \times$ &c. hujuſce termini quædam contenta erunt quæſita ſumma; reliqua ejus contenta per eandem methodum dilui poſſunt. E. G.

Sit æquatio $x^4 - px^3 + qx^2 - rx + s = o$, cujus radices ſint $\alpha, \beta, \gamma, \delta$: invenire aggregatum e ſingulis contentis $\alpha\beta^2\gamma^3 + \alpha\gamma^2\beta^3 + \beta\alpha^2\gamma^3 + \beta\gamma^2\alpha^3 + \gamma\alpha^2\beta^3 + \gamma\beta^2\alpha^3 + \alpha\beta^2\delta^3 + \alpha\delta^2\beta^3 + \beta\alpha^2\delta^3 +$ &c.

Hìc indices a, b, c reſpective ſunt 1, 2, 3; & numerus indicum $v = 3$; & exinde $n - v = 4 - 3 = 1$; & $S = r$, $R = q$, $Q = p$; & $S^a \times R^{b-a} \times Q^{c-b} = rqp$; & reſultantia contenta e multiplicatione factorum $r \times q \times p = \overline{\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta} \times \overline{\alpha\beta + \alpha\gamma + \beta\gamma + \alpha\delta + \beta\delta + \gamma\delta} \times \overline{\alpha + \beta + \gamma + \delta}$ erunt $\alpha\beta^2\gamma^3 + \alpha\gamma^2\beta^3 + \beta\alpha^2\gamma^3 +$ &c. $+ 3 \times$

α^2

$\overline{\alpha^2\beta^2\gamma^2+\alpha^2\beta^2\delta^2+\alpha^2\gamma^2\delta^2+\beta^2\gamma^2\delta^2} + 8\alpha\beta\gamma\delta\times\overline{\alpha\beta+\alpha\gamma+\beta\gamma+\alpha\delta+\beta\delta+\gamma\delta}$ $(8sq) + 3\alpha\beta\gamma\delta \times \overline{\alpha^2+\beta^2+\gamma^2+\delta^2}\,(3sp^2 - 6sq)$; prima contenta $\alpha\beta^2\gamma^3 + \alpha\gamma^2\beta^3 + \beta\alpha^2\gamma^3 +$ &c. erunt quæſita ſumma: ſecundorum contentorum $\alpha^2\beta^2\gamma^2 + \alpha^2\beta^2\delta^2 + \alpha^2\gamma^2\delta^2 + \beta^2\gamma^2\delta^2$ per eandem methodum hìc traditam inveniri poteſt ſumma; indices a, b, c in his contentis ſunt reſpective 2, 2, 2, & eorum numerus $v=3$; unde $n-v=4-3=1$; & $S=r$, & $S^a\times Q^{b-a}\times R^{c-b}=S^2R^0Q^0=S^2=r^2=\overline{\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta}^2=$ $\alpha^2\beta^2\gamma^2+\alpha^2\beta^2\delta^2+\alpha^2\gamma^2\delta^2+\beta^2\gamma^2\delta^2+2\alpha\beta\gamma\delta\times\overline{\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta}$ $(2sq)$; & exinde $\alpha^2\beta^2\gamma^2 + \alpha^2\beta^2\delta^2 + \alpha^2\gamma^2\delta^2 + \beta^2\gamma^2\delta^2 = r^2 - 2qs$; unde ſumma quæſita $\alpha\beta^2\gamma^3 + \alpha\gamma^2\beta^3 + \beta\alpha^2\gamma^3 + \beta\gamma^2\alpha^3 +$ &c. $= rqp - 3 \times \overline{r^2 - 2qs} - 8sq - 3sp^2 + 6sq = rpq + 4qs - 3r^2 - 3sp^2$.

Ex. 2. Datâ æquatione $x^n - px^{n-1} + qx^{n-2} -$ &c. ... $Ox^{n-r+5} \pm Px^{n-r+4} \mp Qx^{n-r+3} \pm Rx^{n-r+2} \mp Sx^{n-r+1} \pm Tx^{n-r} \mp$ &c. ... $\pm ax^{n-m-r} \mp bx^{n-m-r-1} \pm cx^{n-m-r-2} \mp dx^{n-m-r-3} \pm ex^{n-m-r-4} \mp fx^{n-m-r-5} \pm$ &c. $= 0$, cujus radices ſint α, β, γ, δ, ε, ζ, η, θ, &c. invenire ſummam e ſingulis quantitatibus hujuſce generis

$\alpha^2\beta^2\gamma^2\delta^2$ &c. $\times\,\varepsilon\zeta\eta$ &c. $+\,\alpha^2\beta^2\gamma^2\varepsilon^2$ &c. $\times\,\zeta\eta\theta$ &c. $+\,\alpha^2\beta^2\gamma^2\zeta^2$ &c. $\times\,\delta\eta\theta$ &c. $+\,\beta^2\gamma^2\delta^2\varepsilon^2$ &c. $\times\,\zeta\eta\theta$ &c. $+$ &c.

in quibus ſingulis quantitatibus numerus literarum, quæ duas habeant dimenſiones, ſit (r); numerus autem literarum, quæ unam ſolummodo habeant dimenſionem, ſit (m).

Multiplicetur coefficiens (T) termini (x^{n-r}) datæ æquationis $= \alpha\beta\gamma\delta\varepsilon$ &c. $+\,\alpha\beta\gamma\delta\zeta$ &c. $+\,\alpha\gamma\delta\zeta\eta$ &c. $+\,\beta\gamma\delta\eta\theta$ &c. $+$ &c., in quibus ſingulis contentis numerus literarum erit r, in coefficientem (a) termini (x^{n-m-r}) datæ æquationis $= \alpha\beta\gamma\delta\varepsilon\zeta$ &c. $+\,\alpha\beta\gamma\delta\varepsilon\theta$ &c. $+\,\alpha\beta\gamma\varepsilon\zeta\theta$ &c. $+\,\beta\gamma\delta\varepsilon\zeta\theta +$ &c. in quibus ſingulis contentis numerus literarum erit $m+r$; rectangulum erit

$\alpha^2\beta^2\gamma^2\delta^2$ &c. $\times\,\varepsilon\zeta\eta$ &c. $+\,\alpha^2\beta^2\gamma^2\varepsilon^2$ &c. $\times\,\zeta\eta\theta$ &c. $+\,\alpha^2\beta^2\gamma^2\zeta^2$ &c. $\times\,\delta\eta\theta$ &c. $+\,\beta^2\gamma^2\delta^2\varepsilon^2$ &c. $\times\,\zeta\eta\theta +$ &c. (ſumma quæſita)

$$+\overline{m+2}\times A+\overline{m+3}\times\frac{m+4}{2}B+\overline{m+4}\times\frac{m+5}{2}\times\frac{m+6}{3}C+\overline{m+5}\times\frac{m+6}{2}\times\frac{m+7}{3}\times\frac{m+8}{4}D+\overline{m+6}\times\frac{m+7}{2}\times\frac{m+8}{3}\times\frac{m+9}{4}\times\frac{m+10}{5}$$

$E+$

$E +$ &c. ad r terminos ($A, B, C, D, E,$ &c.) ubi
$A = \alpha^2\beta^2\gamma^2$ &c. (ad $r-1$ terminos) $\times\,\delta\varepsilon\zeta$ &c. (ad $m+2$ terminos)
$+\,\alpha^2\beta^2\delta^2$ &c. (ad $r-1$ terminos) $\times\,\gamma\varepsilon\zeta$ &c. (ad $m+2$ terminos)
$+\,\beta^2\gamma^2\delta^2$ &c. $\times\,\zeta\eta\theta$ &c. $+$ &c.
$B = \alpha^2\beta^2$ &c. (ad $r-2$ terminos) $\times\,\gamma\delta\varepsilon\zeta$ &c. (ad $m+4$ terminos)
$+\,\alpha^2\gamma^2$ &c. (ad $r-2$ terminos) $\times\,\beta\delta\varepsilon\zeta\eta$ &c. (ad $m+4$ terminos)
$+\,\beta^2\gamma^2$ &c. $\times\,\alpha\delta\zeta\eta\theta$ &c. $+$ &c.
$C = \alpha^2\beta^2$ &c. (ad $r-3$ terminos) $\times\,\gamma\delta\varepsilon\zeta$ &c. (ad $m+6$ terminos)
$+\,\alpha^2\gamma^2$ &c. (ad $r-3$ terminos) $\times\,\beta\delta\varepsilon\zeta$ &c. (ad $m+6$ terminos) $+$ &c.
$D = \alpha^2\beta^2$ (ad $r-4$ terminos) $\times\,\gamma\delta\varepsilon\zeta\eta$ &c. (ad $m+8$ terminos)
$+\,\alpha^2\gamma^2$ &c. (ad $r-4$ terminos) $\times\,\beta\delta\varepsilon\zeta\eta$ &c. (ad $m+8$ terminos)
$+$ &c.

&c. &c.

Diluantur termini $A, B, C, D,$ &c. ſcribendo pro iis eorum valores $Sb - (\overline{m+4}\,B + \overline{m+5}\times\frac{m+6}{2}\,C + \overline{m+6}\times\frac{m+7}{2}\times\frac{m+8}{3} + \&c.)$, $Rc - (\overline{m+6}\,C + \overline{m+7}\times\frac{m+8}{2}\,D + \&c.)$, $Qd - (\overline{m+8}\,D + \overline{m+9}\times\frac{m+10}{2}\,E + \&c.)$, $Pe - (\overline{m+10}\,E + \&c.)$, &c.; reſultabit ſumma quæſita $Ta - \overline{m+2}\,Sb + \overline{m+1}\times\frac{m+4}{2}\,Rc - \overline{m+1}\times\frac{m+2}{2}\times\frac{m+6}{3}\,Qd + \overline{m+1}\times\frac{m+2}{2}\times\frac{m+3}{3}\times\frac{m+8}{4}\,Pe - \overline{m+1}\times\frac{m+2}{2}\times\frac{m+3}{3}\times\frac{m+4}{4}\times\frac{m+10}{5}\,Of$ $+$ &c. ad $r+1$ terminos.

Generaliter termini coefficiens, cujus diſtantia a primo ſit s, erit $\frac{m+2s}{s}\times\overline{m+1}\times\frac{m+2}{2}\times\frac{m+3}{3}\times\frac{m+4}{4}\,\ldots\ldots\,\frac{m+s-1}{s-1}$.

Cor. Sit a ſumma radicum, b ſumma quadratorum e ſingulis radicibus, c ſumma cuborum, d ſumma quadrato-quadratorum, e ſumma quadrato-cuborum, f ſumma cubo-cuborum, &c.

Summa contentorum ſub ſingulis (m) radicibus erit æqualis fractioni, cujus numerator eſt

$a^m -$

$$a^m - m\times\frac{m-1}{2}\,a^{m-2}b + m\times\overline{m-1}\times\frac{m-2}{3}\,a^{m-3}c - m\times\overline{m-1}\times\overline{m-2}\times\frac{m-3}{4}\,a^{m-4}d + \&c.$$
$$+\,m\times\overline{m-1}\times\frac{m-2}{2}\times\frac{m-3}{2^2}\,a^{m-4}b^2 - \&c.$$

denominator vero $1.2.3.4\ldots m$.

Scribantur pro literis a, b, c, d, &c. literæ p, q, r, s, t, &c. reſpective; hujus ſeriei literalia contenta eadem fient ac literalia contenta ſeriei in problemate primo traditæ.

Coefficiens literalis contenti $(a^{m-1}\,b^{\pi}\,c^{\rho}\,d^{\sigma}$ &c.) erit æqualis fractioni, cujus numerator eſt $m\times\overline{m-1}\times\overline{m-2}\times\overline{m-3}\times\overline{m-4}\times$ &c. in quo tot factores erunt, quot unitates in ſummâ $2\pi+3\rho+4\sigma+$ &c.; denominator vero æqualis eſt facto orto e continuâ multiplicatione contenti $2^{\pi}\,3^{\rho}\,4^{\sigma}\,5^{\tau}$ &c. in ſeries $1\times2\times3\times$ &c., $1\times2\times3\times$ &c., $1\times2\times3\times$ &c., &c. quarum prima continet tot terminos, quot ſunt unitates in π; ſecunda, quot ſunt unitates in ρ; tertia, quot ſunt in σ; & ſic deinceps.

Hoc corollarium ex problemate facile conſtabit, ſi modo in eo pro quantitate $BB\ldots BCC\ldots CDD\ldots D$ &c. (in quâ tot ſint literæ B, quot unitates in π; tot ſint literæ C, quot unitates in ρ; tot ſint literæ D, quot unitates in σ; &c.) ſcribatur ejus valor quantitas

$$\frac{1}{1.2.3..\pi}\times\frac{1}{1.2.3..\rho}\times\frac{1}{1.2.3..\sigma}\times\&c.\times\frac{1}{1^{\pi}\times2^{\pi}}\times\frac{1}{1^{\rho}\times2^{\rho}\times3^{\rho}}\times\frac{1}{1^{\sigma}\times2^{\sigma}\times3^{\sigma}\times4^{\sigma}}\times\&c.\times m\times\overline{m-1}\times\overline{m-2}\times\overline{m-3}\times\ldots\times\overline{m-2\pi-3\rho-4\sigma-\&c.+1}\times a^{m-1}\,b^{\pi}\,c^{\rho}\,d^{\sigma}\ \&c.;$$

prior enim quantitas in hoc caſu in poſteriorem mutatur: quod conſtabit ex animadvertendo, quod in multitudine rerum diſſimilium α, β, γ, δ, ε, ζ, η, θ, ι, κ, λ, μ, ν, o, υ, φ, χ, &c. (quarum numerus eſt m) tot

$$\left(\frac{1}{1.2..\pi\times1.2..\rho\times1.2..\sigma\times\&c.}\times\frac{1}{1^{\pi}\times2^{\pi}\times1^{\rho}\times2^{\rho}\times3^{\rho}\times1^{\sigma}\times2^{\sigma}\times3^{\sigma}\times4^{\sigma}\times\&c.}\times m\,.\,\overline{m-1}\,.\,\overline{m-2}\,.\,\overline{m-3}\ldots\times\overline{m-2\pi-3\rho-4\sigma-\&c.+1}\right)$$

diverſis modis variari poſſunt quantitates hujuſce formulæ $\alpha+\beta$, $\gamma+\delta$,

$\gamma + \delta$, &c. (uſque ad π terminos); $\varepsilon + \zeta + \eta$, $\theta + \iota + \kappa$, &c. (uſque ad ϱ terminos); $\mu + \nu + o + \upsilon$, &c. (uſque ad σ terminos); &c.

Dividatur etiam quantitas per $1.2.3.4 \ldots m$; tot enim diverſi valores contentorum ſub ſingulis m radicibus fiunt inter ſe æquales.

Hoc corollarium etiam ſic demonſtrari poteſt. Sint P, Q, R, S, &c. reſpective coefficientes terminorum $a^{m-l-1} b^{\pi} c^{\varrho} d^{\sigma}$, &c., $a^{m-l} b^{\pi-1} c^{\varrho} d^{\sigma}$, &c., $a^{m-l} b^{\pi} c^{\varrho-1} d^{\sigma}$, &c., $a^{m-l} b^{\pi} c^{\varrho} d^{\sigma-1}$, &c. in aggregatis ſingulorum contentorum ex quibuſque $m-1$, $m-2$, $m-3$, $m-4$, &c. radicibus per hoc cor. deducendis: facile e Neutoni regulâ prius traditâ conſtabit, quod coefficiens V termini $a^{m-l} b^{\pi} c^{\varrho} d^{\sigma}$, &c. in aggregato contentorum ex quibuſque m radicibus deducendo, erit $= P + \overline{m-1} \times Q + \overline{m-1} \times \overline{m-2}\, R + \overline{m-1} \times \overline{m-2} \times \overline{m-3}\, S + \overline{m-1} \times \overline{m-2} \times \overline{m-3} \times \overline{m-4} \times T +$ &c.; hæ autem coefficientes P, Q, R, S, &c. per hoc corollarium erunt $\overline{m-l} \times \frac{\overline{m-1}\,.\,\overline{m-2}\,.\,\overline{m-3} \ldots \overline{m-l+1}}{1.2..\pi \times 1.2..\varrho \times 1.2..\sigma \times \&c. \times 2^{\pi} \times 3^{\varrho} \times 4^{\sigma} \times \&c.}$,

$2\pi \times \frac{\overline{m-2} \times \overline{m-3} \ldots \overline{m-l+1}}{1.2..\pi \times 1.2..\varrho \times 1.2..\sigma \times 2^{\pi} \times 3^{\varrho} \times 4^{\sigma} \times \&c.}$,

$3\varrho \times \frac{\overline{m-3}\,.\,\overline{m-4} \ldots \overline{m-l+1}}{1.2..\pi \times 1.2..\varrho \times 1.2..\sigma \times \&c. \times 2^{\pi} \times 3^{\varrho} \times 4^{\sigma} \times \&c.}$,

$4\sigma \times \frac{\overline{m-4} \times \overline{m-5} \ldots \overline{m-l+1}}{1.2..\pi \times 1.2..\varrho \times 1.2..\sigma \times 2^{\pi} \times 3^{\varrho} \times 4^{\sigma} \times \&c.}$, &c. & conſequenter aggregatum $P + \overline{m-1}\, Q + \overline{m-1} \times \overline{m-2}\, R + \overline{m-1} \times \overline{m-2} \times \overline{m-3} \times S +$ &c. $= \overline{m-l+2\pi+3\varrho+4\sigma+\&c.} \times \frac{\overline{m-1} \times \overline{m-2} \times \overline{m-3} \times \overline{m-4} \ldots \overline{m-l+1}}{1.2..\pi \times 1.2..\varrho \times 1.2..\sigma \times \&c. \times 2^{\pi} \times 3^{\varrho} \times 4^{\sigma} \times \&c.}$; in primo hujus aggregati factore pro l ſcribatur ejus valor $2\pi + 3\varrho + 4\sigma +$ &c.; reſultat aggregatum $= m \times \frac{\overline{m-1} \times \overline{m-2} \times \overline{m-3} \ldots \overline{m-l+1}}{1.2..\pi \times 1.2..\varrho \times 1.2..\sigma \times \&c. \times 2^{\pi} \times 3^{\varrho} \times 4^{\sigma} \times \&c.}$; hoc vero aggregatum eſt coefficiens vel uncia, quæ in datâ ſerie pro unciâ literalis contenti $a^{m-l} b^{\pi} c^{\varrho} d^{\sigma}$ &c. in aggregato ſingulorum contentorum ex quibuſque m radicibus in ſeſe ductis deducendo, ſubſti-

tuitur; ergo, ſi prædicta ſeries, aggregata, ſingulorum inferiorum contentorum ex quibuſque $m-1$, $m-2$, $m-3$, &c. radicibus recte oſtendat; aggregatum proxime ſuperiorum contentorum ex quibuſque m radicibus vere exprimet.

Ex hoc corollario, vice verſâ, ope eorum, quæ prius tradita fuere, facile erui poteſt problema.

PROB. IV.

Datâ æquatione, cujus radix ſit x; invenire ſummam ſingulorum valorum quarumcunque algebraicarum functionum e radicibus compoſitarum.

Sint $\alpha, \beta, \gamma, \delta$, &c. radices x datæ æquationis, & in datâ functione pro x ſcribantur ejus valores $\alpha, \beta, \gamma, \delta$, &c.; & inveniantur ſinguli valores reſultantium quantitatum; tum

1. Si rationales & integræ ſint functiones: methodis ſuperius traditis derivari poteſt ſumma e ſingulis earum valoribus confecta.

2. Si functiones ſint rationaliter fractionales; ad communem denominatorem reducendæ ſunt fractiones; & per methodos prædictas, tam numerator, quam denominator inveſtigari poteſt. E. G.

Sit æquatio $x^3-7x+6=0$, ſintque ejus radices α, β, γ; & fractionalis functio $\frac{3x^2+5}{4x+1}$.

Pro x in datâ functione ſcribantur ejus valores, reſultant fractiones $\frac{3\alpha^2+5}{4\alpha+1}$, $\frac{3\beta^2+5}{4\beta+1}$, $\frac{3\gamma^2+5}{4\gamma+1}$.

Reducantur hæ fractiones ad communem denominatorem, fient

$$\frac{48\times\alpha\beta\gamma\times\overline{\alpha+\beta+\gamma}+12\times\overline{\alpha^2\beta+\alpha^2\gamma+\beta^2\alpha+\beta^2\gamma+\gamma^2\alpha+\gamma^2\beta}+3\times\overline{\alpha^2+\beta^2+\gamma^2}+80\times\overline{\alpha\beta+\alpha\gamma+\beta\gamma}+40\,\overline{\alpha+\beta+\gamma}+15}{64\times\alpha\beta\gamma+16\times\overline{\alpha\beta+\alpha\gamma+\beta\gamma}+4\times\overline{\alpha+\beta+\gamma}+1}.$$

Pro p, q, r, ſcribantur 0, -7 & -6 reſpective; erit denominator $64\times\alpha\beta\gamma\,(64r=64\times-6)+16\,\overline{\alpha\beta+\alpha\gamma+\beta\gamma}\,(16\times q=16\times-7)+4\times\overline{\alpha+\beta+\gamma}\,(4p=0)+1=-495$

&

& numerator $48 \times \alpha\beta\gamma \times \overline{\alpha+\beta+\gamma}\ (48\,rp = o) + 12 \times$ $\overline{\alpha^2\beta+\alpha^2\gamma+\beta^2\alpha+\beta^2\gamma+\gamma^2\alpha+\gamma^2\beta}\ (12\,pq - 3r = 12 \times 3 \times 6)$ $+ 3 \times \overline{\alpha^2+\beta^2+\gamma^2}\ (3 \times p^2 - 2q = 3 \times 2 \times 7) + 80 \times \overline{\alpha\beta+\alpha\gamma+\beta\gamma}$ $(80\,q = 80 \times -7) + 40 \times \overline{\alpha+\beta+\gamma}\ (40 \times p = o) + 15 = -287.$
Et conſequenter ſumma quæſita erit $\frac{287}{495}$. ✓

Ex. 2. Sint $\alpha, \beta, \gamma, \delta, \varepsilon$, &c. radices datæ æquationis $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \&c. = o$; & ſit fractio $\frac{a - bx + cx^2 - dx^3 + \&c.}{A - Bx + Cx^2 - Dx^3 + \&c.}$, in quâ ſint coefficientes (a, b, c, &c. A, B, C, &c.) datæ.

Invenire ſummam fractionum
$\frac{a - b\alpha + c\alpha^2 - d\alpha^3 + \&c.}{A - B\alpha + C\alpha^2 - D\alpha^3 + \&c.}$, $\frac{a - b\beta + c\beta^2 - d\beta^3 + \&c.}{A - B\beta + C\beta^2 - D\beta^3 + \&c.}$, $\frac{a - b\gamma + c\gamma^2 - d\gamma^3 + \&c.}{A - B\gamma + C\gamma^2 - D\gamma^3 + \&c.}$.

Reducantur hæ fractiones ad communem denominatorem, problematis primi & tertii ope inveniri poteſt & numerator & denominator.

3. Si functiones ſint irrationales, tum ſumma e ſingulis valoribus quantitatum irrationalium deducta nihilo erit æqualis; quod ſi ſint fractionales ad communem denominatorem 1mo ſunt reducendæ. E.G.

Sit æquatio $x^7 - 4x^6 + 3x^5 + 10x^4 - 25x^3 + 24x^2 - 11x + 12 = o$; cujus radices ſint $\alpha, \beta, \gamma, \delta, \varepsilon$, &c.; invenire ſummam ſingulorum valorum irrationalis functionis radicum
$\alpha + \sqrt{\alpha^2+\beta^2}$, $\beta + \sqrt{\alpha^2+\beta^2}$, $\alpha + \sqrt{\alpha^2+\gamma^2}$, $\gamma + \sqrt{\alpha^2+\gamma^2}$, &c.
$\alpha - \sqrt{\alpha^2+\beta^2}$, $\beta - \sqrt{\alpha^2+\beta^2}$, $\alpha - \sqrt{\alpha^2+\gamma^2}$, $\gamma - \sqrt{\alpha^2+\gamma^2}$,
ſumma irrationalium quantitatum nihilo ſemper eſt æqualis; ſumma vero rationalium erit $2 \times 6 \times \overline{\alpha+\beta+\gamma+\delta+\varepsilon+\&c.} = 12 \times 4 = 48$.

Ex. 2. Sit æquatio $x^3 - px^2 + qx - r = o$, cujus radices ſint α, β, γ; invenire ſummam irrationalium fractionum $\frac{\alpha}{\alpha + \sqrt{\alpha^2 + 2\alpha}}$,

$$\frac{\alpha}{\alpha-\sqrt{\alpha^2+2\alpha}},\ \frac{\beta}{\beta+\sqrt{\beta^2+2\beta}},\ \frac{\beta}{\beta-\sqrt{\beta^2+2\beta}},\ \frac{\gamma}{\gamma+\sqrt{\gamma^2+2\gamma}},$$

$$\frac{\gamma}{\gamma-\sqrt{\gamma^2+2\gamma}}.$$

Reducantur hæ fractiones ad communem denominatorem, fient

$$\frac{4\beta\gamma\alpha\times\alpha-4\beta\gamma\alpha\sqrt{\alpha^2+2\alpha}}{-8\beta\gamma\alpha}\left(\frac{\alpha-\sqrt{\alpha^2+2\alpha}}{-2}\right),\ \frac{4\alpha\gamma\beta\times\beta-4\alpha\gamma\beta\sqrt{\beta^2+2\beta}}{-8\alpha\beta\gamma}\left(\frac{\beta-\sqrt{}}{}\right.$$

$$\frac{4\beta\gamma\alpha\times\alpha+4\beta\gamma\alpha\sqrt{\alpha^2+2\alpha}}{-8\beta\gamma\alpha}\left(\frac{\alpha+\sqrt{\alpha^2+2\alpha}}{-2}\right),\ \frac{4\alpha\gamma\beta\times\beta+4\alpha\gamma\beta\sqrt{\beta^2+2\beta}}{-8\alpha\beta\gamma}\left(\frac{\beta+\sqrt{}}{}\right.$$

$$\frac{4\alpha\beta\gamma\times\gamma-4\alpha\beta\gamma\sqrt{\gamma^2+2\gamma}}{-8\alpha\beta\gamma}\left(\frac{\gamma-\sqrt{\gamma^2+2\gamma}}{-2}\right).$$

$$\frac{4\alpha\beta\gamma\times\gamma+4\alpha\beta\gamma\sqrt{\gamma^2+2\gamma}}{-8\alpha\beta\gamma}\left(\frac{\gamma+\sqrt{\gamma^2+2\gamma}}{-2}\right).$$ Et conſequenter ſumma quæſita erit $-\overline{\alpha+\beta+\gamma}=-p$.

4. Eadem erit methodus inveniendi ſummam ſingulorum valorum cujuſcunque functionis datæ æquationis radicum, cum plures valores α, β, γ, &c. radicis x in datâ functione contineantur.

Subſtituantur in prædictâ functione omnes (n) diverſæ radices quantitatis x pro ſuis valoribus, & exinde inveniantur ſinguli valores functionis prædictæ: per methodos prius traditas inveniri poteſt aggregatum e ſingulis hiſce valoribus collectum.

Forſan de his rebus haud inutile ſit ſubſequentia adjungere.

In datâ functione contineantur (m) diverſæ radices datæ æquationis; ſit π functio in prædictâ functione contenta, quæ continet ſolummodo (r) diverſas ex (m) prædictis radices, nec in ſimpliciores radices vel quantitates reduci poteſt: inveniatur ob irrationalitatem quot (b) diverſæ radices poſſint eſſe in reliquis functionibus datâ functione contentis, etiamque quot ($\overline{n-r}\times\overline{n-r-1}\times\overline{n-r-2}\times\overline{n-r-3}\ldots\overline{n-m+1}=\text{K}$) ob diverſas ($m-r$) radices in datâ functione contentas: tum in aggregato quæſito ducatur aggregatum (Σ) ex ſingulis valoribus functionis (π) in rectangulum $b\times$

K; tot enim in eo continebuntur iidem valores functionis π: & ſic de reliquis.

Si vero duæ ex $(m - r)$ radices ſimiliter involvantur in datâ functione, tum dividatur contentum $h \times K \times \Sigma$ per $1 . 2$; ſi tres, tum dividatur prædictum contentum per $1 . 2 . 3$; ſi quatuor ſimiliter involvantur, tum dividatur per $1 . 2 . 3 . 4$; & ſic deinceps.

Si duæ radices ſimiliter involvantur in functione π, haud vero in datâ functione, tum multiplicetur prædictum contentum in $1 . 2$; ſi tres, multiplicetur in $1 . 2 . 3$; ſi quatuor, tum multiplicetur in $1 . 2 . 3 . 4$; & ſic deinceps.

Ex. Sit data functio $7\alpha + 11\beta + \frac{3\gamma}{\delta} + \sqrt{4 + \alpha\beta + \gamma^2}$, ubi α, β, γ & δ reſpective denotant diverſas radices æquationis $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. \ldots \pm Rx \mp Qx \pm P = 0$; aggregatum e ſingulis valoribus functionis $\sqrt{4 + \alpha\beta + \gamma^2}$ nihilo erit æquale: aggregatum e ſingulis valoribus functionis $\left(7\alpha + 11\beta + \frac{3\gamma}{\delta}\right)$ erit $= 7 \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{\alpha + \beta + \gamma + \delta + \&c.} + 11 \times \overline{n-1} . \overline{n-2} . \overline{n-3} \times \overline{\alpha + \beta + \gamma + \delta + \&c.} + 3 \times \overline{n-2} \times \overline{n-3} \times \left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{\alpha}{\gamma} + \frac{\gamma}{\alpha} + \frac{\beta}{\gamma} + \frac{\gamma}{\beta} + \frac{\alpha}{\delta} + \frac{\delta}{\alpha} + \frac{\beta}{\delta} + \frac{\delta}{\beta} + \&c.\right) = \overline{11 + 7}$ (18) $\times \overline{n-1} . \overline{n-2} . \overline{n-3}\, p + 3 \times \overline{n-2} \times \overline{n-3} \times \frac{pQ - n \times P}{P}$. Hoc vero aggregatum bis ſumatur, quoniam duo ſunt valores irrationalis quantitatis $\sqrt{4 + \alpha\beta + \gamma^2}$; & conſequenter ſumma quæſita erit $36 \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}\, p + 6 \times \overline{n-2} \times \overline{n-3} \times \frac{pQ - nP}{P}$.

5*. Sint $\alpha, \beta, \gamma, \delta$, &c. radices æquationis $(V = 0)$ $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0$; & A, B, Γ, Δ, &c. radices æquationis $(W = 0)$ $z^m - Pz^{m-1} + Qz^{m-2} - Rz^{m-3} + \&c. \ldots \pm Tz \mp V = 0$; & ſic de pluribus æquationibus: invenire aggregatum ex ſingulis valoribus cujuſcunque algebraicæ functionis prædictarum radicum.

In-

Inveniatur per problema præcedens aggregatum ex ſingulis valoribus datæ functionis; poſito, quod omnes radices ſingularum præter primam æquationum ſint datæ quantitates, & ſit quantitas reſultans a; deinde inveniatur aggregatum ex ſingulis valoribus quantitatis a, poſito, quod omnes radices ſingularum præter ſecundam æquationum ſint datæ quantitates, & ſit quantitas reſultans b: & ſic de reliquis: tum quantitas ultimo reſultans erit aggregatum quæſitum.

Ex. Invenire aggregatum e ſingulis valoribus functionis $\overline{\alpha^2 A + 3\alpha BA} \times \overline{\alpha A + 7\beta\Gamma} = \alpha^3 A^2 + 3\alpha^2 BA^2 + 7\alpha^2\beta A\Gamma + 21\alpha\beta AB\Gamma$; ubi α, β; A, B & Γ denotant eaſdem ac in problemate quantitates.

Inveniatur aggregatum ex ſingulis valoribus hujuſce functionis; poſito, quod A, B, Γ ſint datæ quantitates: aggregatum ex ſingulis valoribus hujuſce functionis ex prædictâ hypotheſi erit $\overline{n-1} \times \overline{\alpha^3 + \beta^3 + \gamma^3 + \&c.} \times A^2 + 3 \times \overline{n-1} \times \overline{\alpha^2 + \beta^2 + \gamma^2 + \&c.} \times BA^2 + 7 \times \overline{\alpha^2\beta + \beta^2\alpha + \alpha^2\gamma + \gamma^2\alpha + \beta^2\gamma + \gamma^2\beta + \&c.} \times A\Gamma + 21 \overline{\alpha\beta + \alpha\gamma + \beta\gamma + \alpha\delta + \beta\delta + \gamma\delta + \&c.} \times AB\Gamma = \overline{n-1} \times \overline{p^3 - 3pq + 3r} \times A^2 + 3 \times \overline{n-1} \times \overline{p^2 - 2q} \times BA^2 + 7 \times \overline{pq - 3r} \times A\Gamma + 21 \times q \times AB\Gamma = a$: deinde fingatur omnes præter A, B & Γ radices æquationis $(W = o)$ eſſe datas quantitates, ex hâc hypotheſi inveniantur ſinguli valores quantitatis a; eorum ſumma $\overline{m-1} \times \overline{m-2} \times \overline{n-1}\, \overline{p^3 - 3pq + 3r} \times \overline{A^2 + B^2 + \Gamma^2 + \&c.} + 3 \times \overline{m-2} \times \overline{n-1}\, \overline{p^2 - 2q} \times \overline{BA^2 + AB^2 + A^2\Gamma + \Gamma^2 A + B^2\Gamma + \Gamma^2 B + \&c.} + 7 \times \overline{m-2} \times \overline{pq - 3r} \times \overline{AB + A\Gamma + B\Gamma + \&c.} + 21q \times AB\Gamma = \overline{m-1} \times \overline{m-2} \times \overline{n-1} \times \overline{p^3 - 3pq + 3r} \times \overline{P^2 - 2Q} + 3 \times \overline{m-2} \times \overline{n-1}\, \overline{p^2 - 2q} \times \overline{PQ - 3R} + 7 \times \overline{m-2} \times \overline{pq - 3r} \times Q + 21q \times R$ erit aggregatum quæſitum.

Ex. 2. Iiſdem poſitis; ſit data functio $\alpha + 2A + \frac{7\beta}{B} + 5\sqrt{A^2 + \gamma^2}$:

primo

primo fingatur A & B eſſe datas quantitates: per caſ. 3. problematis præcedentis conſtat ſummam e ſingulis valoribus cujuſcunque irrationalis quantitatis $\sqrt{A^2+\gamma^2}$ nihilo evadere æqualem: aggregatum e ſingulis valoribus functionis $\alpha + 2A + \frac{7\beta}{B} + \sqrt{A^2+\gamma^2}$ ex hâc hypotheſi erit $2 \times \overline{n-1} \times \overline{n-2} \times \overline{\alpha+\beta+\gamma+\delta+ \&c.} + 4 \times n \times \overline{n-1} \times \overline{n-2} \times A + 2 \times 7 \times \overline{n-1} \times \overline{n-2} \times \frac{\alpha+\beta+\gamma+\delta+\&c.}{B} = 2 \times \overline{n-1} \times \overline{n-2} \times p + 4 \times n \times \overline{n-1} \times \overline{n-2} \times A + 2 \times 7 \times \overline{n-1} \times \overline{n-2} \times \frac{p}{B} = a$; deinde fingatur omnes quantitates aggregati (a) eſſe datas, præter A & B; tum erit aggregatum e ſingulis valoribus quantitatis $(a) = 2 \times m . \overline{m-1} \times \overline{n-1} \times \overline{n-2} \times p + 4 \times \overline{m-1} \times n \times \overline{n-1} \times \overline{n-2} \times \overline{A+B+\Gamma+\Delta+\&c.} + 2 \times 7 \times \overline{m-1} \times \overline{n-1} \times \overline{n-2} \times \overline{\frac{p}{A}+\frac{p}{B}+\frac{p}{\Gamma}+\frac{p}{\Delta}+ \&c.} = 2 \times m . \overline{m-1} \times \overline{n-1} \times \overline{n-2}\, p + 4 \times \overline{m-1} \times n . \overline{n-1} \times \overline{n-2} \times P + 2 \times 7 \times \overline{m-1} \times \overline{n-1} \times \overline{n-2} \times p \times \frac{T}{V}$.

Cor. Hinc inveſtigari poteſt aggregatum e ſingulis valoribus infinitæ poteſtatum ſeriei datæ æquationis radicum; ſi modo illa infinita poteſtatum ſeries per algebraicam radicum functionem exprimi poſſit.

Inveniatur algebraica radicum functio, quæ exprimit infinitam poteſtatum ſeriem; per hocce problema inveniri poteſt aggregatum e ſingulis ejus valoribus, & conſequenter aggregatum e ſingulis valoribus infinitæ poteſtatum ſeriei.

E. G. Sit æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - tx^{n-5} + \&c. = o$. cujus radices ſint $\alpha, \beta, \gamma, \delta, \epsilon$, &c. & ſit a ſumma ejus radicum, b ſumma quadratorum e ſingulis ejus radicibus, c ſumma cuborum, d ſumma

ma quadrato-quadratorum, e ſumma quadrato-cuborum, & ſic in infinitum.

Invenire ſummam $a + b + c + d + e +$ &c. in infinitum. Hæc infinita ſeries exprimi poteſt per fractionum ſummam

$$\frac{1}{1-\alpha} + \frac{1}{1-\beta} + \frac{1}{1-\gamma} + \frac{1}{1-\delta} + \frac{1}{1-\epsilon} + \&c. - n;$$ inveniatur

æquatio, cujus radix eſt $v = \frac{1}{1-x}$; ſumma radicum reſultantis æquationis erit

$$\frac{1}{1-\alpha} + \frac{1}{1-\beta} + \frac{1}{1-\gamma} + \frac{1}{1-\delta} + \frac{1}{1-\epsilon} + \&c. = \frac{n - \overline{n-1}p + \overline{n-2}q - \overline{n-3}r + \overline{n-4}s - \overline{n-5}t + \&c.}{1 - p + q - r + s - t + \&c.}$$

& conſequenter ſumma quæſita

$$\frac{1}{1-\alpha} + \frac{1}{1-\beta} + \frac{1}{1-\gamma} + \frac{1}{1-\delta} + \frac{1}{1-\epsilon} + \&c. - n = \frac{p - 2q + 3r - 4s + 5t - \&c.}{1 - p + q - r + s - t + \&c.}$$

Ex. 2. Invenire ſummam $a + 2b + 3c + 4d + 5e +$ &c. in infinitum.

Hæc ſeries exprimi poteſt per differentiam inter duas ſeries fractionum

$$\frac{1}{\overline{1-\alpha}^2} + \frac{1}{\overline{1-\beta}^2} + \frac{1}{\overline{1-\gamma}^2} + \frac{1}{\overline{1-\delta}^2} + \&c. - \frac{1}{1-\alpha} - \frac{1}{1-\beta} - \frac{1}{1-\gamma} - \frac{1}{1-\delta} -$$

unde ſumma quæſita erit

$$\frac{\overline{n - \overline{n-1}p + \overline{n-2}q - \overline{n-3}r + \overline{n-4}s - \overline{n-5}t + \&c.}^2}{\overline{1 - p + q - r + s - t\ \&c.}^2} -$$

$$\frac{n \times \overline{n-1} - \overline{n-1} \times \overline{n-2}p + \overline{n-2} \times \overline{n-3}q - \overline{n-3} \times \overline{n-4}r + \overline{n-4} \times \overline{n-5}s - \&c.}{1 - p + q - r + s - t + \&c.} - \frac{n - \overline{n-1}p + \overline{n-2}q - n-}{1 - p + q - r + }$$

PROB. V.

Datâ unâ, duabus, vel pluribus æquationibus; invenire æquationem, cujus radices ſint quæcunque algebraica radicum datarum æquationum functio.

Scribantur α, β, γ, δ, &c. pro radicibus unius datæ æquationis; A, B, Γ, Δ, &c. pro radicibus alterius datæ æquationis; & ſic de reliquis.

Ex

Ex datâ algebraicâ relatione inter radices datæ æquationis vel datarum æquationum, & radices quæſitæ æquationis inveniantur omnes quæſitæ æquationis radices; & conſequenter earum numerus, i. e. dimenſiones quæſitæ æquationis; & e problematibus præcedentibus inveniatur e ſingulis hiſce radicibus aggregatum, quod eſt coefficiens ſecundi quæſitæ æquationis termini.

Ducantur quæque duæ radices quæſitæ æquationis in ſe invicem; & e prædictis problematibus inveniatur aggregatum rectangulorum reſultantium; erit coefficiens tertii quæſitæ æquationis termini.

Ducantur quæque tres, quatuor, quinque, &c. radices quæſitæ æquationis continuo in ſe invicem; & ex iiſdem problematibus inveniantur aggregata reſultantium contentorum; hæc aggregata erunt reſpective coefficientes quarti, quinti, ſexti, &c. quæſitæ æquationis terminorum.

Hinc dilucide conſtat methodus inveniendi quæſitæ æquationis coefficientes, & conſequenter ipſam æquationem.

Hujuſce generis problemata ſæpe reſolvi poſſunt e reductione duarum vel plurium æquationum in unam, ita ut incognitæ quantitates exterminentur.

Ex. 1. 1. Sint $\alpha, \beta, \gamma, \delta$, &c. radices datæ æquationis

$$x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \text{\&c.} = 0,$$

invenire æquationem

$$v^n - Pv^{n-1} + Qv^{n-2} - Rv^{n-3} + Sv^{n-4} - \text{\&c.} = 0,$$ cujus radices ſint (α^m, β^m, γ^m, δ^m, &c. reſpective) m poteſtates datæ æquationis radicum.

Scribatur

$$\begin{aligned} a &= \alpha^m + \beta^m + \gamma^m + \delta^m + \text{\&c.} \\ b &= \alpha^{2m} + \beta^{2m} + \gamma^{2m} + \delta^{2m} + \text{\&c.} \\ c &= \alpha^{3m} + \beta^{3m} + \gamma^{3m} + \delta^{3m} + \text{\&c.} \\ d &= \alpha^{4m} + \beta^{4m} + \gamma^{4m} + \delta^{4m} + \text{\&c.} \\ \text{\&c.} & \qquad \text{\&c.} \end{aligned}$$

E problematis tertii corollario inveniantur quæſitæ æquationis coefficientes (P, Q, R, S, &c.); erunt reſpective

 $P =$

$P=\alpha^m+\beta^m+\gamma^m+\delta^m+\&c.=a$

$\mathcal{Q}=\alpha^m\times\beta^m+\alpha^m\times\gamma^m+\beta^m\times\gamma^m+\alpha^m\times\delta^m+\beta^m\times\delta^m+\gamma^m\times\delta^m+\&c.=\frac{a^2-b}{2}.$

$R=\alpha^m\times\beta^m\times\gamma^m+\alpha^m\times\beta^m\times\delta^m+\alpha^m\times\gamma^m\times\delta^m+\beta^m\times\gamma^m\times\delta^m+\&c.=\frac{a^3-3ab+2c}{6}$

$S=\alpha^m\times\beta^m\times\gamma^m\times\delta^m+\&c. \qquad =\frac{a^4-6a^2b+8ac-6d+3b^2}{24}$

& ſic de reliquis coefficientibus.

2. E problemate tertio deduci poteſt alia hujuſce exempli ſolutio.

Sint $b, k, l, o,$ &c. omnes diverſæ radices (m) poteſtatis unitatis: E. G. ſi (m) ſit 3; b, k, l erunt 1, $\frac{-1+\sqrt{-3}}{2}$, $\frac{-1-\sqrt{-3}}{2}$ reſpective.

Ex hypotheſi $x^m=v$, & conſequenter $bv^{\frac{1}{m}}=x$, $kv^{\frac{1}{m}}=x$, $lv^{\frac{1}{m}}=x$, $ov^{\frac{1}{m}}=x$, &c.

Quibus valoribus in datâ æquatione pro quantitate (x) ſubſtitutis, reſultant æquationes

$b^n v^{\frac{n}{m}}-p\times b^{n-1} v^{\frac{n-1}{m}}+q\times b^{n-2} v^{\frac{n-2}{m}}-r\times b^{n-3} v^{\frac{n-3}{m}}+\&c.=0$

$k^n v^{\frac{n}{m}}-p\times k^{n-1} v^{\frac{n-1}{m}}+q\times k^{n-2} v^{\frac{n-2}{m}}-r\times k^{n-3} v^{\frac{n-3}{m}}+\&c.=0$

$l^n v^{\frac{n}{m}}-p\times l^{n-1} v^{\frac{n-1}{m}}+q\times l^{n-2} v^{\frac{n-2}{m}}-r\times l^{n-3} v^{\frac{n-3}{m}}+\&c.=0$

$o^n v^{\frac{n}{m}}-p\times o^{n-1} v^{\frac{n-1}{m}}+q\times o^{n-2} v^{\frac{n-2}{m}}-r\times o^{n-3} v^{\frac{n-3}{m}}+\&c.=0$

& ſic deinceps.

Ducantur hæ æquationes in ſe invicem; æquatio reſultans erit $v^n-Pv^{n-1}+\mathcal{Q}v^{n-2}-Rv^{n-3}+Sv^{n-4}-\&c.=0$, cujus radices ſunt (m) poteſtates datæ æquationis radicum.

E problemate tertio inveſtigari poſſunt cofficientes (P, $\mathcal{Q}$, R, S, &c.); & quo minor eſt numerus (m), eo magis ſimplex eſt lex, quam obſervant quæſitæ æquationis coefficientes. E. G. Sit $m=2$, numerales coefficientes in literis P, $\mathcal{Q}$; R, S, &c. contentæ erunt 1, vel 2; ſit $m=3$, & coefficientes erunt 1, vel 3; ſit $m=4$, & coefficientes erunt 1, vel 2, vel 4; ſit $m=5$, & coefficientes erunt 1, vel 5; &c.

3. E

3. E problemate primo alia conſectatur ſolutio.

Cor. Hinc deleri poſſunt quæcunque ſurdæ quantitates e datâ æquatione.

Sit ſurda quantitas delenda $x^{\frac{1}{n}}$, & data æquatio $Ax^{\frac{n-1}{n}} - Bx^{\frac{n-2}{n}} + Cx^{\frac{n-3}{n}} - Dx^{\frac{n-4}{n}} + \&c. = Q$; aſſumatur $v = x^{\frac{1}{n}}$, & inveniatur ab exemplo æquatio, cujus radix eſt v^n; pro v^n ſubſtituatur x, & quod oportet, factum eſt.

Ex. 2. Invenire æquationem, cujus radices ſint ($\alpha+\beta$, $\alpha+\gamma$, $\beta+\gamma$, $\alpha+\delta$, $\beta+\delta$, $\gamma+\delta$, &c.) ſummæ e quibuſcunque duabus datæ æquationis $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0$ radicibus: numerus quæſitæ æquationis radicum, & conſequenter ejus dimenſiones erit æqualis fractioni $n \times \frac{n-1}{2}$.

Summa radicum eſt $\overline{n-1} \times \overline{\alpha+\beta+\gamma+\delta+\epsilon+\&c.} = \overline{n-1} \times p$; quæ eſt coefficiens ſecundi quæſitæ æquationis termini.

Ducantur quæque duæ radices quæſitæ æquationis in ſe invicem; aggregatum reſultantium rectangulorum ($\overline{\alpha+\beta} \times \overline{\alpha+\gamma}$, $\overline{\alpha+\beta} \times \overline{\alpha+\delta}$, $\overline{\alpha+\gamma} \times \overline{\alpha+\delta}$, $\overline{\alpha+\beta} \times \overline{\gamma+\delta}$, &c.) erit

$$\overline{n-1} \times \frac{n-2}{2} \times \overline{\alpha^2+\beta^2+\gamma^2+\delta^2+\&c.} + n \times \overline{n-2}\,\overline{\alpha\beta+\alpha\gamma+\beta\gamma+\alpha\delta+\beta\delta+\gamma\delta+\&c.}$$

e problemate primo prodit

$$\overline{n-1} \times \frac{n-2}{2} \times \overline{\alpha^2+\beta^2+\gamma^2+\delta^2+\&c.} = \overline{n-1} \times \frac{n-2}{2} \times \overline{p^2-2q}$$

$$n \times \overline{n-2} \times \overline{\alpha\beta+\alpha\gamma+\beta\gamma+\alpha\delta+\beta\delta+\gamma\delta+\&c.} = n \times \overline{n-2} \times q$$

quare aggregatum erit $\overline{n-1} \times \frac{n-2}{2} p^2 + \overline{n-2} \times q$

Hoc aggregatum eſt coefficiens tertii quæſitæ æquationis termini.

Ducantur quæque tres radices quæſitæ æquationis in ſeſe ($\overline{\alpha+\beta} \times \overline{\alpha+\gamma} \times \overline{\alpha+\delta}$, $\overline{\alpha+\beta} \times \overline{\alpha+\gamma} \times \overline{\beta+\delta}$, $\overline{\alpha+\beta} \times \overline{\alpha+\gamma} \times \overline{\gamma+\delta}$, $\overline{\alpha+\beta} \times \overline{\beta+\gamma} \times \overline{\alpha+\delta}$, &c.) & aggregatum horum contentorum eſt

$$\overline{n-1} \times \frac{n-2}{2} \times \frac{n-3}{3} \times \overline{\alpha^3+\beta^3+\gamma^3+\delta^3+\&c.} + \overline{n-2} \times \frac{n^2-2n-1}{2} \times \overline{\alpha^2\beta+\alpha^2\gamma+\alpha^2\delta+\beta^2\alpha+\beta^2\gamma+\beta^2\delta+\gamma^2\alpha+\gamma^2\beta+\&c.} + \overline{n-1} \times n^2-2\,\overline{n-2} \times \overline{\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta+\&c.}$$

e problemate primo & tertio colligitur

 $n-1$

$\overline{n-1} \times \frac{n-2}{2} \times \frac{n-3}{3} \times \overline{\alpha^3 + \beta^3 + \gamma^3 + \delta^3 + \&c.} = \overline{n-1} \times \frac{n-2}{2} \times \frac{n-3}{3} \times \overline{p^3 - 3pq + 3r}$

$\overline{n-2} \times \frac{n^2-2n-1}{2} \times \overline{\alpha^2\beta + \alpha^2\gamma + \beta^2\gamma + \alpha^2\delta + \beta^2\delta + \&c.} = \overline{n-2} \times \frac{n^2-2n-1}{2} \times \overline{pq - 3r}$

$\overline{n-1} \times \overline{n^2-2n-2} \times \overline{\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta + \&c.} = \overline{n-1} \times \overline{n^2-2n-2} \times r$

quare aggregatum eſt $\overline{\overline{n-1} \times \frac{n-2}{2} \times \frac{n-3}{3} p^3 + \overline{n-2}^2 \times pq + \overline{n-4} \times r}$.

Hoc vero aggregatum eſt coefficiens quarti quæſitæ æquationis termini.

Eodem modo inveniri poſſunt reliquæ coefficientes: ſit v radix quæſitæ æquationis; æquatio quæſita erit

$$v^{n \times \frac{n-1}{2}} - \overline{n-1} \times p v^{n \times \frac{n-1}{2} - 1} + \overline{\overline{n-1} \times \frac{n-2}{2} p^2 + \overline{n-2}\, q} \times v^{n \times \frac{n-1}{2} - 2} -$$

$$\overline{\overline{n-1} \times \frac{n-2}{2} \times \frac{n-3}{3} p^3 + \overline{n-2}^2 pq + \overline{n-4}\, r} \times v^{n \times \frac{n-1}{2} - 3} + \&c. = 0.$$

Ex. 3. Data æquatione $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - tx^{n-5} + \&c. = 0$, cujus radices ſint $\alpha, \beta, \gamma, \delta, \epsilon$, &c.; invenire æquationem, cujus radices ſint $\overline{\alpha-\beta}^2, \overline{\alpha-\gamma}^2, \overline{\beta-\gamma}^2, \overline{\alpha-\delta}^2, \overline{\beta-\delta}^2, \overline{\gamma-\delta}^2$, &c.

Summa radicum

$\overline{n-1} \times \overline{\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \&c.}\ (\overline{n-1}\, p^2 - 2 \times \overline{n-1}\, q)$

$-2\, \overline{\alpha\beta + \alpha\gamma + \beta\gamma + \alpha\delta + \beta\delta + \gamma\delta + \&c.}\ (-2q)$ erit $\overline{n-1}\, p^2 - 2nq$; quæ eſt coefficiens ſecundi æquationis termini.

Ducantur quæque duæ radices quæſitæ æquationis in ſeſe, reſultant quantitates $(\overline{\alpha^2 + \beta^2 - 2\alpha\beta} \times \overline{\alpha^2 + \gamma^2 - 2\alpha\gamma}, \overline{\alpha^2 + \delta^2 - 2\alpha\delta} \times \overline{\beta^2 + \delta^2 - 2\beta\delta}, \overline{\alpha^2 + \delta^2 - 2\alpha\delta} \times \overline{\alpha^2 + \gamma^2 - 2\alpha\gamma}, \overline{\beta^2 + \gamma^2 - 2\beta\gamma} \times \overline{\beta^2 + \delta^2 - 2\beta\delta}$, &c.) $\overline{n-1} \times \frac{n-2}{2} \times \overline{\alpha^4 + \beta^4 + \gamma^4 + \delta^4 + \&c.}$

$-2 \times \overline{n-2} \times \overline{\alpha^3 \times \overline{\beta + \gamma + \delta + \&c.} + \beta^3 \times \overline{\alpha + \gamma + \delta + \&c.} + \gamma^3 \times \overline{\alpha + \beta + \delta + \&c.} + \delta^3 \times \overline{\alpha + \beta + \gamma + \&c.} + \&c.} - 2 \times \overline{n-3} \times$

$\overline{\alpha^2 \times \overline{\beta\gamma + \beta\delta + \gamma\delta + \&c.} + \beta^2 \times \overline{\alpha\gamma + \alpha\delta + \gamma\delta + \&c.} + \gamma^2 \times}$

$\gamma^2 \times \overline{\alpha\beta + \alpha\delta + \beta\delta + \&c.} + \delta^2 \times \overline{\alpha\beta + \alpha\gamma + \beta\gamma + \&c.} +$ $\&c. + 12 \times \overline{\alpha\beta\gamma\delta + \&c.} + n \times \overline{n-2} \times \overline{\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2 + \alpha^2\delta^2} \overline{+ \beta^2\delta^2 + \&c.}$

e problematibus præcedentibus erui poſſunt

$$\overline{n-1} \times \frac{n-2}{2} \times \overline{\alpha^4 + \beta^4 + \gamma^4 + \delta^4 + \&c.} = \overline{n-1} \times \frac{n-2}{2} \times \overline{p^4 - 4p^2q + 4pr - 4s + 2q^2}$$

$$-2 \times \overline{n-2} \times \overline{\alpha^3 \times \overline{\beta + \gamma + \delta + \&c.} + \beta^3 \times \overline{\alpha + \gamma + \delta + \&c.} + \gamma^3 \times \overline{\alpha + \beta + \delta + \&c.} \&c.} = -2 \times \overline{n-2} \times \overline{p^2q - pr + 4s - 2q^2}$$

$$-2 \times \overline{n-3} \times \overline{\alpha^2 \times \overline{\beta\gamma + \beta\delta + \gamma\delta + \&c.} + \beta^2 \times \overline{\alpha\gamma + \alpha\delta + \beta\delta + \&c.} + \gamma^2 \times \overline{\alpha\beta + \alpha\delta + \beta\delta + \&c.} \&c.} = -2 \times \overline{n-3} \times \overline{pr - 4s}$$

$$+ n \times \overline{n-2} \times \overline{\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2 + \alpha^2\delta^2 + \beta^2\gamma^2 + \delta^2\gamma^2 + \&c.} = -n \times \overline{n-2} \times \overline{2rp - 2s - q^2}$$

$$+ 12 \times \overline{\alpha\beta\gamma\delta + \alpha\beta\gamma\epsilon + \&c.} = 12 \times s$$

Quare aggregatum eſt $\overline{n-1} \times \frac{n-2}{2} p^4 - 2n \times \overline{n-2}\, p^2 q - 2 \times \overline{n-3}\, pr + 2ns + \overline{2n+3} \times \overline{n-2}\, q^2$, ſed hoc aggregatum eſt coefficiens tertii quæſitæ æquationis termini: ſit v radix quæſitæ æquationis, & æquatio quæſita erit

$$v^{n \times \frac{n-1}{2}} - \overline{n-1\, p^2 - 2nq} \times v^{n \times \frac{n-1}{2} - 1} + \overline{\overline{n-1} \times \frac{n-2}{2} p^4 - 2n \times \overline{n-2}\, p^2 q - 2 \times \overline{n-3}\, pr + 2ns + \overline{2n+3} \times \overline{n-2}\, q^2} \times v^{n \times \frac{n-1}{2} - 2} - \&c. = 0.$$

Ex. 4. Sit æquatio $x^{n-1} - \frac{n-1}{n} \times p x^{n-2} + \frac{n-2}{n} \times q x^{n-3} - \frac{n-3}{n} \times r x^{n-4} + \frac{n-4}{n} \times s x^{n-5} - \frac{n-5}{n} \times t x^{n-6} + \&c. = 0$; cujus radices ſint $\alpha, \beta, \gamma, \delta, \epsilon$, &c. invenire æquationem, cujus radices ſint reſpective

$$\alpha^n - p\alpha^{n-1} + q\alpha^{n-2} - r\alpha^{n-3} + s\alpha^{n-4} - t\alpha^{n-5} + \&c. + \mathcal{Q},$$
$$\beta^n - p\beta^{n-1} + q\beta^{n-2} - r\beta^{n-3} + s\beta^{n-4} - t\beta^{n-5} + \&c. + \mathcal{Q}.$$
$$\gamma^n - p\gamma^{n-1} + q\gamma^{n-2} - r\gamma^{n-3} + s\gamma^{n-4} - t\gamma^{n-5} + \&c. + \mathcal{Q},$$
$$\delta^n - p\delta^{n-1} + q\delta^{n-2} - r\delta^{n-3} + s\delta^{n-4} - t\delta^{n-5} + \&c. + \mathcal{Q}.$$

&c. &c.

Summa

coefficiens quantitatis (p^n) erit $-\frac{\overline{n-1}^{\,n-1}}{n^n}$; in poſteriori caſu coefficiens erit $\pm\frac{\overline{n-1}^{\,n-m}}{n^{n-m}}$. Coefficiens autem quantitatis Q erit $n-1$.

Signum affixum erit affirmativum vel negativum, prout ſumma $m+\mu+\nu+\pi+\varrho+$ &c. fuerit impar vel par numerus. Facile conſtat hoc exemplum ex additione terminorum.

Ex. 5. Sit æquatio

$$x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0,$$

& generaliter per hoc problema inveniri poteſt æquatio

$$v^m + Pv^{m-1} + Qv^{m-2} + \&c. \dots Rv^2 + Sv + T = 0$$

cujus radix $v = x^m + ax^{m-1} + bx^{m-2} + cx^{m-3} + \&c.$

Cor. Sit ultimus terminus (T) reſultantis æquationis nihilo æqualis, tum ſimplicem diviſorem neceſſario habebunt duæ quantitates

$$x^n - px^{n-1} + qx^{n-2} - \&c.$$
$$\&\ x^m + ax^{m-1} + bx^{m-2} + \&c.;$$

ſint duo ultimi reſultantis æquationis termini $(T\ \&\ S)$ nihilo æquales, & ſaltem quadraticum diviſorem habent prædictæ quantitates; ſint tres ultimi termini nihilo æquales, & ſaltem cubicum diviſorem habent prædictæ quantitates; & ſic deinceps.

Et ſic de communibus plurium quantitatum diviſoribus inveniendis.

Ex. 6. Datâ æquatione $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \&c. = 0$, cujus radices ſint $\alpha, \beta, \gamma, \delta$, &c.; invenire æquationem, cujus radices ſint $\alpha-\beta, \alpha-\gamma, \beta-\gamma, \alpha-\delta, \beta-\delta$, &c.

$$\beta-\alpha,\ \gamma-\alpha,\ \gamma-\beta,\ \delta-\alpha,\ \delta-\beta,\ \&c.$$

numerus radicum erit $n \times \overline{n-1}$.

Summa radicum erit nihilo æqualis, quæ eſt coefficiens ſecundi æquationis termini: etiamque conſtat coefficientes quarti, ſexti, & cujuſlibet paris æquationis termini nihilo æquales eſſe.

Ducantur quæque duæ radices quæſitæ æquationis in ſe invicem,

$$2\times\overline{\alpha\beta+\alpha\gamma+\beta\gamma+\alpha\delta+\beta\delta+\&c.}\times\overline{AB+A\Gamma+A\Delta+B\Gamma+B\Delta+\&c.}=2Qq$$

Quorum aggregatum eſt $p^2Q+P^2q-2Qq.$
quod eſt tertii quæſitæ æquationis termini coefficiens.

Ex eâdem ratiocinandi methodo coefficiens quarti quæſitæ æquationis termini invenitur

$P^3r+p^3R-3PQr-3pqR+pqPQ+3Rr$: & ſic deinceps.

Sit v radix quæſitæ æquationis, quæſita æquatio erit

$$v^{nm}-Ppv^{nm-1}+\left.\begin{matrix}P^2q-2Qq\\+p^2Q\end{matrix}\right\rbrace v^{nm-2}-\left.\begin{matrix}P^3r-3PQr+3Rr\\p^3R-3pqR+pqPQ\end{matrix}\right\rbrace v^{nm-3}+=0.$$

Eâdem methodo deduci poſſint quæcunque transformationes hujuſce generis.

SCHOLIUM.

In multis caſibus ſubſequenti methodo inveniri poſſunt facilius æquationis quæſitæ coefficientes; aſſumantur æquationes datæ formulæ, quarum radices cognitæ ſunt; & exinde inveniatur quæſita æquatio. E pluribus hujuſce generis reſultantibus æquationibus inveniri poſſint terminorum coefficientes, & conſequenter æquatio ipſa.

1. Hæc autem methodus plerumque magis uſui inſervit, cum dimenſiones datæ æquationis haud generaliter aſſumantur.

E. G. Sit data æquatio $x^4+qx^2-rx+s=0$; cujus radices ſint $\alpha, \beta, \gamma, \delta$; invenire æquationem, cujus radices ſint $\alpha+\beta$, $\alpha+\gamma$, $\beta+\gamma$, $\alpha+\delta$, $\beta+\delta$, $\gamma+\delta$.

Aſſumo æquationem biquadraticam datæ formulæ $x^4-6x^2+8x-3=0$, cujus radices erunt $1, 1, 1, -3$; radices reſultantis æquationis erunt $1+1, 1+1, 1+1, 1-3, 1-3, 1-3$; vel $2, 2, 2, -2, -2, -2$; & exinde reſultans æquatio erit $v^6-12v^4+48v^2-64=0$: radices ($\alpha+\beta$, $\alpha+\gamma$, &c.) reſultantis æquationis unam ſolummodo habent dimenſionem; & conſequenter earum ſumma, quæ eſt coefficiens ſecundi reſultantis æquationis termini, unam ſolummodo habebit dimenſionem; ſed nullus invenitur terminus in datâ æquatione, quæ non habet plures quam unam dimenſionem; ergo ſumma radicum reſultantis æquationis ſemper nihilo æqualis eſt. Summa rectangulorum ſub ſingulis binis radicibus reſultantis æquationis,

æquationis, quæ eſt coefficiens tertii reſultantis æquationis termini, duas habet dimenſiones; unus autem ſolummodo invenitur terminus (q) in datâ æquatione, qui haud plures quam duas habet dimenſiones: ergo coefficiens tertii termini reſultantis æquationis erit $a \times q$; ſed coefficiens aſſumptæ æquationis, cujus radices cognitæ ſunt, & quæ habet duas dimenſiones, fuit -6; & coefficiens tertii termini ejus reſultantis æquationis fuit -12; ergo $-6\ (q) \times a = -12$; & conſequenter $a = 2$; & quæſita æquatio $v^6 + 2qv^4$ &c. $= o$.

Contentum ſub ſingulis tribus quæſitæ æquationis radicibus ($\overline{\alpha+\beta} \times \overline{\alpha+\gamma} \times \overline{\beta+\gamma} +$ &c.), quod eſt coefficiens quarti reſultantis æquationis termini, tres habet dimenſiones; unus autem ſolummodo invenitur terminus (r) in datâ æquatione, qui tres poſſit conficere dimenſiones; ergo coefficiens tertii reſultantis æquationis termini erit $a \times r$: ſed in aſſumptâ æquatione $r = 8$, & ejus reſultantis æquationis quarti termini coefficiens fuit o, ergo $8 \times a = o$, & conſequenter $a = o$.

Contentum ſub ſingulis quatuor quæſitæ æquationis radicibus ($\overline{\alpha+\beta} \times \overline{\alpha+\gamma} \times \overline{\beta+\gamma} \times \overline{\alpha+\delta} +$ &c.) quatuor habet dimenſiones; e terminis datæ æquationis duobus modis confici poſſunt quatuor dimenſiones, *i. e.* vel coefficiens (s), vel (q^2) quatuor habent dimenſiones; ergo contentum ſub ſingulis quatuor datæ æquationis radicibus, quod eſt coefficiens quinti reſultantis æquationis termini, erit $aq^2 + bs$; & quoniam duæ ſunt quantitates hujuſmodi, quæ conficiunt quatuor dimenſiones, neceſſe eſt, ut totidem aſſumantur æquationes, quarum radices cognitæ ſunt, & exinde inveniantur earum reſultantes æquationes: aſſumo igitur alteram æquationem $x^4 - 2x^2 + 1 = o$; cujus radices ſunt 1, 1, -1, -1; & radices æquationis reſultantis 2, 0, 0, 0, 0, -2; & æquatio ipſa $v^6 - 4v^4 = o$; e priori aſſumptâ æquatione colligitur $a \times 36\ (q^2) + b \times -3\ (s) = 48$, quæ eſt coefficiens quinti termini ejus reſultantis æquationis; e poſteriori vero æquatione ſequitur $a \times 4\ (q^2) + b \times 1\ (s) = o$; & ex his duabus æquationibus conſtat $a = 1$, & $b = -4$; conſequenter coefficiens quinti reſultan-

 tis

tis æquationis termini erit $q^2 - 4s$: & quæſita æquatio

$$v^6 + 2qv^4 + \overline{q^2 - 4s} \times v^2 \text{ \&c.} = o.$$

Contentum ſub ſingulis quinque reſultantis æquationis radicibus, quod eſt coefficiens ſexti reſultantis æquationis termini, quinque habet dimenſiones; ſed quinque dimenſiones unico modo e terminis datæ æquationis oriri poſſunt, *i. e.* rectangulum $q \times r$ ſolummodo quinque habet dimenſiones, & exinde $a \times q \times r$ erit coefficiens ſexti æquationis termini: ergo e priori aſſumptâ æquatione $a \times -6 \times 8\ (a \times qr) = o$, & exinde $a = o$.

Contentum ſub ſex radicibus, quod eſt coefficiens ſeptimi reſultantis æquationis termini, ſex habet dimenſiones; totidem autem dimenſiones habent quantitates q^3, r^2, & sq, & nullæ aliæ in datâ æquatione contentæ; ergo coefficiens ultimi æquationis reſultantis termini erit $aq^3 + br^2 + cqs$: ſed quoniam tres ſint quantitates, quæ habent ſex dimenſiones; neceſſe eſt, ut aſſumantur tres æquationes, quarum radices cognitæ ſint: aſſumo igitur tertiam æquationem $x^4 - 5x^2 + 4 = o$, cujus radices ſunt $1, 2, -1, -2$: & æquationis reſultantis radices erunt $3, 0, -1, +1, 0, -3$, & æquatio ipſa $v^6 - 10v^4 + 9v^2 = o$. E primâ aſſumptâ æquatione colligitur $a \times -216\ (aq^3) + b \times 64\ (br^2) + c \times 18\ (cqs) = -64$ coefficienti ſeptimi ejus reſultantis æquationis termini: e ſecundâ vero $a \times -8\ (aq^3) + b \times o\ (br^2) + c \times -2\ (cqs) = o$: e tertiâ vero $a \times -125\ (aq^3) + b \times o\ (br^2) + c \times -20\ (cqs) = o$; & exinde a, b, c erunt reſpective $o, -1, o$: & æquatio quæſita

$$v^6 + 2qv^4 + \overline{q^2 - 4s} \times v^2 - r^2 = o.$$

2. Datâ æquatione inferioris (n) ordinis, ex ejus reſultante æquatione ſæpe facile inveniri poſſunt multi termini æquationis ſimiliter reſultantis ex æquatione ſuperioris $(n+r)$ ordinis.

Ducatur data æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \text{\&c.} = o$ in æquationem (r) dimenſionum, cujus radices cognitæ ſint; maxime frequenter præſtat, æquationem (r) dimenſionum eſſe $x^r = o$, cujus omnes radices ſunt $= o$; ex æquatione reſultanti inveniatur æquatio, cujus

cujus radices datam habeant relationem ad radices datæ æquationis; & ex hâc reſultanti æquatione ſequuntur ſæpe quam plurimi termini æquationis ſimiliter reſultantis ex æquatione $(n+r)$ dimenſionum.

E.G. Sit æquatio data inferioris ordinis

$x^4 + qx^2 - rx + s = 0$, cujus radices ſint a, β, γ, δ; & æquatio, cujus radices ſint $\alpha+\beta, \alpha+\gamma, \beta+\gamma, \alpha+\delta, \beta+\delta, \gamma+\delta$, erit

$$v^6 + 2qv^4 + \overline{q^2 - 4s}\, v^2 - r^2 = 0.$$

Ducatur data æquatio $x^4 + qx^2 - rx + s = 0$ in ſimplicem æquationem $(x = 0)$, cujus radix eſt (0); & ſit æquatio $x^5 + qx^3 - rx^2 + sx = 0$, cujus radices erunt $\alpha, \beta, \gamma, \delta, 0$; & radices æquationis ſimiliter exinde reſultantis erunt $\alpha+\beta, \alpha+\gamma, \beta+\gamma, \alpha+\delta, \beta+\delta, \gamma+\delta, \alpha, \beta, \gamma, \delta$; & æquatio ipſa

$$\overline{v^4 + qv^2 - rv + s} \times \overline{v^6 + 2qv^4 + \overline{q^2 - 4s} \times v^2 - r^2} = 0;$$

cujus æquationis omnes termini iidem erunt, ac termini æquationis ſimiliter reſultantis e datâ æquatione $x^5 + qx^3 - rx^2 + sx - t = 0$; in quibus continentur ſolummodo literæ q, r, s.

Per æquationem ſimiliter reſultantem in hoc caſu deſignatur æquatio, cujus radices ſunt $\lambda+\mu, \lambda+\nu, \lambda+\pi, \lambda+\rho, \mu+\nu, \mu+\pi, \mu+\rho, \nu+\pi, \nu+\rho, \pi+\rho$; ſi modo $\lambda, \mu, \nu, \pi, \rho$ ſint reſpective radices æquationis $x^5 + qx^3 - rx^2 + sx - t = 0$.

CAP. II.

DE IMPOSSIBILIBUS RADICIBUS.

THEOR. I.

IT æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \&c. = 0$, *cujus radix ſit impoſſibilis quantitas* $a + \sqrt{-b^2}$; *impoſſibilis quantitas* $- \sqrt{-b^2}$ *etiam erit radix datæ æquationis.*

Pro (x) in data æquatione ſcribantur $a + \sqrt{-b^2}$, & $a - \sqrt{-b^2}$ pective; e priori ſubſtitutione reſultat

$$- n \times \frac{n-1}{2} a^{n-2}b^2 \qquad + n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} a^{n-4}b^4 - \&c$$

$$n a^{n-1} \sqrt{-b^2} \qquad - n \times \frac{n-1}{2} \times \frac{n-2}{3} a^{n-3}b^2 \sqrt{-b^2} \qquad + \&c.$$

$$+ \overline{n-1} \times \frac{n-2}{2} pa^{n-3}b^2 \qquad - \overline{n-1} \times \frac{n-2}{2} \times \frac{n-3}{3} \times \frac{n-4}{4} pa^{n-5}b^4 + \&c$$

$$\overline{-1} pa^{n-2} \sqrt{-b^2} \qquad + \overline{n-1} \times \frac{n-2}{2} \times \frac{n-3}{3} pa^{n-4}b^2 \sqrt{-b^2} \qquad - \&c.$$

$$- \overline{n-2} \times \frac{n-3}{2} qa^{n-4}b^2 \qquad + \overline{n-2} \times \frac{n-3}{2} \times \frac{n-4}{3} \times \frac{n-5}{4} qa^{n-6}b^4 - \&c$$

$$\overline{-2} qa^{n-3} \sqrt{-b^2} \qquad - \overline{n-2} \times \frac{n-3}{2} \times \frac{n-4}{3} qa^{n-5}b^2 \sqrt{-b^2} \qquad + \&c.$$

$$+ \overline{n-3} \times \frac{n-4}{2} ra^{n-5}b^2 \qquad - \overline{n-3} \times \frac{n-4}{2} \times \frac{n-5}{3} \times \frac{n-6}{4} ra^{n-7}b^4 + \&c$$

$$\overline{-3} ra^{n-4} \sqrt{-b^2} \qquad + \overline{n-3} \times \frac{n-4}{2} \times \frac{n-5}{3} ra^{n-6}b^2 \sqrt{-b^2} \qquad - \&c.$$

&c. &c.

ſultat

$$- n \times \frac{n-1}{2} a^{n-2}b^2 \qquad + n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} a^{n-4}b^4 - \&$$

$$n a^{n-1} \sqrt{-b^2} \qquad + n \times \frac{n-1}{2} \times \frac{n-2}{3} a^{n-3}b^2 \sqrt{-b^2}$$

$$+ \overline{n-1} \times \frac{n-2}{2} pa^{n-3}b^2 \qquad - \overline{n-1} \times \frac{n-2}{2} \times \frac{n-3}{3} \times \frac{n-4}{4} pa^{n-5}b^4 + \&$$

$$\overline{-1} pa^{n-2} \sqrt{-b^2} \qquad - \overline{n-1} \times \frac{n-2}{2} \times \frac{n-3}{3} pa^{n-4}b^2 \sqrt{-b^2}$$

$$- \overline{n-2} \times \frac{n-3}{2} qa^{n-4}b^2 \qquad + \overline{n-2} \times \frac{n-3}{2} \times \frac{n-4}{3} \times \frac{n-5}{4} qa^{n-6}b^4 -$$

$$\overline{-2} qa^{n-3} \sqrt{-b^2} \qquad + \overline{n-2} \times \frac{n-3}{2} \times \frac{n-4}{3} qa^{n-5}b^2 \sqrt{-b^2}$$

$$+ \overline{n-3} \times \frac{n-4}{2} \times ra^{n-5}b^2 \qquad - \overline{n-3} \times \frac{n-4}{2} \times \frac{n-5}{3} \times \frac{n-6}{4} ra^{n-7}b^4 +$$

$$\overline{-3} \times ra^{n-4} \sqrt{-b^2} \qquad - \overline{n-3} \times \frac{n-4}{2} \times \frac{n-5}{3} ra^{n-6}b^2 \sqrt{-b^2} \qquad + \&c.$$

&c. &c.

E pri

E priori, & poſteriori ſubſtitutione eædem reſultant poſſibiles quantitates, eædemque etiam (ſignis vero mutatis) reſultant impoſſibiles quantitates: conſequenter, ſi prior ſubſtitutio diluat & poſſibiles & impoſſibiles reſultantes terminos, *i. e.* $(a+\sqrt{-b^2})$ ſit radix datæ æquationis; poſterior $(a-\sqrt{-b^2})$ etiam erit radix æquationis.

Cor. 1. Hinc omnis æquatio, vel habet nullas, vel parem numerum impoſſibilium radicum; omnis enim impoſſibilis radix $(a+\sqrt{-b^2})$ alteram habet impoſſibilem radicem $(a-\sqrt{-b^2})$ ſibi ipſi cognatam.

Cor. 2. Ducantur duæ radices $x-a+\sqrt{-b^2}$, & $x-a-\sqrt{-b^2}$ in ſeſe, reſultat quadratica rationalis quantitas $x^2 - 2ax + a^2 + b^2$; & exinde conſtat has radices in æquatione latere ſub ſimilitudine, vel duarum negativarum radicum, vel duarum affirmativarum, & nunquam ſub ſimilitudine unius negativæ, alterius vero affirmativæ, latere.

Cor. 3. Sit æquatio, $x^{2n} - px^{2n-1} + qx^{2n-2} - rx^{2n-3} + sx^{2n-4} - \&c. = 0$, cujus omnes radices ſint impoſſibiles, *i. e.* ſint radices $a+\sqrt{-A^2}$, $a-\sqrt{-A^2}$; $b+\sqrt{-B^2}$, $b-\sqrt{-B^2}$; $c+\sqrt{-C^2}$, $c-\sqrt{-C^2}$; $d+\sqrt{-D^2}$, $d-\sqrt{-D^2}$; $e+\sqrt{-E^2}$, $e-\sqrt{-E^2}$; &c., conſequenter ejus quadratici diviſores erunt $\overline{x-a}^2 + A^2$, $\overline{x-b}^2 + B^2$; $\overline{x-c}^2 + C^2$, $\overline{x-d}^2 + D^2$, $\overline{x-e}^2 + E^2$, &c. Ducantur hi diviſores in ſeſe, reſultat data æquatio,

$$\begin{array}{llll}
\overline{x-a}^2 \times \overline{x-b}^2 \times \overline{x-c}^2 \times \overline{x-d}^2 \times \overline{x-e}^2 \times \&c. & + A^2 \times & \overline{x-b}^2 \times \overline{x-c}^2 \times \overline{x-d}^2 \times \overline{x-e}^2 \times \&c. & + A^2B^2 \times \overline{x-c}^2 \times \overline{x-d}^2 \times \overline{x-e}^2 \times \&c. \\
 & + B^2 \times \overline{x-a}^2 \times & \overline{x-c}^2 \times \overline{x-d}^2 \times \overline{x-e}^2 \times \&c. & + A^2C^2 \times \overline{x-b}^2 \times \overline{x-d}^2 \times \overline{x-e}^2 \times \&c. \\
 & + C^2 \times \overline{x-a}^2 \times \overline{x-b}^2 \times & \overline{x-d}^2 \times \overline{x-e}^2 \times \&c. & + B^2C^2 \times \overline{x-a}^2 \times \overline{x-d}^2 \times \overline{x-e}^2 \times \&c. \\
 & + D^2 \times \overline{x-a}^2 \times \overline{x-b}^2 \times \overline{x-c}^2 \times & \overline{x-e}^2 \times \&c. & + A^2D^2 \times \overline{x-b}^2 \times \overline{x-c}^2 \times \overline{x-e}^2 \times \&c. \\
 & + E^2 \times \overline{x-a}^2 \times \overline{x-b}^2 \times \overline{x-c}^2 \times \overline{x-d}^2 \times & & \&c. \\
 & \&c. & &
\end{array}$$

$$\begin{array}{ll}
+ A^2B^2C^2 \times \overline{x-d}^2 \times \overline{x-e}^2 \times \&c. & + \&c. \\
+ A^2B^2D^2 \times \overline{x-c}^2 \times \overline{x-e}^2 \times \&c. & + \&c. \\
+ \&c. & \&c.
\end{array}$$

&c

& conſequenter conſtat data æquatio e quadratis, *i. e.* erit aggregatum e diverſis quadratis ($\overline{x-a}^2 \times \overline{x-b}^2 \times \overline{x-c}^2 \times \overline{x-d}^2 \times \overline{x-e}^2 \times$ &c., $A^2 \times \overline{x-b}^2 \times \overline{x-c}^2 \times \overline{x-d}^2 \times \overline{x-e}^2 \times$ &c., $B^2 \times \overline{x-a}^2 \times \overline{x-c}^2 \times \overline{x-d}^2 \times \overline{x-e}^2 \times$ &c., $C^2 \times \overline{x-a}^2 \times \overline{x-b}^2 \times \overline{x-d}^2 \times \overline{x-e}^2 \times$ &c., $D^2 \times \overline{x-a}^2 \times \overline{x-b}^2 \times \overline{x-c}^2 \times \overline{x-e}^2 \times$ &c., &c., $A^2 B^2 \times \overline{x-c}^2 \times \overline{x-d}^2 \times \overline{x-e}^2 \times$ &c., &c.)
quorum numerus ſecundum hanc deductionem erit 2^n, ni aliqua quadrata nihilo ſint æqualia.

Cor. Æquatio, quæ habet poſſibilem radicem nullo modo poteſt eſſe quadratorum aggregatum; ni omnis poſſibilis radix alteram habet ſibi ipſi æqualem.

2. 1. In genere ſit æquatio quæcunque functio quantitatis x, in quâ haud continetur irrationalis quantitas hujuſce formulæ $\sqrt{-b^2}$; tum, ſi $a + \sqrt{-c^2}$ ſit radix datæ æquationis, $a - \sqrt{-c^2}$ etiam erit radix ejuſdem æquationis: 2. ſi vero contineatur quantitas $\sqrt{-b^2}$ in datâ æquatione, in eâ pro $\sqrt{-b^2}$ ſcribatur $-\sqrt{-b^2}$, & reſultet æquatio $B = o$; tum ſi $a + \sqrt{-b^2}$ ſit radix datæ æquationis, erit $a - \sqrt{-b^2}$ radix reſultantis æquationis $B = o$.

Conſtat hic caſus e ſubſtitutione & expanſione terminorum.

3. 1. Omnis finita æquatio, quæ habet unam radicem (α) ab irrationalibus quantitatibus haud in eâ contentis expreſſam; habebit etiam tot radices ejuſdem formulæ, quot ex earum multiplicatione in prædictam α rationalem quantitatem præbebunt.

THEOR.

Omnis æquatio parium ($2n$) dimenſionum ($x^{2n} - px^{2n-1} + qx^{2n-2} - rx^{2n-3} +$ &c. $= o$), cujus radices ſunt poſſibiles, poteſt eſſe differentia inter duo poſſibilia quadrata; quorum alterum habeat $2n$ dimenſiones, alterum vero $2n-2$ dimenſiones.

Sint n radices datæ æquationis α, β, γ, δ, &c.; n vero reliquæ π, ρ, σ, τ, &c.; ducantur n radices in ſeſe $\overline{x-\alpha} \times \overline{x-\beta} \times \overline{x-\gamma} \times \overline{x-\delta} \times$ &c. $= x^n - ax^{n-1} + bx^{n-2} - cx^{n-3} + dx^{n-4} -$ &c.; etiamque n reliquæ in ſeſe

ſeſe $\overline{x-\pi}\times\overline{x-\rho}\times\overline{x-\sigma}\times\overline{x-\tau}\times$ &c. $= x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} -$ &c.: nunc fingatur differentia inter duo quadrata quæſita $(x^n - a'x^{n-1} + b'x^{n-2} - c'x^{n-3} + d'x^{n-4} -$ &c.$)^2 - (A'x^{n-1} - B'x^{n-2} + C'x^{n-3} - D'x^{n-4} +$ &c.$)^2 = 0$, cujus duo diviſores erunt

$$\begin{matrix} x^n & - a'x^{n-1} & + b'x^{n-2} & - c'x^{n-3} & + d'x^{n-4} & - \text{\&c.} = 0, \text{ \&} \\ & - A' & + B' & - C' & + D' & \end{matrix}$$

$$\begin{matrix} x^n & - a'x^{n-1} & + b'x^{n-2} & - c'x^{n-3} & + d'x^{n-4} & - \text{\&c.} = 0, \\ & + A' & - B' & + C' & - D' & \end{matrix}$$

aſſumantur hi duo diviſores duobus prædictis contentis $x^n - ax^{n-1} + bx^{n-2} - cx^{n-3} +$ &c. $= 0$ & $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} +$ &c. $= 0$ reſpective æquales; conſequenter eorum coefficientes correſpondentes inter ſe evadent æquales, *i. e.* $a' + A' = a$ & $a' - A' = A$ (unde $a' = \frac{a+A}{2}$ & $A' = \frac{a-A}{2}$); $b' + B' = b$ & $b' - B' = B$ (unde $b' = \frac{b+B}{2}$ & $B' = \frac{b-B}{2}$); $c' + C' = c$ & $c' - C' = C$ (unde $c' = \frac{c+C}{2}$ & $C' = \frac{c-C}{2}$), & ſic deinceps; & exinde data æquatio erit æqualis differentiæ inter duo poſſibilia quadrata, quorum alterum habeat $2n$ dimenſiones, alterum vero $2n - 2$ dimenſiones.

Cor. Prædicta æquatio confici poteſt e differentiâ inter duo quadrata prædicti generis, quæ variari poſſint $\frac{1}{2} \times 2n \times \frac{2n-1}{2} \times \frac{2n-2}{3} \times \frac{2n-3}{4} \times \ldots\ldots \times \frac{n+1}{n}$ diverſis modis; tot enim diverſis modis reduci poteſt æquatio $2n$ dimenſionum in duas æquationes n dimenſionum.

Cor. 2. Si vero duæ ejus radices fiant inter ſe æquales, tum $\overline{2n-3} \times \frac{2n-4}{2} \times \frac{2n-5}{3} \ldots \frac{n}{n-1}$ e prædictis variationibus bis evadent inter ſe eædem: facile etiam conſtat numerus variationum, quæ eædem evadent; cum tres, vel duæ bis, vel quatuor, &c. radices datæ æquationis fiant inter ſe æquales.

2. Sint

2. Sint $2m$ radices datæ æquationis impoſſibiles & $2n - 2m$ poſſibiles; aſſumantur (r) impoſſibiles datæ æquationis radices $g + h\sqrt{-1}$, $k + l\sqrt{-1}$, &c., & earum (r) correſpondentes impoſſibiles radices $g - h\sqrt{-1}$, $k - l\sqrt{-1}$, &c.; etiamque $n - 2r$ poſſibiles radices α, β, γ, δ, &c.; & $(n - 2m + 2r)$ poſſibiles radices π, ρ, σ, τ, &c.: ſint reliquæ $(2m - 2r)$ impoſſibiles radices reſpective $G + H\sqrt{-1}$, $G - H\sqrt{-1}$; $K + L\sqrt{-1}$, $K - L\sqrt{-1}$; &c.; ducantur $(2r)$ prædictæ impoſſibiles correſpondentes & $(n - 2r)$ poſſibiles, nihilo æquales redditæ, continuo in ſeſe; & evadet $\overline{x - g + h\sqrt{-1}} \times \overline{x - g - h\sqrt{-1}} \times \overline{x - k + l\sqrt{-1}} \times \overline{x - k - l\sqrt{-1}} \times$ &c. $\times\, \overline{x - \alpha} \times \overline{x - \beta} \times \overline{x - \gamma} \times \overline{x - \delta} \times$ &c. $= x^n - ax^{n-1} + bx^{n-2} - cx^{n-3} +$ &c. $= 0$; etiamque ducantur $2m - 2r$ reliquæ impoſſibiles radices & $(n - 2m + 2r)$ reliquæ poſſibiles, nihilo æquales redditæ, in ſeſe, & evadet $\overline{x - G + H\sqrt{-1}} \times \overline{x - G - H\sqrt{-1}} \times \overline{x - K + L\sqrt{-1}} \times \overline{x - K - L\sqrt{-1}} \times$ &c. $\times\, \overline{x - \pi} \times \overline{x - \rho} \times \overline{x - \sigma} \times \overline{x - \tau} \times$ &c. $= x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} +$ &c. $= 0$: nunc, ut prius, fingatur differentia inter duo quadrata quæſita $(x^n - a'x^{n-1} + b'x^{n-2} - c'x^{n-3} + \&c.)^2 - (A'x^{n-1} - B'x^{n-2} + C'x^{n-3} - D'x^{n-4} + \&c.)^2 = 0$, cujus diviſores erunt

$$\begin{array}{l} x^n \begin{array}{c} - a' \\ - A' \end{array} x^{n-1} \begin{array}{c} + b' \\ + B' \end{array} x^{n-2} \begin{array}{c} - c' \\ - C' \end{array} x^{n-3} + \&c. = 0, \ \& \\ x^n \begin{array}{c} - a' \\ + A' \end{array} x^{n-1} \begin{array}{c} + b' \\ - B' \end{array} x^{n-1} \begin{array}{c} - c' \\ + C' \end{array} x^{n-3} + \&c. = 0, \end{array}$$

aſſumantur hi duo diviſores duobus prædictis contentis $x^n - ax^{n-1} + bx^{n-2} - cx^{n-3} +$ &c. $= 0$, & $x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} +$ &c. $= 0$ reſpective æquales; conſequenter eorum coefficientes correſpondentes evadent inter ſe æquales, i. e. $a' + A' = a$ & $a' - A' = A$ (unde $a' = \frac{a + A}{2}$ & $A' = \frac{a - A}{2}$); $b' + B' = b$ & $b' - B' =$ (unde $b' = \frac{b + B}{2}$ & $B' = \frac{b - B}{2}$), & ſic deinceps.

Cor.

Cor. Hinc conſtat, omnem æquationem parium $(2n)$ dimenſionum poſſe differentiam eſſe inter duo poſſibilia quadrata prædictarum dimenſionum: ni æquatio ſit $(x^n + ax^{n-1} + bx^{n-2} + \&c.)^2 = o$; in quo caſu unum quadratum evaneſcit: vel ni $2n = 4r + 2$, ubi r ſit integer numerus, & omnes radices datæ æquationis ſint impoſſibiles.

3. Sit æquatio $x^{2n} - Px^{2n-1} + Qx^{2n-2} - \&c. = o$, cujus quadratum $(x^r - ax^{r-1} + bx^{r-2} - cx^{r-3} + \&c.)^2 = o$, in quo omnes coefficientes a, b, c, &c. ſunt poſſibiles, ſit diviſor: ſint $2n - 2r$ reliquæ radices datæ æquationis impoſſibiles, quarum $(n-r)$ impoſſibiles radices ſint $\alpha + \beta\sqrt{-1}$, $\gamma + \delta\sqrt{-1}$, $\varepsilon + \zeta\sqrt{-1}$, &c.; earum vero $(n-r)$ correſpondentes radices erunt $\alpha - \beta\sqrt{-1}$, $\gamma - \delta\sqrt{-1}$, $\varepsilon - \zeta\sqrt{-1}$, &c.: æquatio $x^{2n} - Px^{2n-1} + Qx^{2n-2} - \&c. = o$ erit ſumma duorum poſſibilium quadratorum, quorum alterum habet n dimenſiones, alterum vero $n-1$ dimenſiones: ſint enim duo quadrata poſſibilia $(x^n - d'x^{n-1} + b'x^{n-2} - c'x^{n-3} + \&c.)^2 + (A'x^{n-1} - B'x^{n-2} + C'x^{n-3} - \&c.)^2 = o$, ubi d', b', c', &c. A', B', C', &c. ſint poſſibiles quantitates, cujus æquationis diviſores erunt

$$\begin{array}{llll} x^n - d' & x^{n-1} + b' & x^{n-2} - c' & x^{n-3} + \&c. = o, \ \& \\ \quad - A'\sqrt{-1} & \quad + B'\sqrt{-1} & \quad - C'\sqrt{-1} & \end{array}$$

$$\begin{array}{llll} x^n - d' & x^{n-1} + b' & x^{n-2} - c' & x^{n-3} + \&c. = o, \\ \quad + A'\sqrt{-1} & \quad - B'\sqrt{-1} & \quad + C'\sqrt{-1} & \end{array}$$

ducatur $x^r - ax^{r-1} + bx^{r-2} - cx^{r-3} + \&c.$ in $(n-r)$ impoſſibiles radices nihilo æquales redditas $\overline{x - \alpha - \beta\sqrt{-1}} \times \overline{x - \gamma - \delta\sqrt{-1}} \times \overline{x - \varepsilon - \zeta\sqrt{-1}} \times \&c. = o$, & evadet $\overline{x^r - ax^{r-1} + bx^{r-2} - \&c.} \times \overline{x - \alpha - \beta\sqrt{-1}} \times \overline{x - \gamma - \delta\sqrt{-1}} \times \&c. = x^n - \overline{\varkappa + \pi\sqrt{-1}}\, x^{n-1} + \overline{\lambda + \varrho\sqrt{-1}}\, x^{n-2} - \overline{\mu + \sigma\sqrt{-1}}\, x^{n-3} + \overline{\nu + \tau\sqrt{-1}}\, x^{n-4} - \&c. = o$; ducatur etiam quantitas prædicta $x^r - ax^{r-1} + bx^{r-2} - \&c.$ in $(n-r)$ impoſſibiles radices præcedentibus correſpondentes nihilo æquales redditas $\overline{x - \alpha + \beta\sqrt{-1}} \times \overline{x - \gamma + \delta\sqrt{-1}} \times \overline{x - \varepsilon + \zeta\sqrt{-1}} \times \&c.$, & evadet $\overline{x^r - ax^{r-1} + bx^{r-2} - \&c.} \times \overline{x - \alpha + \beta\sqrt{-1}} \times$

$x - \gamma$

$\overline{x-\gamma+\delta\sqrt{-1}} \times \overline{x-\varepsilon+\zeta\sqrt{-1}} \times$ &c. $= x^n - \overline{\varkappa-\pi\sqrt{-1}}\, x^{n-1} + \overline{\lambda-\varrho\sqrt{-1}}\, x^{n-2} - \overline{\mu-\sigma\sqrt{-1}}\, x^{n-3} + \overline{\nu-\tau\sqrt{-1}}\, x^{n-4} -$ &c. $= 0$: aſſumantur duo diviſores prædicti duobus contentis $x^n - \overline{\varkappa+\pi\sqrt{-1}}\, x^{n-1} + \overline{\lambda+\varrho\sqrt{-1}}\, x^{n-2} -$ &c. $= 0$, & $x^n - \overline{\varkappa-\pi\sqrt{-1}}\, x^{n-1} + \overline{\lambda-\varrho\sqrt{-1}}\, x^{n-2} -$ &c. $= 0$ reſpective æquales, & conſequenter eorum coefficientes correſpondentes evadent inter ſe æquales, *i. e.* $a' = \varkappa$, $b' = \lambda$, $c' = \mu$, $d' = \nu$, &c.; etiamque $A' = \pi$, $B' = \varrho$, $C' = \sigma$, $D' = \tau$, &c., & exinde data æquatio æqualis erit ſummæ duorum quadratorum ſic inventorum.

Si æquatio ſit hujuſce generis, tum erit aggregatum e duobus vel pluribus poſſibilibus quadratis; ſin aliter non: & vice versâ, ſi correſpondens valor e ſingulis quantitatibus a', b', c', &c., A', B', C', &c. ſit poſſibilis; tum vel omnes radices datæ æquationis erunt impoſſibiles; vel in eâ ſemper continebitur par numerus e ſingulis poſſibilibus radicibus inter ſe æqualibus, *i. e.* haud continebitur poſſibilis radix, quæ haud plures habet ſibi ipſi æquales, quarum numerus eſt par.

Cor. Sit æquatio $x^4 + qx^2 + rx + s = \overline{x^2+\alpha}^2 + \overline{\beta x+\gamma}^2 = x^4 + \overline{2\alpha+\beta^2}\, x^2 + 2\beta\gamma x + \alpha^2 + \gamma^2 = 0$; unde reſultant tres æquationes $2\alpha + \beta^2 = q$, $2\beta\gamma = r$, $\alpha^2 + \gamma^2 = s$; quibus in unam reductis, ita ut exterminentur incognitæ quantitates γ & α, reſultat æquatio $\left(\frac{q-\beta^2}{2}\right)^2 + \frac{r^2}{4\beta^2} = s$, unde $\beta^6 - 2q\beta^4 + \overline{q^2-4s}\,\beta^2 + r^2 = 0$. Si modo detur correſpondens poſſibilis radix e tribus quantitatibus α, β & γ; tum erunt omnes radices datæ æquationis impoſſibiles; ni duæ radices (π) datæ æquationis ſint æquales, tum duæ reliquæ erunt $-\pi + \sqrt{-\varrho^2}$ & $-\pi - \sqrt{-\varrho^2}$.

2. Ita reducantur tres prædictæ reſultantes æquationes in unam, ut exterminentur incognitæ quantitates β & γ, etiamque α & β: reſultant duæ æquationes $\alpha^3 - \frac{q}{2}\alpha^2 - sa + \frac{1}{2}qs - \frac{r^2}{8} = 0$, & $\gamma^6 + (\frac{1}{4}q^2 - s)$

$-s)\,\gamma^4 - \frac{1}{8} r^2 q \gamma^2 + \frac{1}{64} r^4 = 0$. Sint π, ρ, σ & τ reſpectivæ radices datæ æquationis, tum erunt ſex valores quantitatis β reſpective, $\pm\overline{\pi+\rho}\sqrt{-1}$, $\pm\overline{\pi+\sigma}\sqrt{-1}$, $\pm\overline{\pi+\tau}\sqrt{-1}$; tres vero valores quantitatis α erunt $\frac{1}{2}(-\pi^2 - \pi\times\overline{\rho+\sigma}+\rho\sigma) = \frac{1}{2}\overline{\pi\tau+\rho\sigma}$, $\frac{1}{2}(-\rho^2 - \rho\times\overline{\pi+\sigma}+\pi\sigma) = \frac{1}{2}\overline{\rho\tau+\pi\sigma}$ & $\frac{1}{2}(-\sigma^2-\sigma\times\overline{\pi+\rho}+\pi\rho) = \frac{1}{2}\times\overline{\sigma\tau+\pi\rho}$; ſex vero valores quantitatis γ erunt $\pm\frac{1}{2}(\pi\rho+\pi\sigma+\rho\sigma+\pi^2)\sqrt{-1} = \pm\frac{1}{2}(\rho\sigma-\pi\tau)\sqrt{-1}$, $\pm\frac{1}{2}(\rho\tau-\pi\sigma)\sqrt{-1}$ & $\pm\frac{1}{2}(\sigma\tau-\pi\rho)\sqrt{-1}$.

THEOR. 2.

Sit æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} -$ &c. $= 0$, cujus poſſibiles radices ſint α, β, γ, δ, ε, &c. *i. e.* $\overline{x-\alpha}\times\overline{x-\beta}\times\overline{x-\gamma}\times\overline{x-\delta}\times\overline{x-\varepsilon}\times$ &c. $= x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} -$ &c. $= 0$, quarum radicum (α) major ſit quam β, β quam γ, γ quam δ, δ quam ε, &c.: ſubſtituatur in datâ æquatione pro (x), quantitas (A) major quam maxima affirmativa radix: omnes factores $A-\alpha$, $A-\beta$, $A-\gamma$, $A-\delta$, &c. erunt affirmativi; ergo eorum continuum contentum $(\overline{A-\alpha}\times\overline{A-\beta}\times\overline{A-\gamma}\times\overline{A-\delta}\times\overline{A-\varepsilon}\times$ &c.) erit affirmativa quantitas.

Sit autem A limes inter α & β, *i. e.* minor quam α, major autem quam β, & conſequenter $A-\alpha$ erit negativa quantitas; ſed $A-\beta$, $A-\gamma$, $A-\delta$, &c. erunt affirmativæ quantitates: ſubſtituatur (A) pro incognitâ quantitate (x) in datâ æquatione; quantitas reſultans $\overline{A-\alpha}\times\overline{A-\beta}\times\overline{A-\gamma}\times\overline{A-\delta}\times$ &c. erit negativa quantitas.

Sit A limes inter β & γ; & $A-\alpha$, & $A-\beta$ erunt negativi factores, cæteri vero omnes $A-\gamma$, $A-\delta$, &c. affirmativi; & eorum continuum contentum erit affirmativa quantitas, & ſic deinceps: conſequenter, ſi modo radices ſint poſſibiles, & ſubſtituantur limites inter ſucceſſivas radices pro incognitâ quantitate x, quantitates reſultantes continuo mutabuntur de + in —, & — in +.

 Cor.

Cor. Si duæ quantitates in datâ æquatione pro incognitâ quantitate (x) ſubſtitutæ mutent quantitates reſultantes de $+$ in $-$, vel $-$ in $+$; inter quantitates ſubſtitutas ſemper continetur impar radicum poſſibilium numerus: ſi vero quantitates reſultantes progrediantur de $+$ in $+$, vel $-$ in $-$; inter quantitates ſubſtitutas, vel nulla, vel par radicum numerus invenitur.

PROB. VI.

Sit æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} -$ &c. $= 0$: *cujus radices ſucceſſivæ poſſibiles ſint* $\alpha, \beta, \gamma, \delta,$ *&c. reſpective; invenire æquationem, cujus radices ſint limites inter ſucceſſivas poſſibiles datæ æquationis radices* α *&* β, β *&* γ, γ *&* δ, *&c.*

Ducantur termini ſucceſſivi datæ æquationis in ſucceſſivos arithmeticæ ſeriei terminos, n, $n-1$, $n-2$, $n-3$, &c. reſultat æquatio quæſita

$$nx^{n-1} - \overline{n-1}\, px^{n-2} + \overline{n-2} \times qx^{n-3} - \overline{n-3} \times rx^{n-4} + \text{\&c.} = 0.$$

Scribatur enim pro (x) in datâ æquatione $(z+a)$, & reſultat

$$\begin{array}{lllll}
x^n & = z^n + naz^{n-1} + n \times \frac{n-1}{2} a^2 z^{n-2} + \text{------} + & na^{n-1}z & + a^n \\
-px^{n-1} = & -pz^{n-1} - \overline{n-1} \times paz^{n-1} - \text{------} & - \overline{n-1}\, pa^{n-2}z & - pa^{n-1} \\
+qx^{n-2} = & + \qquad qz^{n-2} + \text{------} & + \overline{n-2}\, qa^{n-3}z & + qa^{n-2} \\
-rx^{n-3} = & - \qquad rz^{n-3} - \text{------} & - \overline{n-3}\, ra^{n-4}z & - ra^{n-3} \\
\text{\&c.} & & \text{\&c.} &
\end{array}$$

Sit 1.mo $+a = \alpha$ maxima poſſibilis radix datæ æquationis; & ultimus terminus reſultantis æquationis $(a^n - pa^{n-1} + qa^{n-2} - ra^{n-3} +$ &c. $= 0)$; coefficiens autem penultimi termini $nx^{n-1} - \overline{n-1}\, px^{n-2} + \overline{n-2} \times qx^{n-3} - \overline{n-3}\, rx^{n-4} +$ &c. $= \overline{\alpha - \beta} \times \overline{\alpha - \gamma} \times \overline{\alpha - \delta} \times \overline{\alpha - \varepsilon} \times$ &c.; quodſi omnes radices $\beta, \gamma, \delta, \varepsilon,$ &c. ſint poſſibiles, tum omnes hæ quantitates erunt affirmativæ, ergo earum contentum erit affirmativum: ſi igitur ſubſtituatur in æquatione $nx^{n-1} - \overline{n-1}\, px^{n-2} +$

$n-2$

$\overline{n-2}\times qx^{n-3}-\overline{n-3}\times rx^{n-4}+$ &c. $=0$ pro (x) maxima radix (α) datæ æquationis, reſultans quantitas erit affirmativa; ſi vero duæ vel quatuor vel plures radices ſint impoſſibiles, viz. $\pi+\sqrt{-\varrho^2}$ & $\pi-\sqrt{-\varrho^2}$; $\sigma+\sqrt{-\tau^2}$ & $\sigma-\sqrt{-\tau^2}$, &c.; tum etiam erit $na^{n-1}-\overline{n-1}\,pa^{n-2}+\overline{n-2}\,qa^{n-3}-$ &c. $=\overline{\alpha-\pi+\sqrt{-\varrho^2}}\times\overline{\alpha-\pi-\sqrt{-\varrho^2}}\times\overline{\alpha-\sigma+\sqrt{-\tau^2}}\times\overline{\alpha-\sigma-\sqrt{-\tau^2}}\times$ &c. $\times\overline{\alpha-\beta}\times\overline{\alpha-\gamma}\times\overline{\alpha-\delta}\times$ &c. $=(\overline{\alpha-\pi}^2+\varrho^2)\times(\overline{\alpha-\sigma}^2+\tau^2)\times$ &c. $\times\overline{\alpha-\beta}\times\overline{\alpha-\gamma}\times\overline{\alpha-\delta}\times$ &c. affirmativa; ea enim oritur ex multiplicatione quantitatum $\overline{\alpha-\pi}^2+\varrho^2$, $\overline{\alpha-\sigma}^2+\tau^2$, &c.; quarum ſingula eſt affirmativa quantitas.

Sit autem $+a=\beta$, & ultimus terminus $a^n-pa^{n-1}+qa^{n-2}-ra^{n-3}+$ &c. erit nihilo æqualis; penultimi termini coefficiens $(na^{n-1}-\overline{n-1}\,pa^{n-2}+\overline{n-2}\,qa^{n-3}-\overline{n-3}\,ra^{n-4}+$ &c.) erit æqualis contento $\overline{\beta-\alpha}\times\overline{\beta-\gamma}\times\overline{\beta-\delta}\times\overline{\beta-\varepsilon}\times$ &c.: hujus contenti factor $\beta-\alpha$ erit negativa quantitas, omnes vero reliqui affirmativæ quantitates; ergo eorum contentum erit negativum: ita quoque ſe res habet, cum duæ vel quatuor vel plures datæ æquationis radices ſint impoſſibiles: ſit $+a=\gamma$, & $na^{n-1}-\overline{n-1}\,pa^{n-2}+\overline{n-2}\,qa^{n-3}-\overline{n-3}\,ra^{n-4}+$ &c. $=\overline{\gamma-\alpha}\times\overline{\gamma-\beta}\times\overline{\gamma-\delta}\times\overline{\gamma-\varepsilon}\times$ &c. erit affirmativa quantitas; duo enim factores $\gamma-\alpha$ & $\gamma-\beta$ erunt negativæ quantitates, reliqui autem ($\gamma-\delta$, $\gamma-\varepsilon$, &c.) affirmativæ; & conſequenter eorum contentum $\overline{\gamma-\alpha}\times\overline{\gamma-\beta}\times\overline{\gamma-\delta}\times\overline{\gamma-\varepsilon}\times$ &c. erit affirmativum; ita quoque ſe res habet, cum duæ vel quatuor vel plures datæ æquationis radices ſint impoſſibiles: eodem modo ſubſtituantur reliquæ radices ſucceſſive datæ æquationis pro (x) in quantitate $nx^{n-1}-\overline{n-1}\,px^{n-2}+\overline{n-2}\,qx^{n-3}-\overline{n-3}\,rx^{n-4}+$ &c.; & quantitatum reſultantium ſigna continuo mutantur de $+$ in $-$, & $-$ in $+$; ergo poſſibiles radices datæ æquationis $x^n-px^{n-1}+qx^{n-2}-rx^{n-3}+$ &c. $=0$ erunt limites inter poſſibiles radices æquationis

nx

$nx^{n-1} - \overline{n-1}\,px^{n-2} + \overline{n-2}\,qx^{n-3} - \overline{n-3}\,rx^{n-4} + \&c. = 0$; & vice versâ radices æquationis $nx^{n-1} - \overline{n-1}\,px^{n-2} + \overline{n-2}\,qx^{n-3} - \overline{n-3} \times rx^{n-4} + \&c. = 0$ erunt limites inter radices æquationis $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0$.

Cor. 1. Hinc radices æquationis $nx^{n-1} - \overline{n-1}\,px^{n-2} + \&c. = 0$, erunt limites inter poſſibiles radices æquationis $x^n - px^{n-1} + \ldots Px - Q = 0$; utcunque variatur litera Q, quæ in priorem æquationem haud ingreditur.

Cor. 2. Hinc conſtat æquationem $(B = 0)$ $nx^{n-1} - \overline{n-1}\,px^{n-2} + \overline{n-2}\,qx^{n-3} - \&c. = 0$ habere unam vel tres vel quinque, &c. vel in genere imparem numerum poſſibilium radicum inter quaſcunque duas proxime ſucceſſivas radices æquationis $(A = 0)$ $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0$.

Cor. 3. Hinc conſtat, ſi modo æquatio $A = 0$ habeat m poſſibiles radices, æquationem $B = 0$ vel habere $m - 1$, vel $m + 1$, vel $m + 3$, vel $m + 5$, &c. poſſibiles radices.

Cor. 4. Hinc facile deduci poteſt, ſi modo detur mutatio ſignorum de $+$ in $-$ vel $-$ in $+$ a penultimo σx ad ultimum τ datæ æquationis $A = x^n - px^{n-1} + qx^{n-2} \ldots \pm \sigma x \mp \tau = 0$ terminum, tum inter minimam affirmativam & minimam negativam datæ æquationis $A = 0$ radicem, poni unam, tres vel quinque, &c. negativas radices; etiamque vel nullam vel duas vel quatuor, &c. affirmativas radices æquationis $B = nx^{n-1} - \overline{n-1}\,px^{n-2} + \overline{n-2}\,qx^{n-3} - \&c. = 0$: ſi vero in prædicto penultimo σx & ultimo τ terminis eadem dentur ſigna, tum inter prædictas minimam affirmativam & minimam negativam radicem contineri unam vel tres vel quinque, &c. affirmativas radices, etiamque vel nullam vel duas vel quatuor, &c. negativas radices.

THEOR.

1. Sint duæ æquationes $A = 0$ & $B = 0$, ubi A & B ſint functiones quantitatis x; in duabus quantitatibus bA & mB, ubi b & m ſunt

ſunt datæ quantitates, pro x ſcribatur eadem quantitas $\varkappa$, & duæ reſultantes quantitates eadem habeant ſigna; ſit α radix vel æquationis $A = o$ vel $B = o$ proxime major vel minor quam $\varkappa$; tum inter $\varkappa$ & α nulla datur radix æquationis $bA + mB = o$.

Si enim reſultantes quantitates bA & mB eadem habeant ſigna, tum earum ſumma $bA + mB$ haud poteſt eſſe nihilo æqualis, & conſequenter nulla radix æquationis $bA + mB = o$ inter $\varkappa$ & α continetur.

2. Sit α radix æquationis $A = o$, & π radix æquationis $B = o$, inter quas (α & π) nulla interponatur radix æquationum $A = o$ & $B = o$; ſubſtituatur quæcunque quantitas inter α & π contenta pro x in quantitatibus bA & mB, & ſi duæ quantitates reſultantes diverſa habeant ſigna, tum inter radices α & π continetur vel una vel tres vel quinque, &c. radices æquationis $bA + mB = o$.

Primo, cum $x = \alpha$, tum quantitas $bA = o$; deinde cum $x = \pi$, tum quantitas $mB = o$; ergo inter valores α & π quantitatis x, erit $bA = -mB$, & quoniam diverſa habent ſigna erit $bA + mB = o$, & conſequenter inter α & π continetur radix æquationis $bA + mB = o$.

Et ſic probari poteſt vel unam vel tres vel quinque, &c. radices æquationis $bA + mB = o$, inter α & π contineri: in quibuſdam caſibus duæ vel plures radices evadere poſſunt inter ſe æquales.

Ex iiſdem principiis demonſtrari poteſt, quodſi α & β ſint duæ proxime ſucceſſivæ radices æquationis $A = o$; inter quantitates (α & β) ponetur par vel impar radicum æquationis $bA + mB = o$ numerus, prout par vel impar fuerit numerus radicum (π, ϱ, σ, &c.) æquationis $B = o$ inter radices α & β poſitarum. Si in quantitatibus bA & mB pro x ſcribatur quantitas $\varkappa$ jacens inter prædictam radicem α & ei proximam radicem (π) æquationis $B = o$ inter α & β poſitam, & duæ quantitates reſultantes idem habeant ſignum; tum forſan inter α & β nulla radix æquationis $bA + mB = o$ ponitur, cum par ſit radicum prædictarum (π, ϱ, σ, &c.) inter α & β poſitarum numerus.

Cor. Sit æquatio (A) $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = o$, cujus radices ſint α, β, γ, δ, &c., quarum α major ſit quam β, β quam

γ,

γ, γ quam δ, & ſic deinceps: ſint π, ρ, σ, τ, &c. radices æquationis $(B)\ nx^{n-1} - \overline{n-1}\,px^{n-2} + \overline{n-2}\,qx^{n-3} - \overline{n-3}\,rx^{n-4} +$ &c. $= o$ inter α & β, β & γ, γ & δ, &c. reſpective poſitæ; tum, ſi b & m eadem habeant ſigna, erunt radices æquationis $bA + mB = o$ inter α & π, β & ρ, γ & σ, δ & τ, &c.; reſpective poſitæ: ſi quantitates b & m diverſa habeant ſigna, tum una radix æquationis $bA + mB = o$ major erit quam α; cæteræ vero inter π & β, ρ & γ, σ & δ, &c. reſpective ponentur.

Cor. 2. Ducatur quantitas B in x, & evadet $Bx = nx^n - \overline{n-1}\,px^{n-1} + \overline{n-2}\,qx^{n-2} -$ &c. $= o$, cujus radices erunt π, ρ, σ, τ, &c. & o; jam aſſumantur omnes poſſibiles radices vel affirmativæ vel negativæ; tum, iiſdem ac in præcedente coroll. poſitis, erunt radices æquationis $bA + mBx = o$ limites inter eaſdem radices æquationum $A = o$ & $B = o$, ac radices æquationis $bA + mB = o$.

Hinc, ſi modo ducantur ſucceſſivi termini datæ æquationis $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} +$ &c. $= o$ in ſucceſſivos arithmeticæ ſeriei a, $a+b$, $a+2b$, $a+3b$, &c. terminos, reſultat æquatio $ax^n - \overline{a+b}\,px^{n-1} + \overline{a+2b}\,qx^{n-2} - \overline{a+3b}\,rx^{n-3} +$ &c. $= bA + mBx = \overline{a+nb}\,A - bx \times B = o$, cujus radices erunt prædicti limites inter radices datæ æquationis, cum omnes poſſibiles radices datæ æquationis vel ſint affirmativæ, vel omnes negativæ.

Cor. 2. Æquationis reſultantis $ax^n - \overline{a+b}\,px^{n-1} + \overline{a+2b} \times qx^{n-2} -$ &c. $= o$ (n) erunt radices; ni vel $a = o$, vel $a + nb = o$; i. e. vel evaneſcat primus, vel ultimus reſultantis æquationis terminus: & in genere ita aſſumi poſſunt rationes inter a & b, ut quilibet numerus A ſit radix reſultantis æquationis, erit

$$\frac{b}{a} = \frac{A^n - pA^{n-1} + qA^{n-2} - rA^{n-3} + \&c.}{pA^{n-1} - 2qA^{n-2} + 3rA^{n-3} - \&c.}.$$

Cor. 3. Ducantur termini datæ æquationis in (m) ſucceſſivas arithmeticas ſeries, & dilui poſſunt quicunque (m) termini datæ æquationis: multiplicatio enim arithmeticæ uniuſcujuſque ſeriei diluere poteſt quemlibet datæ æquationis terminum: ſit enim terminus arith-

arithmeticæ ſeriei, qui in terminum æquationis diluendum ducatur, nihilo æqualis; & diluetur terminus datæ æquationis quæſitus: & ſic de reliquis.

E. G. Sit æquatio, $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \&c. = o$, & requiratur eliminatio termini rx^{n-3}; ducatur hæc æquatio in arithmeticam ſeriem $-3b$; $-2b$, $-b$, o, b, $2b$, $3b$, &c. cujus terminus ſit (o), qui ducatur in (rx^{n-3}) terminum datæ æquationis deſtruendum; reſultabit æquatio quæſita $-3bx^n + 2bpx^{n-1} - bqx^{n-2} * + bsx^{n-4} - \&c. = o$, cujus terminus ($rx^{n-3}$) deeſt, & cujus radices ſunt limites inter reſpectivas datæ æquationis radices.

Cor. 4. Sit æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \&c. = o$, cujus radices ſint α, β, γ, δ, ε, &c.: radices æquationis $ax^n - \overline{a+b}\, px^{n-1} + \overline{a+2b}\, qx^{n-2} - \overline{a+3b}\, rx^{n-3} + \&c. = o$ erunt limites inter α & β, β & γ, γ & δ, δ & ε, &c. reſpective: ducantur hujus æquationis ſucceſſivi termini in terminos alterius arithmeticæ ſeriei a^1, $a^1 + b^1$, $a^1 + 2b^1$, $a^1 + 3b^1$, &c. ſucceſſivos, & reſultat æquatio $a \times a^1x^n - \overline{a+b} \times \overline{a^1+b^1}\, px^{n-1} + \overline{a+2b} \times \overline{a^1+2b^1}\, qx^{n-2} - \overline{a+3b} \times \overline{a^1+3b^1}\, rx^{n-3} + \&c. = o$; cujus radices erunt limites inter α & γ, β & δ, γ & ε, &c. & ſic deinceps: & ſimiliter ſit data æquatio $x^n + px^{n-1} + qx^{n-2} + rx^{n-3} + sx^{n-4} \ldots Px^{n-r} + Qx^{n-r-1} + Rx^{n-r-2} + Sx^{n-r-3} + \&c. = o$, cujus radices ſint α, β, γ, δ, &c. π, ϱ, σ, τ, &c. quarum α major ſit quam β, β quam γ, γ quam δ, &c. π quam ϱ, ϱ quam σ, &c. & inter radices α & π, β & ϱ, γ & σ, δ & τ, &c. interponantur ($m-1$) radices; hujus æquationis ducantur termini ſucceſſivi in terminos duarum arithmeticarum ſerierum ſucceſſivos reſpective

$$n,\ n-1,\ n-2,\ n-3,\ n-4,\ \&c.$$
$$0,\ 1,\ 2,\ 3,\ 4,\ \&c.$$

& reſultabunt duæ æquationes ($n-1$) dimenſionum

$$nx^{n-1} + \overline{n-1}\, px^{n-2} + \overline{n-2}\, qx^{n-3} + \overline{n-3}\, rx^{n-4} + \&c. = o;$$
$$\&\quad + px^{n-1} + 2qx^{n-2} + 3rx^{n-3} + \&c. = o.$$

quarum radices erunt limites reſpective inter radices α & β, β & γ, γ & δ, δ & ε, &c. datæ æquationis.

Ducantur termini ſucceſſivi duarum æquationum reſultantium in ſucceſſivos terminos duarum prædictarum arithmeticarum ſerierum

$$n-1,\ n-2,\ n-3,\ \&c.$$
$$0,\quad 1,\quad 2,\quad 3,\ \&c.$$

& reſultabunt tres æquationes $(n-2)$ dimenſionum

$$n\times\overline{n-1}\,x^{n-2}+\overline{n-1}\times\overline{n-2}\,px^{n-3}+\overline{n-2}\times\overline{n-3}\,qx^{n-4}+\overline{n-3}\times\overline{n-4}\,rx^{n-5}+\&c.=0$$
$$\&\quad +\quad 1\times\overline{n-1}\,px^{n-2}+2\times\overline{n-2}\,qx^{n-3}+3\times\overline{n-3}\,rx^{n-4}+4\times\overline{n-4}\,sx^{n-5}+\&c.=0$$
$$\&\quad 1\times 2\,qx^{n-2}+2\times 3\,rx^{n-3}+3\times 4\,sx^{n-4}+4\times 5\,tx^{n-5}+\&c.=0$$

quarum radices erunt reſpective limites inter α & γ, β & δ, γ & ε, &c.

Et generaliter eodem prorſus modo inveniri poſſunt $(m+1)$ æquationes $(n-m)$ dimenſionum, $n\times\overline{n-1}\times\overline{n-2}\times..\overline{n-m+1}\,x^{n-m}+$ $\overline{n-1}\times\overline{n-2}\times\overline{n-3}\,..\,\overline{n-m}\,px^{n-m-1}+\overline{n-2}\times\overline{n-3}\times\overline{n-4}\,..$ $\overline{n-m-1}\,qx^{n-m-2}+\overline{n-3}\times\overline{n-4}\times\overline{n-5}\,..\,\overline{n-m-2}\,rx^{n-m-3}+$ $\overline{n-4}\times\overline{n-5}\,..\,\overline{n-m-3}\times sx^{n-m-4}+\&c.=0$, & $+1\times\overline{n-1}\times$ $\overline{n-2}\,..\,\overline{n-m+1}\,px^{n-m}+2\times\overline{n-2}\times\overline{n-3}\,..\,\overline{n-m}\,qx^{n-m-1}+3$ $\times\overline{n-3}\times\overline{n-4}\,..\,\overline{n-m-1}\times rx^{n-m-2}+4\times\overline{n-4}\times\overline{n-5}\,..$ $\overline{n-m-2}\,sx^{n-m-3}+5\times\overline{n-5}\times\overline{n-6}\,..\,\overline{n-m-3}\,tx^{n-m-4}+\&c.$ $=0$, & $1\times 2\times\overline{n-2}\times\overline{n-3}\,..\,\overline{n-m+1}\,qx^{n-m}+2\times 3\times\overline{n-3}$ $\times\overline{n-4}\,..\,\overline{n-m}\,rx^{n-m-1}+3\times 4\times\overline{n-4}\times\overline{n-5}\,..\,\overline{n-m-1}$ $sx^{n-m-2}+\&c.=0$; & in genere æquatio, cujus diſtantia e prima ſit (r), evadit $1.2.3\,..\,r\times\overline{n-r}\times\overline{n-r-1}\times\overline{n-r-2}\,..\,\overline{n-m+1}$ $Px^{n-m}+2.3.4\,..\,\overline{r+1}\times\overline{n-r-1}\times\overline{n-r-2}\times\overline{n-r-3}\,..\,\overline{n-m}$ $Qx^{n-m-1}+3.4.5\,..\,\overline{r+2}\times\overline{n-r-2}\times\overline{n-r-3}\times\overline{n-r-4}\,..\,\overline{n-m-1}$ $Rx^{n-m-2}+4.5.6\,..\,\overline{r+3}\times\overline{n-r-3}\times\overline{n-r-4}\times\overline{n-r-5}\,..\,\overline{n-m-2}$ $S\times x^{n-m-3}+5\times 6\times 7\,..\,\overline{r+4}\times\overline{n-r-4}\times\overline{n-r-5}\times\overline{n-r-6}\,..\,\overline{n-m-3}$ $Tx^{n-m-4}+\&c.=0$, quarum $(m+1)$ æquationum radices erunt limites inter α & π, β & ρ, γ & σ, δ & τ, &c.

Cor. 5. Si data æquatio $x^n-px^{n-1}+qx^{n-2}-\&c.\ (A)=0$, habeat & affirmativas & negativas radices, tum (a) inter minimam affirmativam

tivam ($+\theta$) & minimam negativam ($-\iota$) ponitur; ergo duæ erunt radices (o & $\pm\upsilon$) æquationis $B \times x = nx^n - \overline{n-1}\,px^{n-1} + \overline{n-2}\,qx^{n-2} -$ &c. $= o$ inter minimam affirmativam & minimam negativam datæ æquationis radicem: unde, ſi modo quantitates b & m eadem habeant ſigna, tum duæ radices æquationis $b \times A + m \times Bx = o$ inter prædictas radices $+\theta$ & $-\iota$ ponuntur; quarum una affirmativa, altera vero negativa erit: etiamque per cor. 4. prob. 6. ſi modo detur mutatio ſignorum de $+$ in $-$ vel $-$ in $+$ a penultimo ad ultimum datæ æquationis terminum; tum affirmativa radix inter θ & o, negativa vero inter $-\upsilon$ & $-\iota$ ponetur; ſin aliter, tum affirmativa radix inter θ & υ, negativa vero inter o & $-\iota$ ponentur.

2do. Si literæ b & m diverſa habeant ſigna, & b major ſit quam $m \times n$; tum nulla radix æquationis $bA - mBx = o$, inter θ & $-\iota$ ponitur; ſed ea habet affirmativam & negativam reſpective majorem maximâ affirmativâ & maximâ negativâ datæ æquationis radice: ſi $b = m \times n$, tum haud plures quam $n-1$ radices habet æquatio $bA - mBx = o$; nullam habet radicem inter θ & $-\iota$ poſitam, habet vero unam radicem majorem vel maximâ negativâ vel maximâ affirmativâ, prout detur mutatio ſignorum de $+$ in $-$ & $-$ in $+$ a primo ad ſecundum datæ æquationis terminum, necne: ſi mn major ſit quam b; tum vel 1mo duæ radices æquationis $bA - mBx = o$ inter θ & $-\iota$, ponuntur, quæ erunt utræque negativæ vel utræque affirmativæ, prout detur mutatio ſignorum de $+$ in $-$ & $-$ in $+$ a penultimo ad ultimum datæ æquationis terminum, necne; hoc vero fit, cum permagna ſit ratio quantitatis mn ad b: vel 2do utræque prædictæ radices evadent majores quam maxima negativa vel maxima affirmativa radix datæ æquationis; in hoc caſu ratio quantitatis $mn : b$ propior erit rationi æqualitatis: vel tertio utræque prædictæ radices evadent impoſſibiles, in quo caſu ratio quantitatis $mn : b$ erit inter duas prædictas rationes poſita.

Omnia ea, quæ prius tradita fuere de limitibus inter radices æquationum, quæ vel ſunt omnes affirmativæ vel omnes negativæ, æque

applicari poſſunt ad omnes limites inter radices harum æquationum, quæ nunquam evadant limites inter minimam affirmativam & minimam negativam datæ vel reſultantium æquationum radicem.

Cor. 6. Sit data æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0$: ducantur termini hujus æquationis ſucceſſive in terminos arithmeticæ ſeriei a, $a + b$, $a + 2b$, $a + 3b$, &c.: & ſi (m) radices ſint (α) in datâ æquatione, $(m - 1)$ radices etiam erunt (α) in reſultanti æquatione: ducatur æquatio reſultans in alteram arithmeticam ſeriem, & $(m - 2)$ radices etiam erunt (α) in æquatione deductâ; & ſic deinceps.

Æquationes, quarum radices ſunt limites inter radices datæ æquationis, etiam inveniri poſſunt; vel ducendo terminos datæ æquationis in terminos harmonicæ ſeriei, vel augendo, vel diminuendo, vel utcunque mutando datæ æquationis radices per quantitatem minorem quam differentiam inter duas quaſcunque ſucceſſivas datæ æquationis radices.

Cor. 7. Ducantur termini datæ æquationis (A) $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0$ ſucceſſive in arithmeticam ſeriem n, $n - 1$, $n - 2$, $n - 3$, &c. reſultat æquatio (B) $nx^{n-1} - \overline{n - 1}\,px^{n-2} + \overline{n - 2}\,qx^{n-3} - \&c. = 0$; ſi duæ, tres vel plures (m) radices (α) æquationis (B) $nx^{n-1} - \overline{n - 1}\,px^{n-2} + \&c. = 0$ ſint inter ſe æquales; & una radix datæ æquationis ſit α; ejus $(m + 1)$ radices etiam erunt α; poteſt vero nulla radix datæ æquationis eſſe α; hæ enim (m) æquales radices poſſunt eſſe limites inter duas, quarum una eſt poſſibilis, altera impoſſibilis; vel ſolummodo inter duas impoſſibiles. e. g. Sit data æquatio $\overline{x - a}^{m+1} \times \overline{x - b} + A = 0$, ubi m ſit integer numerus; tum erit $\overline{x - a}^{m+1} \times \overline{x - b} + A = x^{m+2} - (\overline{m + 1}\,a + b)\,x^{m+1} + (\overline{m + 1} \cdot \frac{m}{2} a^2 + \overline{m + 1}\,ab)\,x^m + \&c. = 0$; ducantur termini ejus ſucceſſivi in arithmeticam ſeriem $m + 2$, $m + 1$, m, $m - 1$, &c. & reſultabit $\overline{m + 2}\,x^{m+1} - \overline{m + 1}\,(\overline{m + 1}\,a + b)\,x^m + m\,(\overline{m + 1} \cdot \frac{m}{2} a^2 + \overline{m + 1}\,ab)\,x^{m-1} + \&c. = \frac{1}{x}(\overline{x - a}^{m+1}$

$\times \overline{x - b}$

$\times\ \overline{x-b}+A)$; unde, ſi $A=o$, erunt $(m+1)$ radices datæ æquationis $=\alpha$; ſin aliter nullæ.

Cor. 8. Sit æquatio $\overline{x-a}^{m+1}\times\overline{x^s-px^{s-1}+qx^{s-2}-\&c.}+A=o$; & ſaltem ejus $m-1$ radices erunt impoſſibiles, niſi ſit $A=o$.

Eadem principia etiam applicari poſſunt ad æquationes hujuſce generis $\overline{x-a}^{m+1}\times\overline{x-b}^{n+1}\times\overline{x-c}^{b+1}\times\overline{x^s-Px^{s-1}+Qx^{s-2}-\&c.}+A=o$; ſaltem enim $m-1+n-1+b-1+\&c.$ impoſſibiles radices continent.

Hoc cor. pendet ex hoc principio; quod æquatio $nx^{n-1}-\overline{n-1}\,px^{n-2}+\overline{n-2}\,qx^{n-3}-\&c.=o$, cujus radices ſunt limites inter radices datæ æquationis $x^n-px^{n-1}+qx^{n-2}-\&c.=o$, haud continet duas vel plures m radices (α) inter ſe æquales, ni $(m+1)$ radices datæ æquationis ſint α; α poteſt eſſe limes inter duas poſſibiles radices, in quo caſu reliquæ $(m-1)$ radices (α) ſemper erunt limites inter impoſſibiles & poſſibiles vel inter impoſſibiles ſolummodo radices.

Omnis regula, quæ generaliter invenit limites inter radices vel quantitates; neceſſario etiam inveniet, quando prædictæ radices vel quantitates fiant æquales.

THEOR.

1. Sit A quæcunque functio algebraica rationalis vel irrationalis quantitatis x; tum $\frac{\dot{A}}{\dot{x}}=o$ erit æquatio, quæ habet quaſdam radices inter radices æquationis $A=o$ limites; forſan vero habet omnes vel nonnullas radices, eaſdem, quas habet data æquatio $A=o$: duo enim vel plures valores e ſingulis radicibus æquationis $A=o$, poſſunt eſſe inter ſe æquales, in quo caſu æquatio $\frac{\dot{A}}{\dot{x}}=o$ habet omnes radices æquationis $A=o$.

2. Ita reducatur æquatio $A=o$, ut exterminentur omnes irrationales quantitates, & reſultat æquatio $P=o$, cujus radices eædem ſunt

ac

ac radices æquationis $A = o$; deinde ita reducatur æquatio $\frac{A}{\dot{x}} = o$, ut exterminentur omnes irrationales quantitates, & reſultabit æquatio ($\mathcal{Q} = o$), cujus radices eædem ſunt ac radices æquationis $\frac{A}{\dot{x}} = o$; hæ duæ reſultantes æquationes habebunt radices limites inter ſe.

Ex. Sit (A) $\overline{x^n - px^{n-1} + qx^{n-2} - \&c.}^{\frac{b}{m}} = o$; tum erit $\left(\frac{\dot{A}}{\dot{x}}\right) \frac{b}{m} \times \overline{nx^{n-1} - \overline{n-1}\, px^{n-2} + \&c.} \times \overline{x^n - px^{n-1} + qx^{n-2} - \&c.}^{\frac{b-m}{m}} = o$; ita reducantur hæ æquationes ut exterminentur irrationales quantitates, & reſultant æquationes (P) $\overline{x^n - px^{n-1} + \&c.}^{b} = o$ & ($\mathcal{Q}$) $\overline{x^n - px^{n-1} + \&c.}^{b-m} \times \overline{nx^{n-1} - \overline{n-1}\, px^{n-2} + \&c.}^{m} = o$; & quoniam æquatio $A = o$ habet plures valores e ſingulis ejus radicibus inter ſe æquales; ergo æquatio $\frac{\dot{A}}{\dot{x}} = o$ habet ſingulas radices, quas habet æquatio $A = o$; habet etiam radices æquationis $nx^{n-1} - \overline{n-1}\, px^{n-2} + \&c. = o$ limites inter ſucceſſivas radices datæ æquationis $x^n - px^{n-1} + \&c. = o$.

Cor. 1. Si duæ vel plures (m) radices (α) æquationis $A = o$ vel $P = o$ ſint inter ſe æquales; tum æquationes $\frac{A}{\dot{x}} = o$ & $\mathcal{Q} = o$ habent pauciores radices quam habet data æquatio $A = o$ vel $P = o$, quæ ſunt (α).

Hinc erui poſſunt quamplurimæ propoſitiones haud inelegantes de limitibus æquationum.

THEOR.

Datis duabus æquationibus $A = o$ & $B = o$, quarum quædam radices ſunt reſpective inter ſe limites; ex iis ſæpe detegi poſſunt duæ aliæ æquationes $P = o$ & $\mathcal{Q} = o$, quarum quædam radices ſunt inter ſe limites.

1. In-

1. Inveniantur duæ æquationes $P=0$ & $Q=0$, quarum incognitæ quantitates ſint reſpective z & v, ubi z ſit eadem algebraica functio (σ) incognitæ quantitatis (x) datæ æquationis $A=0$, ac v ſit incognitæ quantitatis (y) datæ æquationis $B=0$; haud raro quædam radices æquationum $P=0$ & $Q=0$ erunt inter ſe limites.

Sint α & β duæ radices æquationis $A=0$, inter quas ponitur radix (π) æquationis $B=0$; fiat functio $\sigma=0$, & inveniantur radices incognitæ quantitatis $(x$ vel $y)$ in æquatione reſultante contentæ; tum, ſi æquatio $\sigma=0$ omnes habeat ſuas radices ex eâdem parte quantitatum α & β poſitas, *i. e.* vel omnes majores vel omnes minores quam α & β, & in eâ pro $(x$ vel $y)$ ſcribantur reſpective α, π & β; quantitas reſultans e ſubſtitutione literæ π erit limes inter quantitates reſultantes e ſubſtitutione literarum α & β; *i. e.* radix æquationis $Q=0$ erit limes inter duas radices æquationis $P=0$ poſitus. Idem principium etiam applicari poteſt, cum impoſſibiles habeant prædictæ æquationes radices. Et ſic de pluribus hujuſmodi caſibus.

2. Si duæ vel plures diverſæ radices quantitatis $(x$ vel $y)$ in functione σ contineantur; tum ex principiis in priori caſu traditis aſſignari poſſunt æquationes, quarum quædam radices dati generis erunt inter ſe limites.

Ex. 1. Sint duæ æquationes $A=0$ & $B=0$, quarum radices ſunt inter ſe limites reſpective; tum, ſi m & n ſint integri numeri, & $x^{2m+1}=z$ & $y^{2n+1}=v$; vel $x^{\frac{2m+1}{2n+1}}=z$ & $y^{\frac{2m+1}{2n+1}}=v$; erunt radices æquationum $P=0$ & $Q=0$ reſpective inter ſe limites: ſi vero $x^{2m}=z$ & $y^{2m}=v$, & α & $-\beta$ ſint duæ radices æquationis $A=0$; tum forſan inter radices α^{2m} & β^{2m} ſibi ipſis proxime ſucceſſivas reſultantis æquationis $P=0$ haud datur radix in æquatione $Q=0$ contenta; ſin autem radices æquationum $A=0$ & $B=0$ vel ſint omnes affirmativæ vel omnes negativæ, tum erunt omnes radices æquationum $P=0$ & $Q=0$ reſpective inter ſe limites; &c.

Ex. 2. Sint duæ æquationes $1+px+qx^2 \ldots + Sx^{n-1} + Tx^n\ (A) = 0$ & $p+2qy+3ry^2 \ldots + \overline{n-1}\,Sy^{n-2} + nTy^{n-1}\ (B) = 0$, quarum

rum radices ſunt reſpective inter ſe limites. Inveniantur duæ æquationes, quarum radices ſint reſpective $\frac{1}{x} = z$ & $\frac{1}{y} = v$; tum radices æquationum reſultantium $x^n + px^{n-1} + qx^{n-2} +$ &c. $(P) = o$ & $py^{n-1} + 2qy^{n-2} + 3ry^{n-3} \ldots + \overline{n-1}\,Sy + nT\,(Q) = o$ erunt reſpective inter ſe limites, ni in æquatione $Q = o$ nullus contineatur limes inter minimam affirmativam & minimam negativam æquationis $P = o$ radicem; habet vero æquatio $Q = o$ radicem majorem maximâ affirmativâ vel negativâ radice æquationis $P = o$, prout radix æquationis $B = o$ poſita inter minimam affirmativam & minimam negativam æquationis $A = o$ radicem ſit affirmativa vel negativa: radix quæ eſt reciproca minimæ erit maxima. Radices duarum æquationum $x^n + px^{n-1} + qx^{n-2} + rx^{n-3} +$ &c. $= o$, &c. & $py^n + 2qy^{n-1} + 3ry^{n-2} +$ &c. erunt reſpective limites inter ſe, (o) enim erit limes inter minimam affirmativam & minimam negativam radicem.

Idem etiam aſſeri poteſt, ſi modo $x^{-2m-1} = z$ & $y^{-2m-1} = v$; &c.

Ex. 3. Sint duæ æquationes $(A)\ x^n + px^{n-1} + qx^{n-2} +$ &c. $= o$ & $(B)\ ny^{n-1} + \overline{n-1}\,py^{n-2} + \overline{n-2}\,qy^{n-3} +$ &c. $= o$, quarum radices ſunt limites inter ſe reſpective; ſcribantur $\frac{x^2}{1+x^2} = z^2$ & $\frac{y^2}{1+y^2} = v^2$; *i. e.* pro x & y in datis æquationibus ſcribantur $\frac{z}{\sqrt{1-z^2}}$ & $\frac{v}{\sqrt{1-v^2}}$ reſpective; reſultantes æquationes ſemper habebunt radices limites inter ſe, ni æquatio $(A = o)$ habeat negativam radicem $(-\alpha)$, ubi α affirmative ſumpta propior ſit ad affirmativam radicem β, quam quævis alia affirmativa radix; &c. Si duæ æquationes $A = o$ & $B = o$ communem habeant diviſorem, & conſequenter duo vel plures radices æquationis $A = o$ ſint inter ſe æquales; tum duæ æquationes $P = o$ & $Q = o$ communem habent diviſorem, & duæ radices æquationis $P = o$ ſunt inter ſe æquales; ſin aliter non, niſi

u æquatio

æquatio $A = 0$ habeat negativam & affirmativam radicem inter ſe æquales, in quo caſu æquatio $P = 0$ habet duas radices æquales, at non habebunt communem diviſorem æquationes $A = 0$ & $B = 0$; in nullo caſu habent hæ æquationes communem diviſorem, niſi cum duæ radices æquationis $A = 0$ ſint inter ſe æquales.

PROB. VII.

Datâ æquatione $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} -$ &c. $= 0$, *cujus radices ſint* $\alpha, \beta, \gamma, \delta, \epsilon$, *&c. ſucceſſive; datis etiam limitibus inter* α *&* β, β *&* γ, γ *&* δ, δ *&* ϵ, *&c. poſitis; invenire numerum poſſibilium vel impoſſibilium radicum in datâ æquatione contentarum.*

Sit π major quam maxima radix (α), ϱ limes inter α & β, σ inter β & γ, τ inter γ & δ, &c.

Subſtituantur limites (π, ϱ, σ, τ, &c.) pro incognitâ quantitate (x) in datâ æquatione; tot erunt radices poſſibiles, quot ſunt mutationes ſignorum quantitatum reſultantium ex $+$ in $-$, & $-$ in $+$; cæteræ autem impoſſibiles ſunt.

Hoc conſtat e theoremate ſecundo.

Cor. 1. Sit æquatio (A) $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} -$ &c. $= 0$; radices æquationis $nx^{n-1} - \overline{n-1}\,px^{n-2} + \overline{n-2}\,qx^{n-3} - \overline{n-3}\,rx^{n-4} +$ &c. $= 0$ erunt limites inter radices datæ æquationis: inveniantur radices hujus æquationis, & ſubſtituantur pro radice (x) in priori æquatione; e numero mutationum ſignorum ex $+$ in $-$, & $-$ in $+$ quantitatum reſultantium conſtabit poſſibilium & impoſſibilium radicum æquationis $A = 0$ numerus.

Cor. 2. Hinc extractione radicum æquationis ($n-1$) dimenſionum $nx^{n-1} - \overline{n-1}\,px^{n-2} + \overline{n-2}\,qx^{n-3} - \overline{n-3}\,rx^{n-4} +$ &c. $= 0$; vel ſi modo deleatur ſecundus datæ æquationis terminus, *i.e.* fiat æquatio $x^n + qx^{n-2} - rx^{n-3} +$ &c. $= 0$, extractione radicum æquationis ($n-2$) dimenſionum (in hoc enim caſu unus limes inter radices datæ æquationis nihilo erit æqualis) erui poteſt impoſſibilium radicum numerus.

E. G. Sit æquatio $x^2 + q = 0$, limes inter ejus duas radices erit nihilo æqualis; conſequenter ejus duæ radices erunt poſſibiles, vel non; prout q ſit negativa, vel affirmativa quantitas.

Ex. 2. Sit $x^3 + qx - r = 0$, cujus radices ſint α, β, γ, ſucceſſive; ducantur ſucceſſivi hujus æquationis termini in ſucceſſivos arithmeticæ ſeriei 0, 1, 2, 3 terminos, reſultat ſimplex æquatio $2qx - 3r = 0$; cujus radix $\left(\frac{3r}{2q}\right)$ ſcribatur pro incognitâ quantitate (x) in datâ æquatione, & reſultat $\frac{27r^3 + 4rq^3}{8q^3}$; ſcribatur etiam (0) pro incognitâ quantitate in datâ æquatione, & reſultat $-r$: 1mo ſit $\frac{r}{q}$ affirmativa quantitas, tum $\frac{3r}{2q}$ erit limes inter α & β, (0) vero limes inter β & γ; & tres radices datæ æquationis erunt poſſibiles, vel non, prout $\frac{27r^3 + 4rq^3}{8q^3}$, vel quod ad idem ſemper redit, $27r^2 + 4q^3$ ſit negativa, vel affirmativa quantitas: 2do ſit $\frac{r}{q}$ negativa quantitas, & conſequenter (0) limes inter α & β, & $\frac{3r}{2q}$ inter β & γ; & tres radices datæ æquationis erunt poſſibiles, vel non; prout $\frac{27r^3 + 4rq^3}{8q^3}$ ſit affirmativa vel negativa quantitas, *i. e.* prout $27r^2 + 4q^3$ ſit negativa vel affirmativa quantitas.

Ex. 3. Sit æquatio $x^4 + qx^2 - rx + s = 0$; æquatio, cujus radices ſint limites inter radices datæ æquationis, erit $2qx^3 - 3rx^2 + 4sx = 0$, cujus tres radices erunt 0, $\frac{3r \pm \sqrt{9r^2 - 32sq}}{4q}$.

1mo. Sit $9r^2 - 32sq$ negativa quantitas; quatuor, vel duæ datæ æquationis radices erunt impoſſibiles, prout s ſit affirmativa, vel negativa quantitas.

2do. Sit

2do. Sit $9r^2 - 32sq$ affirmativa quantitas, & s & q diverſa habeant ſigna; ſubſtituantur pro (x) in datâ æquatione $\frac{3r + \sqrt{9r^2 - 32sq}}{4q}$, o, $\frac{3r - \sqrt{9r^2 - 32sq}}{4q}$; & ſi quantitatum reſultantium prima ſit negativa, ſecunda vero affirmativa, tertia autem negativa; tum omnes datæ æquationis radices erunt poſſibiles; ſin autem quantitates reſultantes omnes fuerint affirmativæ, tum nullam habebit poſſibilem radicem data æquatio; ſin aliter, duas.

Si s & q eadem habeant ſigna, & r ſit negativa quantitas; ſubſtituantur quantitates o, $\frac{-3r + \sqrt{9r^2 - 32sq}}{4q}$, $\frac{-3r - \sqrt{9r^2 - 32sq}}{4q}$ ſucceſſive; ſi vero r ſit affirmativa quantitas, ſubſtituantur o, $\frac{3r - \sqrt{9r^2 - 32sq}}{4q}$, $\frac{3r + \sqrt{9r^2 - 32sq}}{4q}$ ſucceſſive pro radice (x) in datâ æquatione; & ſi quantitates reſultantes reſpectivè habeant ſigna —, +, —; tum quatuor radices poſſibiles habet data æquatio: ſi vero prædicta ſigna ſint +, +, +; nullam poſſibilem radicem habet data æquatio; ſin aliter, duas.

Et ſic ad æquationes majorum dimenſionum progredi licet.

Cor. Datâ æquatione $x^n + px^{n-1} + qx^{n-2} + rx^{n-3} + \&c. + Px^{n-m} + Qx^{n-m-1} + Rx^{n-m-2} + Sx^{n-m-3} + Tx^{n-m-4} + \&c. = o$, cujus radices ſint α, β, γ, δ, &c. ν, π, ρ, σ, τ; quarum α ſit maxima, & τ minima radix, & β major quam γ, γ quam δ, &c. ν quam π, π quam ρ, ρ quam σ, &c. conſtat e corollario 4. theor. 4. tres eſſe æquationes $n-2$ dimenſionum, (ſi modo omnes radices vel ſint affirmativæ vel omnes negativæ; ſin aliter, reducantur hæ æquationes, ita ut omnes radices evadant negativæ vel affirmativæ, quod facile fieri poteſt augendo vel minuendo radicem per quantitatem majorem maximâ affirmativâ vel negativâ radice)

$$n \times \overline{n-1}\, x^{n-2} + \overline{n-1} \times \overline{n-2}\, px^{n-3} + \overline{n-2} \times \overline{n-3} \times qx^{n-4} + \overline{n-3} \times \overline{n-4} \times rx^{n-5} + \&c. = o$$

$$\& \quad 1 \times \overline{n-1}\, px^{n-2} + 2 \times \overline{n-2} \times qx^{n-3} + 3 \times \overline{n-3} \times rx^{n-4} + 4 \times \overline{n-4}\, sx^{n-5} + \&c. = o$$

$$\& \quad 1 \times 2qx^{n-2} + 2 \times 3rx^{n-3} + 3 \times 4sx^{n-4} + \&c. = o$$

quarum radices erunt reſpective limites inter α & β, β & δ, γ & ε, &c.; at forſan nullum habebunt limitem hæ tres æquationes inter quaſcunque duas ſucceſſivas radices E. G. (α & β); poteſt enim maxima radix harum trium æquationum eſſe limes inter β & γ; & conſequenter reſolutio harum trium æquationum reſultantium haud neceſſario invenit numerum impoſſibilium radicum in datâ æquatione contentarum; & a fortiori reſolutio nullius inferioris æquationis ſemper verum præbebit numerum datæ æquationis radicum. E. G. Si per corollarium 4. theor. 4. diluantur $(n-2)$ termini datæ æquationis, conſequenter reſultant $(n-1)$ quadraticæ æquationes $n \times \overline{n-1}\, x^2 + 2 \times \overline{n-1}\, px + 2q = 0$, $\overline{n-1} \times \overline{n-2}\, px^2 + 2 \times 2 \times \overline{n-2}\, qx + 2 \cdot 3 r = 0$, $\overline{n-2} \times \overline{n-3}\, qx^2 + 2 \times 3 \times \overline{n-3}\, rx + 3 \cdot 4 s = 0$, &c. & generaliter quadratica æquatio, cujus diſtantia a primâ ſit m, erit $\overline{n-m} \times \overline{n-m-1} \times Px^2 + 2 \times \overline{m+1} \times \overline{n-m-1}\, Qx + \overline{m+1} \times \overline{m+2}\, R = 0$, quarum omnium radices erunt limites inter α & σ, & β &c. τ; & exinde e reſolutione harum omnium æquationum frequenter haud inveniri poteſt impoſſibilium radicum numerus, plures enim harum æquationum radices poſſunt eſſe limites inter eaſdem duas datæ æquationis radices.

Diluantur per prædictum cor. omnes præter quatuor ſucceſſivos datæ æquationis terminos, reſultabunt $(n-2)$ cubicæ æquationes $n \times \overline{n-1} \times \overline{n-2}\, x^3 + 3 \times \overline{n-1} \times \overline{n-2}\, px^2 + 3 \times 2 \times \overline{n-2}\, qx + 1 \times 2 \times 3 \times r = 0$, $\overline{n-1} \times \overline{n-2} \times \overline{n-3}\, px^3 + 3 \times 2 \times \overline{n-2} \times \overline{n-3}\, qx^2 + 3 \times 2 \times 3 \times \overline{n-3}\, rx + 2 \times 3 \times 4 s = 0$; & generaliter cubica æquatio, cujus diſtantia a primâ ſit (m), erit $\overline{n-m} \times \overline{n-m-1} \times \overline{n-m-2} \times Px^3 + 3 \times \overline{m+1} \times \overline{n-m-1} \times \overline{n-m-2}\, Qx^2 + 3 \times \overline{m+1} \times \overline{m+2} \times \overline{n-m-2} \times Rx + \overline{m+1} \times \overline{m+2} \times \overline{m+3}\, S = 0$; harum $(n-2)$ æquationum cubicarum radices erunt limites inter α & ϱ, β & σ, γ & τ; & conſequenter ex earum reſolutione haud generaliter invenitur impoſſibilium radicum numerus: & ſic de omnibus æquationibus, in quibus haud contineantur limites inter α & β, β & γ, γ & δ, &c.

PROB.

PROB. VIII.

Invenire limites inter α & β, β & γ, γ & δ, &c.

Transformetur per exem. 6. prob. 5. data æquatio in alteram, cujus radices ſint $\frac{1}{\alpha-\beta}$, $\frac{1}{\alpha-\gamma}$, $\frac{1}{\alpha-\delta}$, $\frac{1}{\beta-\gamma}$, $\frac{1}{\beta-\delta}$, $\frac{1}{\gamma-\delta}$, &c. reſpective; ſit $\frac{1}{A}$ major quam maxima reſultantis æquationis radix; erit A minor quam differentia, vel inter α & β, β & γ, γ & δ, &c. & ſit S major quam maxima datæ æquationis radix.

Inventâ igitur quantitate S, quæ major eſt quam maxima radix (α); & A, quæ minor erit quam minima differentia inter ullas duas ſucceſſivas radices; continuo auferantur A, $2A$, $3A$, $4A$, $5A$, &c. reſpective de S; tandem reſultabunt limites a, b, c, d, &c. inter ſucceſſivas radices α & β, β & γ, γ & δ, &c.

Scribantur pro x in data æquatione a, b, c, &c. ſucceſſive; & e mutatione ſignorum facile dici poteſt, quot radices vel poſſibiles vel impoſſibiles in datâ æquatione contineantur.

THEOR. 7.

1. *Sint* α, β, γ, δ, ϵ, ζ, *&c. quæcunque poſſibiles quantitates, quarum numerus ſit* (n); *& ſint* 2a, 2b, 2c, 2d, 2e, 2f, *&c. quicunque indices pares numeri, & quorum numerus ſit* (m), *& quorum* a *æqualis vel major ſit quam* b, b *quam* c, c *quam* d, *&c. ſint etiam* p, q, r, s, t, *&c. quicunque indices, quorum numerus ſit* (M), *& ſit* p *æqualis vel major quam* q, q *quam* r, r *quam* s, s *quam* t, *&c. & ſit etiam* 2a *æqualis vel major quam* p, 2b *æqualis vel major quam* q, *ni* 2a *æqualis vel major ſit quam* p + q; 2c *æqualis vel major quam* r, *ni* 2a + 2b, *vel æqualis vel major ſit quam* p + q + r; *& ſic deinceps; & ſit* 2a + 2b + 2c + 2d + 2e + &c. = p + q + r + s + t + &c.

Summa

Summa omnium hujuſcemodi contentorum $\alpha^{2a}\beta^{2b}\gamma^{2c}\delta^{2d}\varepsilon^{2e}$ &c. $+\ \alpha^{2a}\beta^{2b}\gamma^{2c}\delta^{2d}\zeta^{2e}$ &c. $+\ \beta^{2a}\gamma^{2b}\delta^{2c}\varepsilon^{2d}\zeta^{2e}$ &c. + &c. = P *habet vel eandem vel majorem rationem, ad ſummam omnium hujuſcemodi contentorum* $\alpha^{p}\beta^{q}\gamma^{r}\delta^{s}\varepsilon^{t}$ &c. $+\ \alpha^{p}\beta^{q}\gamma^{r}\delta^{s}\zeta^{t}$ &c. $+\ \beta^{p}\gamma^{q}\delta^{r}\varepsilon^{s}\zeta^{t}$ &c. + &c. = Q, *quam numerus contentorum in priori ſummâ ad numerum contentorum in poſteriori; i. e. quam* $n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\ldots\overline{n-m+1}$ *ad* $n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\ldots\overline{n-M+1}$: *vel* $1:\overline{n-m}\times\overline{n-m-1}\times\overline{n-m-2}\ldots\overline{n-M+1}$.

Hoc theorema, ſcilicet, quod $\overline{n-m}\times\overline{n-m-1}\times\overline{n-m-2}\ldots\overline{n-M+1}\times P$ æqualis vel major erit quam Q, ſequitur ex hâc ratione, nempe quod quantitas $\overline{n-m}\times\overline{n-m-1}\times\overline{n-m-2}\ldots\overline{n-M+1}\times P-Q$ ſemper probari poteſt æqualis aggregato quadratorum e compluribus quantitatibus. Hæc enim ſola eſt methodus, e quâ concludi poteſt datam quantitatem ſemper eſſe affirmativam.

Si *(b)* indices in priori quantitate *(P)* ſint *a*, *k* indices vero *b*, *l* & *o* indices reſpective *c* & *d*, &c.; & ſint etiam in poſteriori quantitate *(Q)* *(b′)* indices *p*, *k′* indices *q*, *l′* & *o′* indices reſpective *r* & *s*, &c. & $1\times2\times3\ldots b\times1\times2\times3\times\ldots k\times1\times2\times3\times\ldots l\times1\times2\times3\times\ldots o\times$ &c. $\times P : 1\times2\times3\times\ldots b'\times1\times2\times3\times\ldots k'\times1\times2\times3\times\ldots l'\times1\times2\times3\times\ldots o'\times$ &c. $\times Q$ habet vel eandem vel majorem rationem, quam $1:\overline{n-m}\times\overline{n-m-1}\times\overline{n-m-2}\times\ldots\overline{n-M+1}$.

Cor. 1. Sit æquatio $x^{n}+px^{n-1}+qx^{n-2}+rx^{n-3}+sx^{n-4}+$ &c. … $+\,gx^{n-m+l+6}+fx^{n-m+l+5}+ex^{n-m+l+4}+dx^{n-m+l+3}+cx^{n-m+l+2}+bx^{n-m+l+1}+ax^{n-m+l}+$ &c. … $Kx^{n-m+5}+Lx^{n-m+4}+Mx^{n-m+3}+Nx^{n-m+2}+Ox^{n-m+1}+Px^{n-m}+Qx^{n-m-1}+Rx^{n-m-2}+Sx^{n-m-3}+Tx^{n-m-4}+Vx^{n-m-5}+$ &c. … $Ax^{n-m-l}+Bx^{n-m-l-1}+Cx^{n-m-l-2}+Dx^{n-m-l-3}+Ex^{n-m-l-4}+Fx^{n-m-l-5}+$ &c. $=0$; cujus radices ſint omnes poſſibiles, & per α, β, γ, δ, ε, ζ, &c. reſpective deſignatæ.

Per theorema ſumma (π) omnium hujuſcemodi contentorum $\alpha^{2}\beta^{2}\gamma^{2}\delta^{2}\varepsilon^{2}$ &c. $+\ \alpha^{2}\beta^{2}\gamma^{2}\delta^{2}\zeta^{2}$ &c. $+\ \alpha^{2}\beta^{2}\gamma^{2}\varepsilon^{2}\zeta^{2}$ &c. + &c. $+\ \beta^{2}\gamma^{2}\delta^{2}\varepsilon^{2}\zeta^{2}$ &c. + &c. (in quibus ſingulis contentis numerus literarum

rum $\alpha, \beta, \gamma, \delta, \varepsilon$, &c. contentarum ſit m) multiplicata in contentum $m \times \overline{m-1} \times \overline{m-2} \times \overline{m-3} \times \ldots \overline{m-l+1} \times \overline{n-m} \times \overline{n-m-1} \times \overline{n-m-2} \times \ldots \overline{n-m-l+1} = X$ æqualis vel major erit quam ſumma (σ) omnium contentorum $\alpha^2\beta^2\gamma^2$ &c. $\times\, \varepsilon\zeta$ &c. $+\, \alpha^2\beta^2\delta^2$ &c. $\times\, \gamma\zeta$ &c. $+\, \beta^2\gamma^2\delta^2$ &c. $\times\, \alpha\varepsilon$ &c. $+\, \beta^2\gamma^2\varepsilon^2$ &c. $\times\, \alpha\zeta$ &c. $+$ &c. (in ſingulis hiſce contentis numerus literarum, in quibus duæ inveniuntur dimenſiones, ſit $m-l$; & exinde literarum, in quibus una ſolummodo continetur litera, erit $2l$) multiplicata in contentum $1 \times 2 \times 3 \times 4 \times \ldots 2l = Z$.

Per exemp. 2. caſ. 3. prob. 3. $\pi = P^2 - 2QO + 2RN - 2SM + 2TL - 2VK +$ &c. & $\sigma = Aa - \overline{2l+2}\, Bb + \overline{2l+1} \times \frac{2l+4}{2} Cc - \overline{2l+1} \times \frac{2l+2}{2} \times \frac{2l+6}{3} Dd + \overline{2l+1} \times \frac{2l+2}{2} \times \frac{2l+3}{3} \times \frac{2l+8}{4} Ee -$ &c. ducantur hæ duæ ſeries (π & σ) reſpective in X & Z, & reſultant æquationes $X\pi = XP^2 - 2XQO + 2XRN - 2XSM + 2XTL - 2XVK +$ &c. & $Z\sigma = ZAa - \overline{2l+2}\, ZBb + \overline{2l+1} \times \frac{2l+4}{2} ZCc - \overline{2l+1} \times \frac{2l+2}{2} \times \frac{2l+6}{3} ZDd +$ &c. quarum differentia erit $X \times \overline{P^2 - 2QO + 2RN - 2SM + 2TL - \text{\&c. ad } l \text{ terminos}} - \overline{Z \pm 2X} \times Aa + \overline{2l+2} \times \overline{Z \mp 2X} \times Bb - \overline{2l+1} \times \frac{2l+4}{2} \overline{Z \pm 2X} \times Cc + \overline{\overline{2l+1} \times \frac{2l+2}{2} \times \frac{2l+6}{3} Z \mp 2X} \times Dd - \overline{\overline{2l+1} \times \frac{2l+2}{2} \times \frac{2l+3}{3} \times \frac{2l+8}{4} Z \pm 2X} \times Ee +$ &c. quæ ſemper, cum radices ſint poſſibiles, erit affirmativa quantitas.

Si l ſit par numerus, tum erit $\overline{Z - 2X} \times Aa$; ſin aliter $\overline{Z + 2X} \times Aa$.

Sit $l = 1$, & Z erit 1×2; & differentia erit $XP^2 - 2 \times \overline{X+1} \times$

$QO + 2$

$QO + 2\times\overline{X+4}\times RN - 2\times\overline{X+9}\,SM + 2\times\overline{X+16}\,TL - 2\times\overline{X+25}\,VK +$ &c.

Sit $l=2$, & Z erit $1\times 2\times 3\times 4 = 24$; & differentia erit $X\times P^2 - 2X\times QO + 2\times\overline{X-12}\times RN - 2\times\overline{X-72}\times SM + 2\times\overline{X-240}\times TL - 2\times\overline{X-600}\times VK +$ &c.

Sit $l=3$, & Z erit $1\times 2\times 3\times 4\times 5\times 6 = 720$; & differentia erit $XP^2 - 2XQO + 2XRN - 2\times\overline{X+360}\,SM + 2\times\overline{X+2880}\,TL - 2\times\overline{X+12600}\,VK +$ &c.

THEOR. 8.

Sint $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$, &c. *quæcunque poſſibiles quantitates, quarum omnes idem habeant ſignum, & quarum numerus ſit* n; *& ſint* a, b, c, d, e, f, &c. *quicunque indices integri numeri; quorum* a *vel æqualis vel major ſit quam* b, b *quam* c, c *quam* d, d *quam* e, e *quam* f, *&c. & quorum numerus ſit* m.

Sint etiam p, q, r, s, t, &c. *quicunque indices, quorum numerus ſit* M, *& ſit* p *æqualis vel major quam* q, q *quam* r, r *quam* s, s *quam* t, *&c. ſit etiamque* a *vel æqualis vel major quam* p, b *vel æqualis vel major quam* q, *ni* a *vel æqualis vel major ſit quam* p + q; c *vel æqualis vel major quam* r, *ni* a + b *vel æqualis vel major ſit quam* p + q + r; d *vel æqualis vel major quam* s, *ni* a + b + c *vel æqualis vel major ſit quam* p + q + r + s; *&c. ſic deinceps; &* a + b + c + d + e + *&c.* = p + q + r + s + t + *&c.*

Summa (P) *omnium hujuſcemodi contentorum* $\alpha^a\beta^b\gamma^c\delta^d\epsilon^e$ *&c.* + $\alpha^b\beta^a\gamma^c\delta^d\epsilon^e$ *&c.* + $\alpha^a\beta^b\gamma^c\delta^d\zeta^e$ *&c.* + *&c. habet vel eandem vel majorem rationem ad ſummam* (Q) *contentorum* $\alpha^p\beta^q\gamma^r\delta^s\epsilon^t$ *&c.* + $\alpha^q\beta^p\gamma^r\delta^s\epsilon^t$ *&c.* + $\alpha^p\beta^q\gamma^r\delta^s\zeta^t$ *&c.* + *&c. quam numerus contentorum in priori ſummâ ad numerum contentorum in poſteriori, i, e. quam* $1 : \overline{n-m}\times\overline{n-m-1}\times\overline{n-m-2}\times.\times\overline{n-M+1}$.

Quantitas $\overline{n-m}\times\overline{n-m+1}\times\overline{n-m+2}\,..\,\overline{n-M+1}\times P$ ſemper

per

per probari poteſt eſſe major quam $\mathcal{Q}$; ex eo, quod probari poteſt quantitas $\overline{n-m} \times \overline{n-m-1} \times \overline{n-m-2} \,..\, \overline{n-M+1} \times P - \mathcal{Q}$, æqualis eſſe aggregato quadratorum e compluribus quadratis.

Sint (b) indices in priori quantitate (P) æquales per (a) deſignati, k indices æquales per (b), l & o indices æquales per c & d reſpective deſignati, &c. Sint etiam b^1 indices æquales per (p) in poſteriori quantitate deſignati, k^1 indices æquales per (q), l^1 & o^1 indices æquales per r & s reſpective deſignati, &c. & $1.2.3 \,..\, b \times 1.2.3 \,..\, k \times 1.2.3 \,..\, l \times 1.2.3 \,..\, o \times$ &c. $\times P : 1 \times 2 \times 3 \,..\, b^1 \times 1 \times 2.3 \,..\, k^1 \times 1.2.3 \,..\, l^1 \times 1.2.3 \,..\, o^1 \times \mathcal{Q}$ habet æqualem vel majorem rationem quam $1 : \overline{n-m-1} \times \overline{n-m-2} \,..\, \overline{n-M+1}$.

Cor. Sit æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} +$ &c. $\pm gx^{n-l+6} \mp fx^{n-l+5} \pm ex^{n-l+4} \mp dx^{n-l+3} \pm cx^{n-l+2} \mp bx^{n-l+1} \pm ax^{n-l} \mp$ &c. $\pm Kx^{n-m+5} \mp Lx^{n-m+4} \pm Mx^{n-m+3} \mp Nx^{n-m+2} \pm Ox^{n-m+1} \mp Px^{n-m} \pm \mathcal{Q}x^{n-m-1} \mp Rx^{n-m-2} \pm Sx^{n-m-3} \mp Tx^{n-m-4} \pm$ &c. $\mp Ax^{n-m-r} \pm Bx^{n-m-r-1} \mp Cx^{n-m-r-2} \pm Dx^{n-m-r-3} \mp Ex^{n-m-r-4} \pm Fx^{n-m-r-5} \mp Gx^{n-m-r-6} \pm$ &c. $\pm bx^{n-2m+l-r} \mp ix^{n-2m+l-r-1} \pm kx^{n-2m+l-r-2} \mp l'x^{n-2m+l-r-3} \pm$ &c. $= 0$, cujus radices ſint omnes poſſibiles, & reſpective α, β, γ, δ, ε, &c. & ſint omnes hæ radices vel affirmativæ vel negativæ.

Per theorema ſumma (π) omnium hujuſcemodi contentorum $\alpha^2\beta^2\gamma^2$ &c. $\times \delta\varepsilon$ &c. $+ \alpha^2\gamma^2\delta^2$ &c. $\times \beta\varepsilon$ &c. $+ \alpha^2\beta^2\gamma^2$ &c. $\times \delta\zeta$ &c. $+$ &c. (in quibus ſingulis contentis numerus literarum, quæ duas habent dimenſiones, ſit m; numerus autem literarum, quæ unam ſolummodo habent dimenſionem, ſit r) vel eandem vel majorem rationem habet ad ſummam (σ) omnium contentorum hujuſcemodi $\alpha^2\beta^2$ &c. $\times \gamma\delta\varepsilon\zeta$ &c. $+ \alpha^2\gamma^2$ &c. $\times \beta\delta\varepsilon\zeta$ &c. $+ \beta^2\gamma^2$ &c. $\times \alpha\delta\varepsilon\zeta$ &c. $+$ &c. (in quibus ſingulis contentis numerus literarum, quæ duas habent dimenſiones, ſit l; & exinde numerus literarum, quæ unam ſolummodo habent dimenſionem, erit $2m + r - 2l$) quam $\overline{r+1} \times \overline{r+2} \times \overline{r+3} \times .. \times \overline{2m-2l+r} = Z : \overline{l+1} \times \overline{l+2} \times \overline{l+3} \times .. m \times$

$\overline{n-m-r}\times\overline{n-m-r-1}\times\overline{n-m-r-2}\ldots\overline{n-2m+l-r+1}$ $=X$; & conſequenter $\overline{l+1}\times\overline{l+2}\ldots m\times\overline{n-m-r}\times\overline{n-m-r-1}$ $\ldots\overline{n-2m+l-r+1}\times\pi$ vel æqualis vel major erit quam $\overline{r+1}\times$ $\overline{r+2}\ldots\overline{2m-2l+r}\times\sigma$, modo m major ſit quam l.

Ex. 1. Si $m=l+1$, tum erit $m\times\overline{n-m-r}\times\pi$ æqualis vel major quam $\overline{r+1}\times\overline{r+2}\times\sigma$.

2. Per exemplum 2. caſ. 3. problem. 3.

$$\pi = PA-\overline{r+2}\,OB+\overline{r+1}\times\frac{r+4}{2}NC-\overline{r+1}\times\frac{r+2}{2}\times\frac{r+6}{3}MD+\&c.$$ & $$\sigma = ba-\overline{2m-2l+r+2}\,ib+\overline{2m-2l+r+1}\times\frac{2m-2l+r+4}{2}kc-\overline{2m-2l+r+1}\times\frac{2m-2l+r+2}{2}\times\frac{2m-2l+r+6}{3}l'd+\&c.$$

Ducantur hæ quantitates reſpectivè in X & Z; reſultant $$X\pi = XPA-\overline{r+2}\,XOB+\overline{r+1}\times\frac{r+4}{2}XNC-\overline{r+1}\times\frac{r+2}{2}\times\frac{r+6}{3}\times XMD+\&c.$$ & $$Z\sigma = Zba-\overline{2m-2l+r+2}\,Zib+\overline{2m-2l+r+1}\times\frac{2m-2l+r+4}{2}Zkc-\overline{2m-2l+r+1}\times\frac{2m-2l+r+2}{2}\times\frac{2m-2l+r+6}{3}Zl'd+\&c.$$ quarum differentia erit $XPA-\overline{r+2}\,XOB+\overline{r+1}\times\frac{r+4}{2}\times XNC-$ &c. ad $m-l$ terminos $$-\overline{Z\pm\frac{\overline{r+1}\times\overline{r+2}\times\ldots\overline{r+m-l-1}\times\overline{r+2m-2l}\times X}{1.2.3\ldots\overline{m-l}}}$$ $$\times ba+\overline{2m-2l+r+2}\,\overline{Z\mp\frac{\overline{r+1}\times\overline{r+2}\ldots\overline{r+m-l}\times\overline{r+2m-2l+2}\times X}{1.2.3.4\ldots\overline{m-l+1}}}$$

$\times ib$

$$\times\, \dot{i}b + \overline{2m-2l+r+1} \times \frac{2m-2l+r+4}{2} Z \pm$$

$$\frac{\overline{r+1} \times \overline{r+2} \times \overline{r+3} \,..\, \overline{r+m-l+1} \times \overline{r+2m-2l+4} \times X}{1.2.3.4 \ldots \overline{m-l+2}}$$

$\times\, kc +$ &c. $= X\pi - Z\sigma$.

Si $m-l$ ſit par numerus, tum ſignum quantitati $\left(\frac{\overline{r+1} \times \overline{r+2} \,..\, \overline{r+m-l-1} \times \overline{r+2m-2l} \times X}{1.2.3 \,..\, \overline{m-l}}\right)$ affixum erit negativum; ſin aliter affirmativum.

Sit $m = l+1$; & X & Z evadunt reſpective $m \times \overline{n-m-r}$, & $\overline{r+1} \times \overline{r+2}$; ſit etiam $r = 0$; & Px^{n-m} & Ax^{n-m-r} iidem fiunt termini: & exinde reſultat prædicta differentia $XP^2 - 2 \times \overline{X+1} \times QO + 2 \times \overline{X+4} \times RN - 2 \times \overline{X+9} \times SM + 2 \times \overline{X+16} \times TL - 2 \times \overline{X+25}\, VK +$ &c. $= a^1$: Hæc quantitas ſemper erit affirmativa, cum radices $\alpha, \beta, \gamma, \delta$, &c. ſint poſſibiles.

Sit $r = 2$, tum X & Z erunt reſpective $\overline{n-m-1} \times \overline{m-1}$, & 3×4; & differentia prædicta $XQO - 4 \times \overline{X+3}\, RN + 9 \times \overline{X+8}\, SM - 16 \times \overline{X+15} \times TL + 25 \times \overline{X+24} \times VK -$ &c. $= b^1$.

Sit $r = 4$, tum X & Z erunt reſpective $\overline{m-2} \times \overline{n-m-2}$, & $5 \times 6 = 30$; & reſultat $X \times RN - 6 \times \overline{X+5} \times SM + 20 \times \overline{X+12}\, TL - 50 \times \overline{X+21}\, VK +$ &c. $= c^1$, & ſic deinceps.

Quantitates a^1, b^1, c^1, d^1, e^1, &c. reſultantes ſemper erunt affirmativæ, cum radices $\alpha, \beta, \gamma, \delta, \varepsilon$, &c. ſint affirmativæ.

Ducantur quantitates a^1, b^1, c^1, d^1, &c. in affirmativas quantitates; tum aggregatum e ſingulis quantitatibus reſultantibus etiam erit affirmativa quantitas. Exhinc deduci poſſunt quam plurimæ quantitates; quæ ſemper erunt affirmativæ, cum radices datæ æquationis ſint poſſibiles & affirmativæ.

PROB. IX.

E theorematibus præcedentibus plures invenire regulas, e quibus detegi poſſunt impoſſibiles datæ æquationis radices.

Sint α, β, γ, δ, ϵ, &c. quæcunque datæ æquationis radices, & ex iis per duo theoremata præcedentia deduci poſſunt ſeries quantitatum a, b, c, d, e, f, &c. (in quibus ſimiliter inter ſe involvuntur radices α, β, γ, δ, ϵ, &c.) quæ ſemper erunt affirmativæ, cum radices α, β, γ, δ, &c. ſint poſſibiles; & quæ ſecundum dimenſiones radicum α, β, γ, δ, &c. regulariter aſcendent, vel deſcendent.

Sit numerus radicum α, β, γ, δ, ϵ, ζ, &c. $= n$, & numerus quantitatum (a, b, c, d, e, &c.) $= n - 1$; & ſcribantur quantitates a, b, c, d, e, &c. hoc modo $1 + a + b + c + d + e +$ &c. $+ 1$; tot vel plures erunt radices impoſſibiles, quot ſunt mutationes ſignorum terminorum de $+$ in $-$, & $-$ in $+$.

Ex. 1. Sit æquatio, $x^n + px^{n-1} + qx^{n-2} + rx^{n-3} + sx^{n-4} +$ &c. . . $Px^{n-m+1} + Qx^{n-m} + Rx^{n-m-1} +$ &c. $= 0$, cujus radices ſint α, β, γ, δ, ϵ, ζ, η, θ, ι, &c. per theoremata præcedentia (ſi modo radices ſint poſſibiles) $\frac{n-1}{2n}p^2$ major erit quam q; $\frac{2\times\overline{n-2}}{3\times\overline{n-1}}q^2$ major eric quam pr; & generaliter $\frac{m\times\overline{n-m}}{\overline{m+1}\times\overline{n-m+1}}\times Q^2$ major erit quam PR.

Ergo invenietur ſeries $(n-1)$ quantitatum $\frac{n-1}{2n}p^2 - q$, $\frac{2\times\overline{n-2}}{3\times\overline{n-1}}q^2 - pr$, $\frac{3\times\overline{n-3}}{4\times\overline{n-2}}r^2 - qs$, &c. regulariter aſcendentium, in quibus ſimiliter involvuntur radices α, β, γ, δ, &c. quæ ſemper erunt affirmativæ quantitates, cum radices ſint poſſibiles; exinde tot vel plures radices erunt impoſſibiles, quot ſunt mutationes ſignorum quantitatum in datâ ſerie contentarum de $+$ in $-$, & $-$ in $+$; i. e. ſub primo & ultimo datæ æquationis termino $x^n + px^{n-1} + qx^{n-2} +$

rx

$rx^{n-3} + sx^{n-4} + \&c. = 0$ colloca ſignum $+$; ſub quolibet mediorum terminorum, ſi ejus quadratum ductum in prædictam fractionem majus ſit quam rectangulum terminorum utrinque conſiſtentium, colloca ſignum $+$, ſin minus ſignum $-$; i.e. ſi $\frac{n-1}{2n}p^2$ majus ſit quam q, colloca ſignum $+$ ſub termino px^{n-1}; & eodem modo, ſi $\frac{\overline{2n-2}}{3\,\overline{n-1}}q^2$ majus ſit quam pr, & $\frac{\overline{3n-3}}{4\times\overline{n-2}}r^2$ majus quam qs; colloca ſub reſpectivis terminis qx^{n-2}, rx^{n-3}, ſignum $+$; ſin minus, ſignum $-$; & ſic de reliquis: tot vel plures radices erunt impoſſibiles, quot ſunt mutationes ſignorum reſultantium de $+$ in $-$, & $-$ in $+$.

In hâc & conſimilibus regulis, ubi termini duo vel plures deſint, ſub primo terminorum deficientium collocandum eſt ſignum $-$, ſub ſecundo ſignum $+$, ſub tertio ſignum $-$, & ſic deinceps, niſi quod ſub ultimo terminorum ſimul deficientium ſemper collocandum eſt ſignum $+$, ubi termini deficientibus utrinque proximi habent ſigna contraria.

Hæc methodus in quadraticis æquationibus verum præbet numerum impoſſibilium radicum: in cubicis autem ejus probabilitas inveniendi impoſſibiles radices non videtur majorem habere rationem ad probabilitatem fallendi quam 2 : 1. In æquationibus autem multo ſuperiorum dimenſionum hæc methodus verum impoſſibilium radicum numerum perraro deteget.

Quod $\frac{m\times\overline{n-m}}{\overline{m+1}\times\overline{n-m+1}}Q^2$ majus erit quam PR, cum radices datæ æquationis ſint poſſibiles, hâc methodo probari poteſt. Notum eſt $Q = \alpha\beta\gamma\delta\epsilon\ \&c. + \alpha\beta\gamma\delta\zeta\ \&c. + \alpha\beta\gamma\epsilon\zeta\ \&c. + \&c.$ Sit Y ſumma quadratorum e ſingulis differentiis inter quoſcunque duos coefficientis Q terminos.

Sit A ſumma ex omnibus prædictis quadratis, quorum duo termini in radice contenti a ſe invicem unam habent literam diverſam.

Sint B, C & D, &c. ſumma omnium quadratorum, quorum termini a ſe

$\times \overline{n-m+2} \times \frac{n-m+1}{m}$ & $n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdots \frac{n-m+2}{m-1} \times \overline{n-m+1}$ $\times \frac{n-m}{m+1}$; ex hinc formulis quantitatum prædictarum rite perpenſis deduci poteſt quantitatem $\frac{\overline{m-2} \times m^2}{\overline{m-1}^2 \times \overline{m+1}} \times \frac{n-m}{n-m+1} \times Q^2 - PR$ $= \pi$ ſemper evadere affirmativam, cum radices datæ æquationis ſint poſſibiles; hinc ſub primo & ultimo datæ æquationis termino colloca ſignum +; ſub quolibet mediorum terminorum Qx^{n-m} colloca ſignum +, ſi π ſit affirmativa quantitas; ſin minus ſignum —; tot vel plures erunt impoſſibiles radices, quot ſunt mutationes ſignorum reſultantium de + in —, & — in +.

Idem principium ad quamplurimos alios caſus applicari poteſt.

Ex. 3. Sit æquatio $x^n + px^{n-1} + qx^{n-2} + rx^{n-3} + sx^{n-4} + tx^{n-5} + vx^{n-6} + wx^{n-7} - - - - Px^{n-m} + Qx^{n-m-1} + Rx^{n-m-2} + Sx^{n-m-3} + Tx^{n-m-4} +$ &c. $= 0$. Colloca ſub primo & ultimo termino ſignum +; ſub ſecundo datæ æquationis termino px^{n-1} colloca ſignum +, ſi $\frac{9r^2}{\overline{n-1} \cdot \overline{n-2}^2} - \frac{18pqr}{n \cdot \overline{n-1} \cdot \overline{n-2}} - \frac{3p^2q^2}{n^2 \cdot \overline{n-1}} + \frac{6rp^3}{n^2 \cdot \overline{n-2}} + \frac{8q^3}{n \cdot \overline{n-1}^2}$ ſit negativa quantitas, ſin minus ſignum —; colloca ſub tertio datæ æquationis termino qx^{n-2} ſignum +, ſi $\frac{4s^2p^2}{\overline{n-2} \cdot \overline{n-3}^2}$ $- \frac{12qrsp}{\overline{n-1} \cdot \overline{n-2} \cdot \overline{n-3}} - \frac{3q^2r^2}{\overline{n-1}^2 \cdot \overline{n-2}} + \frac{16sq^3}{3 \times \overline{n-1}^2 \cdot \overline{n-3}} + \frac{6r^3p}{\overline{n-1} \cdot \overline{n-2}^2}$ ſit negativa quantitas, ſin minus ſignum —; & generaliter ſub quolibet termino Qx^{n-m-1} colloca ſignum +, ſi $\frac{\overline{m+2} \cdot \overline{m+3}^2 S^2P^2}{\overline{n-m-1} \cdot \overline{n-m-2}^2} - \frac{6 \times \overline{m+1} \cdot \overline{m+2} \cdot \overline{m+3} QRSP}{\overline{n-m} \cdot \overline{n-m-1} \cdot \overline{n-m-2}} -$

$\frac{3 . \overline{m+1}^2 . \overline{m+2} Q^2 R^2}{\overline{n-m}^2 . \overline{n-m-1}} + \frac{4 . \overline{m+1}^2 . \overline{m+3} S Q^3}{\overline{n-m}^2 . \overline{n-m-2}} + \frac{4 . \overline{m+1} . \overline{m+2}^2 R^3 P}{\overline{n-m} . \overline{n-m-1}^2}$ $= \zeta$ ſit negativa quantitas, ſin minus ſignum $-$; & tot vel plures erunt radices impoſſibiles, quot ſunt mutationes ſignorum de $+$ in $-$, & $-$ in $+$.

Hæc regula e ſubſequentibus obſervationibus deduci poteſt: 1ma, tot vel plures impoſſibiles radices habet data æquatio $x^n + px^{n-1} + qx^{n-2} + rx^{n-3} + sx^{n-4} + tx^{n-5} + vx^{n-6} + wx^{n-7} +$ &c. $= 0$, quot habet vel æquatio $nx^{n-1} + \overline{n-1}\, px^{n-2} + \overline{n-2}\, qx^{n-3} + \overline{n-3}\, rx^{n-4} +$ &c. $= 0$ vel $px^{n-1} + 2qx^{n-2} + 3rx^{n-3} + 4sx^{n-4} +$ &c. $= 0$: 2da, duæ radices æquationis $ax^3 + bx^2 + cx + d = 0$ evadunt impoſſibiles, necne; prout $4db^3 - b^2c^2 - 18abcd + 27a^2d^2 + 4c^3a = \pi$ ſit affirmativa vel negativa quantitas: 3^tia^, ſi ita ducantur termini ſucceſſivi datæ æquationis in terminos ſucceſſivos arithmeticarum ſerierum $n, n-1, n-2$, &c.; $n-1, n-2, n-3$, &c.; $n-2, n-3, n-4$, &c.; $n-3, n-4, n-5$, &c. &c.; 0, 1, 2, 3, &c. &c.; 0, 1, 2, 3, &c. &c.; ut tandem evadat æquatio $n . \overline{n-1} . \overline{n-2}\, x^3 + 3 \times \overline{n-1} \times \overline{n-2}\, px^2 + 3 . 2 \times \overline{n-2}\, qx + 6r = 0$ vel generaliter $\overline{n-m} \times \overline{n-m-1} \times \overline{n-m-2} \times Px^2 + 3 \times \overline{n-m-1} \times \overline{n-m-2} \times \overline{m+1} \times Qx^2 + 3 \times \overline{n-m-2} \times \overline{m+1} \times \overline{m+2} \times R + \overline{m+1} \times \overline{m+2} \times \overline{m+3} \times S = 0$; & ſcribantur $\overline{n-m} \times \overline{n-m-1} \times \overline{n-m-2} \times P$, $3 \times \overline{n-m-1} \times \overline{n-m-2} \times \overline{m+1} \times Q$, $3 \times \overline{n-m-2} \times \overline{m+1} \times \overline{m+2} \times R$, & $\overline{m+1} \times \overline{m+2} \times \overline{m+3} \times S$ reſpective pro a, b, c & d in quantitate π prius datâ, & ducatur quantitas exinde reſultans in $\frac{1}{27 \times \overline{n-m}^2 \times \overline{n-m-1}^3 \times \overline{n-m-2}^4 \times \overline{m+1}^2 \times \overline{m+2}}$; tum evadet quantitas ζ prædicta: unde ſingulæ quantitates reſultantes (ζ) erunt negativæ, ſi omnes radices datæ æquationis ſint poſſibiles; &c.

Cor. Sit æquatio (A) $x^n + px^{n-1} + qx^{n-2} \ldots + Px^{n-m} + Qx^{n-m-1} + Rx^{n-m-2} + Sx^{n-m-3} +$ &c. $= 0$, aſſumatur æquatio $n . \overline{n-1} . \overline{n-2}$

$\ldots n-m$

$\ldots(m+1)\times x^m + (n-1).(n-2).(n-3)\ldots m p x^{m-1} + (n-2).(n-3).(n-4)\ldots(m-1)\times q x^{m-2} + (n-3).(n-4).(n-5)\ldots(m-2)\times r x^{m-3}\ldots + 1.2.3\ldots\overline{n-m}P = 0$, cujus radices ſint π, ρ, σ, τ, &c.; ſubſtituantur hæ quantitates π, ρ, σ, τ, &c. pro x in quantitate $n.(n-1).(n-2)\ldots\overline{m+2}\,x^{m+1} + (n-1).(n-2).(n-3)\ldots\overline{m+1}\,x^m + (n-2).(n-3)\ldots m x^{m-1} + (n-3).(n-4)\ldots\overline{m-1}\,x^{m-2}\ldots + 2.3.4\ldots(n-m)\times P x$: primo ſit $m+1$ par numerus; tum, ſi maxima negativa quantitas reſultans minor ſit quam $1.2.3\ldots\overline{n-m-1}\,Q = \pi$, ubi Q eſt affirmativa quantitas, ſaltem $m+1$ radices datæ æquationis erunt impoſſibiles: ſi vero $m+1$ ſit impar numerus, & maxima affirmativa quantitas reſultans minor ſit quam π, cum Q ſit negativa; vel maxima negativa quantitas reſultans minor ſit quam π, cum Q ſit affirmativa quantitas; tum ſaltem erunt in datâ æquatione (m) impoſſibiles radices.

Hæc methodus ſemper impoſſibiles radices deteget, quando præcedens regula in Ex. 1. data eas inveniet; & ſæpe impoſſibiles radices inveniet, quando prædicta fallit. e. g. In cubicis æquationibus impoſſibiles, ſi modo ullæ ſint, radices ſemper deteget; in æquationibus n dimenſionum, quarum $(n$ vel $n-1)$ radices ſint impoſſibiles, probabilitas verum impoſſibilium radicum numerum e præcedente regulâ detegendi videtur eſſe ad probabilitatem verum impoſſibilium radicum numerum ex hâc regulâ detegendi prope in ratione $2^{n-2} : 3^{n-2}$.

In æquationibus multo ſuperiorum dimenſionum hæc regula perraro verum impoſſibilium radicum numerum deteget.

Hæc regula ſemper inveniet, utrum ulla contineatur impoſſibilitas in quatuor ſucceſſivis datæ æquationis terminis, necne.

Et ſic progredi licet ad regulas, quæ inveniunt, cum ulla impoſſibilitas contineatur in quintis, ſextis, &c. ſucceſſivis terminis; & exinde conſtabunt diverſæ regulæ de inveniendo impoſſibilium radicum numero.

Sæpe augendo vel diminuendo datæ æquationis radices per datas quantitates facile ſe manifeſtant impoſſibiles radices, quæ prius la-

tuere; & hoc conſtabit ex eo, quod, poſito radicem impoſſibilem eſſe $a+b\sqrt{(-1)}$, maxime pendet e ratione poſſibilis partis (a) ad impoſſibilem b, utrum hæ regulæ impoſſibilem radicem inveniant, necne.

Ex. 4. Sit æquatio $x^n + px^{n-1} + qx^{n-2} + rx^{n-3} + sx^{n-4} + tx^{n-5} + vx^{n-6} + wx^{n-7} + zx^{n-8} + \&c. \ldots . Ax^{n-m+4} + Bx^{n-m+3} + Cx^{n-m+2} + Dx^{n-m+1} + Ex^{n-m} + Fx^{n-m-1} + Gx^{n-m-2} + Hx^{n-m-3} + Ix^{n-m-4} + \&c. = 0$, cujus radices ſint $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \theta$, &c.

Colloca ſub primo & ultimo datæ æquationis termino ſignum $+$; ſub ſecundo ſignum $+$, ſi $\frac{n-1}{2n}p^2$ major ſit quam q, ſin minus ſignum $-$; ſub tertio ſignum $+$, ſi $\frac{n\times(n-1)-2}{2n.(n-1)}q^2$ major ſit quam $pr-s$, ſin minus ſignum $-$; colloca ſub quarto termino rx^{n-3} ſignum $+$, ſi $\frac{n\times(n-1)\times(n-2)-2.3}{2.n.(n-1).(n-2)}r^2$ major ſit quam $qs-pt+v$; & colloca ſub quinto termino (sx^{n-4}) ſignum $+$, ſi $\frac{n\times(n-1)\times(n-2)\times(n-3)-2.3.4}{2n.(n-1).(n-2).(n-3)}s^2$ major ſit quam $rt-qv+pw-z$; & in genere colloca ſub quolibet termino (Ex^{n-m}) ſignum $+$, ſi $\frac{n\times(n-1)\times(n-2)\ldots(n-m+1)-1.2.3.4.5\ldots m}{2.n.(n-1).(n-2).(n-3)..(n-m+1)}E^2$ major ſit quam $DF-CG+BH-AI+\&c.$, ſin minus ſignum $-$; tot vel plures erunt impoſſibiles radices, quot reſultant mutationes ſignorum de $+$ in $-$, & $-$ in $+$.

Facile e ſubſtitutione conſtabit $\overline{1-\frac{1}{n.\frac{n-1}{2}.\frac{n-2}{3}..\frac{n-m+1}{m}}}\times E^2$ majorem eſſe quam $2DF-2CG+2BH-2AI+\&c.$ ſummâ $\frac{\Upsilon}{n.\frac{n-1}{2}.\frac{n-2}{3}..\frac{n-m+1}{m}}$; ſi modo Υ ſit ſumma quadratorum e ſingulis differentiis inter quoſcunque duos coefficientis E terminos, i. e. $\Upsilon = \overline{\alpha\beta\gamma\delta\varepsilon\ \&c. - \alpha\beta\gamma\delta\zeta\ \&c.}^2 + \overline{\alpha\beta\gamma\delta\varepsilon\ \&c. - \alpha\beta\gamma\delta\eta\ \&c.}^2 +$

$\beta\gamma\delta\varepsilon\zeta$

$\overline{\beta\gamma\delta\epsilon\zeta \text{ \&c.} - \beta\gamma\delta\eta\theta \text{ \&c.}}^2 +$ &c. ubi $E = \alpha\beta\gamma\delta\epsilon$ &c. $+ \alpha\beta\gamma\delta\zeta$ &c. $+ \alpha\beta\gamma\delta\eta$ &c. $+ \beta\gamma\delta\epsilon\zeta$ &c. $+ \beta\gamma\delta\eta\theta$ &c. $+$ &c.

Multæ regulæ hujuſmodi etiam inveniri poſſunt e corollar. theor. 7. exinde enim inveniri poſſunt diverſæ quantitatum ſeries, quæ ſemper erunt affirmativæ, quando radices ſint poſſibiles; & haud raro negativæ, cum radices ſint impoſſibiles.

E coroll. theorem. 8. conſtant ſeries quantitatum, quæ ſemper erunt affirmativæ, cum omnes datæ æquationis radices vel ſint affirmativæ, vel negativæ.

Omnes hæ regulæ etiam generaliores reddi poſſunt calculum inſtituendo de quantitatibus ſuperiorum dimenſionum, quæ neceſſario erunt affirmativæ, quando radices ſint poſſibiles: ſed hæ methodi fient multo magis operoſæ.

Hæc principia etiam in alia problemata promovere licet. E. G. Sit ſeries quantitatum

$$\begin{array}{ll} \alpha + \beta + \gamma + \delta + \epsilon + \zeta + \text{\&c.} & = p \\ \alpha\beta + \alpha\gamma + \beta\gamma + \alpha\delta + \beta\delta + \gamma\delta + \alpha\epsilon + \text{\&c.} & = q \\ \alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta + \alpha\gamma\epsilon + \text{\&c.} & = r \\ \alpha\beta\gamma\delta + \alpha\beta\gamma\epsilon + \alpha\beta\gamma\zeta + \text{\&c.} & = s \\ \alpha\beta\gamma\delta\epsilon + \alpha\beta\gamma\delta\zeta + \text{\&c.} & = t \\ \text{\&c.} & \text{\&c.} \end{array}$$

Hæ autem quantitates continuo regulariter aſcendunt; & omnes erunt affirmativæ, ſi modo radices datæ æquationis ſint omnes affirmativæ; ergo conſtat ſaltem tot radices eſſe negativas vel impoſſibiles, quot ſunt mutationes ſignorum de + in —, & — in + ſucceſſive. i. e.

Collocentur termini hoc modo $1 + p + q + r + s + t +$ &c. & tot vel plures erunt radices negativæ vel impoſſibiles, quot ſunt mutationes ſignorum de + in —, & — in +.

Collocentur termini autem ſubſequenti methodo $1 - p + q - r + s - t +$ &c. & tot vel plures erunt negativæ vel impoſſibiles radices, quot ſunt continui progreſſus de + in +, & — in —.

Eodem modo hæc ratiocinandi methodus ad quamplurimos caſus

 appli-

applicari poteſt: datâ enim ſerie quantitatum regulariter aſcendentium $1 + a + b + c + d + e + f +$ &c. quæ neceſſario fient affirmativæ, quando radices ſint dati generis; ſæpe e mutatione ſignorum de $+$ in $-$, & $-$ in $+$ dici poteſt, quot ſunt quantitates diverſi generis.

Haud neceſſario in hâc argumentandi methodo, literæ $a, b, c, d, e,$ &c. quantitates affirmativas vel negativas deſignant; ſed etiam, cujuſcunque aſſignabilis generis quantitates denotare poſſunt.

PROB. X.

In multis caſibus e datis quantitatibus π, &c. quæ ſemper erunt affirmativæ vel negativæ, cum radices ſint poſſibiles; plures conſimiles deducere.

1. Sit π quæcunque rationalis functio coefficientium $a, b, c, d,$ &c. terminorum $x^{n-\alpha}$, $x^{n-\alpha-\beta}$, $x^{n-\alpha-\beta-\gamma}$, $x^{n-\alpha-\beta-\gamma-\delta}$, &c. datæ æquationis $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} \dots ax^{n-\alpha} \dots bx^{n-\alpha-\beta} \dots cx^{n-\alpha-\beta-\gamma} \dots$ &c. $= 0$, quæ ſemper erit affirmativa, cum radices datæ æquationis ſint poſſibiles; transformentur prædictæ coefficientes $a, b, c, d,$ &c. in alias $A, B, C, D,$ &c. coefficientes terminorum $x^{n-\mu}$, $x^{n-\mu-\beta}$, $x^{n-\mu-\beta-\gamma}$, $x^{n-\mu-\beta-\gamma-\delta}$, &c.; in quibus $\beta, \gamma, \delta,$ &c. ſint eædem ac in prædictis terminis, quorum coefficientes fuerunt $a, b, c, d,$ &c.: In datâ functione π pro a ſcribatur $n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \dots \frac{n-\alpha+1}{\alpha} \times \frac{1}{n} \times \frac{2}{n-1} \times \frac{3}{n-2} \times \frac{4}{n-3} \dots \frac{\mu}{n-\mu+1} \times A$, & pro b ſubſtituatur $n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \dots \frac{n-\alpha-\beta+1}{\alpha+\beta} \times \frac{1}{n} \times \frac{2}{n-1} \times \frac{3}{n-2} \dots \times \frac{\mu+\beta}{n-\mu-\beta+1} \times B$; & ſic pro c & d ſcribantur reſpective $n \cdot \frac{n-1}{2} \times \frac{n-2}{3} \dots \frac{n-\alpha-\beta-\gamma+1}{\alpha+\beta+\gamma} \times \frac{1}{n} \times \frac{2}{n-1} \times \frac{3}{n-2} \dots \times \frac{\mu+\beta+\gamma}{n-\mu-\beta-\gamma+1} \times C$ & $n \cdot \frac{n-1}{2} \times \frac{n-2}{3} \dots$

$n-\alpha$

$\frac{n-\alpha-\beta-\gamma-\delta+1}{\alpha+\beta+\gamma+\delta} \times \frac{1}{n} \times \frac{2}{n-1} \times \frac{3}{n-2} \times \frac{4}{n-3} \dots \frac{\mu+\beta+\gamma+\delta}{n-\mu-\beta-\gamma-\delta+1}$ $\times D$, &c. & functio quantitatum A, B, C, D, &c. ex hujuſmodi ſubſtitutione reſultans erit quantitas deſiderata: ſed hìc animadvertendum eſt, ut reſpectivæ dimenſiones e ſingulis terminis datæ & reſultantis quantitatis ſint eædem, *i.e.* eaſdem dimenſiones radicum datæ æquationis reſpective contineant ſinguli earum termini; aliter haud vera eſt reſolutio.

2. Ex æquatione $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} \dots ax^{n-\alpha} \dots bx^{n-\alpha-\beta} \dots cx^{n-\alpha-\beta-\gamma} \dots dx^{n-\alpha-\beta-\gamma-\delta} +$ &c. $= 0$, & functione π coefficientium a, b, c, d, &c. in quâ eædem continentur datæ æquationis radicum dimenſiones, & quæ eſt vel affirmativa vel negativa, cum omnes radices prædictæ æquationis ſint poſſibiles, datis; erui poteſt functio ejuſdem generis quantitatum A, B, C, D, &c. correſpondentium terminorum $x^{m-\alpha}$, $x^{m-\alpha-\beta}$, $x^{m-\alpha-\beta-\gamma}$, $x^{m-\alpha-\beta-\gamma-\delta}$, &c. æquationis $x^m - Px^{m-1} + Qx^{m-2} \dots Ax^{m-\alpha} \dots Bx^{m-\alpha-\beta} \dots Cx^{m-\alpha-\beta-\gamma} \dots Dx^{m-\alpha-\beta-\gamma-\delta} \dots +$ &c. $= 0$. In datâ functione (π) pro a, b, c, d, &c. ſcribantur reſpective $\frac{n}{m} \times \frac{n-1}{m-1} \cdot \frac{n-2}{m-2} \cdot \frac{n-3}{m-3} \dots \frac{n-\alpha+1}{m-\alpha+1} \times A$, $\frac{n}{m} \cdot \frac{n-1}{m-1} \cdot \frac{n-2}{m-2} \dots \frac{n-\alpha-\beta+1}{m-\alpha-\beta+1} \times B$, $\frac{n}{m} \cdot \frac{n-1}{m-1} \cdot \frac{n-2}{m-2} \dots \frac{n-\alpha-\beta-\gamma+1}{m-\alpha-\beta-\gamma+1} \times C$, & ſic deinceps; reſultabit functio quantitatum A, B, C, &c. quæſita.

Ex. 1. Sit quadratica æquatio $x^2 - px + q = 0$; $p^2 - 4q$ erit affirmativa quantitas, ſi modo radices datæ æquationis ſint poſſibiles; in hoc caſu $n = 2$; aſſumatur æquatio $x^m - Px^{m-1} + Qx^{m-2} -$ &c. $= 0$; pro p & q in functione $p^2 - 4q$ ſcribantur reſpective $\frac{2}{m} P$ & $\frac{2 \cdot 1 Q}{m \cdot m-1}$; reſultat quantitas $\frac{4}{m^2} P^2 - \frac{8Q}{m \cdot m-1}$, quæ ſemper erit affirmativa, cum omnes radices æquationis $x^m - Px^{m-1} + Qx^{m-2} -$ &c.

&c. $= o$ ſint poſſibiles; ducatur quantitas reſultans in $\frac{m \cdot \overline{m-1}}{8}$, & exorietur quantitas $\frac{m-1}{2m} P^2 - Q$, quæ erit affirmativa, cum omnes radices prædictæ ſint poſſibiles.

Ex. 2. Sit æquatio $x^m - px^{m-1} + qx^{m-2} \ldots \pm Ax^{m-\alpha+1} \mp Bx^{m-\alpha} \pm Cx^{m-\alpha-1} \mp \ldots + ax^2 - bx + c = o$: In hâc æquatione pro x ſcribatur $\frac{1}{z}$, reſultat æquatio $z^m - \frac{b}{c} z^{m-1} + \frac{a}{c} z^{m-2} \ldots \pm \frac{C}{c} z^{\alpha+1} \mp \frac{B}{c} z^{\alpha} \pm \frac{A}{c} z^{\alpha-1} \mp$ &c. $= o$; ſed per præcedens Ex. conſtat $\frac{m-1}{2m} P^2 - Q$, quæ in hoc exemplo evadit $\frac{m-1}{2m} \frac{b^2}{c^2} - \frac{a}{c}$, & conſequenter $\frac{m-1}{2m} b^2 - ac$ eſſe affirmativam quantitatem, cum omnes radices datæ æquationis ſint poſſibiles: deinde per caſ. 1[mum] in hâc quantitate pro a, b & c ſcribantur reſpective $m \cdot \frac{m-1}{2} \times \frac{1}{m} \cdot \frac{2}{m-1} \cdot \frac{3}{m-2} \cdots \frac{\alpha-1}{m-\alpha+2} \times A$, $m \times \frac{1}{m} \cdot \frac{2}{m-1} \cdot \frac{3}{m-2} \cdots \frac{\alpha}{m-\alpha+1} \times B$ & $\frac{1}{m} \cdot \frac{2}{m-1} \cdot \frac{3}{m-2} \cdots \frac{\alpha+1}{m-\alpha} \times C$; & exorietur $\frac{m-1}{2m} \times \frac{2^2}{\overline{m-1}^2} \times \frac{3^2}{\overline{m-2}^2} \cdots \frac{\alpha^2}{(m-\alpha+1)^2} \times B^2 - \frac{1}{m} \cdot \frac{2}{m-1} \cdot \frac{3^2}{\overline{m-2}^2} \cdot \frac{4}{\overline{m-3}^2} \cdots \frac{\overline{\alpha-1}^2}{\overline{m-\alpha+2}^2} \times \frac{\alpha}{m-\alpha+1} \times \frac{\alpha+1}{m-\alpha} \times AC$; exinde $\frac{\alpha}{m-\alpha+1} \times B^2 - \frac{\alpha+1}{m-\alpha} \times AC$, & conſequenter $\frac{\alpha \times \overline{m-\alpha}}{\overline{\alpha+1} \times \overline{m-\alpha+1}} B^2 - AC$ erit affirmativa quantitas, ſi modo omnes radices datæ æquationis ſint poſſibiles. Et ſic de pluribus exemplis.

PROB.

PROB. XI.

E regulis de impoſſibilibus radicibus detegendis prius datis; alias invenire, quæ magis generalem præbeant ſolutionem.

Sit quæcunque æquatio

$$x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \&c. = 0$$

ducatur hæc æquatio in $x - a$, reſultat

$$\begin{array}{llllll} x^{n+1} & - px^n & + qx^{n-1} & - rx^{n-2} & + sx^{n-3} & - \&c. = 0. \\ & - a & + ap & - aq & + ar & \end{array}$$

Inveniatur, ſi modo facile poſſit, annon ita aſſumi poteſt incognita quantitas (a), ut præbeat datam impoſſibilium radicum notam; exinde ſequitur regula.

Hæc regula eodem modo adhuc magis generalis reddi poteſt, ſi modo reſultans æquatio ducatur in $x - b$; & inveniatur, annon ita aſſumi poteſt quantitas b, ut præbeat notam impoſſibilium radicum e regulis datis inveſtigatam; & ſic deinceps.

Ex. 1. Datâ æquatione $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. + Px^{n-m+1} - Qx^{n-m} + Rx^{n-m-1} - Sx^{n-m-2} + \&c. = 0$, quæ ducta in $x - a$, fit

$$\begin{array}{llllllll} x^{n+1} & - px^n & + qx^{n-1} & - rx^{n-2} & + \&c. & - Qx^{n-m+1} & + Rx^{n-m} & - Sx^{n-m-1} + \&c. = 0; \\ & - a & + ap & - aq & & - aP & + aQ & - aR \end{array}$$

quicunque aſſumatur incognitæ quantitatis (a) valor, erit per regulam prius traditam $\overline{p + a}^2$ major quam $\overline{q + ap}$, $\overline{q + ap}^2$ major quam $\overline{p + a} \times \overline{r + aq}$, & ſic deinceps; ſi modo radices datæ æquationis ſint poſſibiles.

Sed quoniam per hypotheſin $\overline{p + a}^2$ ſemper major erit quam $\overline{q + ap}$; $\overline{p + a}^2 - ap = \overline{a + \frac{1}{2}p}^2 + \frac{3}{4}p^2$ ſemper major erit quam q, quicunque ſit valor quantitatis (a); & conſequenter $\frac{3}{4}p^2$ major erit quam q: & eodem modo, quoniam $\overline{q + ap}^2$ major erit quam $\overline{p + a} \times \overline{r + aq}$, & conſequenter $\overline{p^2 - q}\, a^2 + \overline{qp - r}\, a + \overline{q^2 - pr}$ ſemper erit affirmativa quantitas; exinde invenietur $4 \times \overline{p^2 - q} \times \overline{q^2 - pr}$ major quam

Cor. Sit æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = o$, in eâ pro x ſcribatur $z - a$, & ſit æquatio reſultans $z^n - Pz^{n-1} + Qz^{n-2} - Rz^{n-3} + \&c. = o$, cujus radices ſint α, β, γ, δ, &c.; deinde inveniatur æquatio $v^n - Av^{n-1} + Bv^{n-2} - Cv^{n-3} + Dv^{n-4} - \&c. = o$, cujus radices ſint α^2, β^2, γ^2, δ^2, &c. vel magis generaliter rationalis functio radicum reſultantis æquationis, quæ erit affirmativa, cum ejus correſpondens radix datæ æquationis ſit poſſibilis quantitas: fingantur $A = o$, $B = o$, $C = o$, $D = o$, &c.; &, ſi radices datæ æquationis $x^n - px^{n-1} + qx^{n-2} - \&c. = o$ ſint omnes poſſibiles, tum omnes radices (a) æquationum $A = o$, $B = o$, $C = o$, $D = o$, &c. erunt impoſſibiles; ni duæ vel quatuor, &c. evadant inter ſe æquales; & vice verſâ, ſi $v = \alpha^2$ vel $= \beta^2$ vel $= \gamma^2$, &c. & omnes radices æquationum $A = o$, $B = o$, $C = o$, $D = o$, &c. ſint impoſſibiles, tum radices datæ æquationis $x^n - px^{n-1} + qx^{n-2} - \&c. = o$ erunt omnes poſſibiles.

Et ſic de pluribus hujuſce caſûs exemplis.

PROB. XII.

Datâ æquatione $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \&c. = o$; *invenire, utrum ullas habet impoſſibiles radices, necne.*

Sint α, β, γ, δ, ϵ, &c. radices datæ æquationis, per problema quintum transformetur data æquatio in alteram, cujus radices (v) ſint $\alpha^2 + \beta^2 - 2\alpha\beta$, $\alpha^2 + \gamma^2 - 2\alpha\gamma$, $\alpha^2 + \delta^2 - 2\alpha\delta$, $\beta^2 + \gamma^2 - 2\beta\gamma$, $\beta^2 + \delta^2 - 2\beta\delta$, &c. æquatio reſultans eſt

$$v^{n\times\frac{n-1}{2}} - \overline{\overline{n-1}\,p^2 - 2nq}\,v^{n\times\frac{n-1}{2}-1} + \overline{\overline{n-1} \times \frac{n-2}{2}p^4 - 2n \times \overline{n-2}\,p^2q - 2 \times \overline{n-3}\,pr + 2ns + \overline{2n+3} \times \overline{n-2} \times q^2}\;v^{n\times\frac{n-1}{2}-2} - \&c. = o,$$

ſi mutentur hujuſce æquationis reſultantis ſigna alternatim de $+$ in $-$, & $-$ in $+$; nullam radicem impoſſibilem habet data æquatio: ſin hujuſce æquationis ſigna haud continuo mutentur de $+$ in $-$, & $-$ in $+$; impoſſibiles radices habet data æquatio.

Sint

a

Sint enim omnes datæ æquationis radices (α, β, γ, δ, &c.) poſſibiles; reſultantis æquationis radices ($\overline{\alpha-\beta}^2$, $\overline{\alpha-\gamma}^2$, $\overline{\alpha-\delta}^2$, $\overline{\beta-\gamma}^2$, &c.) erunt affirmativæ, & æquationis ſigna alternatim de + in —, & — in + mutabuntur: ſint vero duæ datæ æquationis radices impoſſibiles, viz. $a+\sqrt{-b^2}$ & $a-\sqrt{-b^2}$; & una radix reſultantis æquationis erit $\overline{a+\sqrt{-b^2}-a+\sqrt{-b^2}}^2 = \overline{2\sqrt{-b^2}}^2 = -4b^2$; quæ eſt negativa quantitas; & conſequenter ſignorum haud continuo erit mutatio de + in —, & — in +.

Cor. Sit $n.\frac{n-1}{2}$ par numerus; numerus impoſſibilium datæ æquationis radicum erit 0, 4, 8, 12, 16, &c. vel 2, 6, 10, 14, &c. prout ultimus reſultantis æquationis terminus ſit affirmativa vel negativa quantitas: ſi vero $n.\frac{n-1}{2}$ ſit impar; tum 0, 4, 8, 12, &c. vel 2, 6, 10, &c. radices erint impoſſibiles, prout prædictus ultimus terminus ſit negativa vel affirmativa quantitas.

Cor. 2. Si m ultimi reſultantis æquationis termini nihilo ſint æquales, & $n.\frac{n-1}{2}-m$ ſit par numerus; tum 0, 4, 8, 12, &c. vel 2, 6, 10, 14, &c. radices datæ æquationis erunt impoſſibiles, prout ultimus reliquorum reſultantis æquationis terminorum ſit affirmativa vel negativa: ſit $n.\frac{n-1}{2}-m$ impar numerus; tum 0, 4, 8, &c.; vel 2, 6, 10, &c. radices erunt impoſſibiles, prout prædictus ultimus terminus ſit negativa vel affirmativa quantitas.

Ex. 1. Sit data æquatio $x^4-10x^3+35x^2-50x+24=0$, & æquatio reſultans erit $v^6-20v^5+142v^4-460v^3+713v^2-520v+144=0$: hujus æquationis ſigna mutantur alternatim de + in —, & — in +; & conſequenter nullam habet impoſſibilem radicem data æquatio.

Ex. 2. Sit æquatio $x^3+qx-r=0$, cujus radices ſint α, β, γ;

 æqua-

æquatio, cujus radices ſunt $\overline{\alpha-\beta}^2$, $\overline{\alpha-\gamma}^2$, $\overline{\beta-\gamma}^2$, erit

$$v^3+6qv^2+9q^2v+4q^3+27r^2=0.$$

Sint q & $4q^3+27r^2$ negativæ quantitates, ſigna mutantur alternatim de + in —, & — in +; & conſequenter nullam impoſſibilem radicem habet data æquatio: ſit hæc vel illa affirmativa, & haud continuo mutantur ſigna de + in —, & — in +; & conſequenter duas impoſſibiles radices habet data æquatio; tertia vero radix erit affirmativa vel negativa, prout ejus ultima coefficiens (r) ſit affirmativa vel negativa quantitas.

Sit $4q^3+27r^2=0$; & omnes radices erunt poſſibiles, duæ vero æquales.

Ex. Sit æquatio $x^4+qx^2-rx+s=0$, cujus radices ſint $\alpha, \beta, \gamma, \delta$; æquatio, cujus radices ſunt ($\overline{\alpha-\beta}^2$, $\overline{\alpha-\gamma}^2$, $\overline{\beta-\gamma}^2$, $\overline{\alpha-\delta}^2$, $\overline{\beta-\delta}^2$, &c.) erit

$$v^6+8q\,v^5\left.\begin{matrix}+22q^2\\+8s\end{matrix}\right\}v^4\left.\begin{matrix}+28q^3\\+26r^2\\+16qs\end{matrix}\right\}v^3\left.\begin{matrix}+17q^4\\+48r^2q\\-112s^2\\+24q^2s\end{matrix}\right\}v^2\left.\begin{matrix}+4q^5\\+18q^2r^2\\+32q^3s\\-192qs^2\\+216r^2s\end{matrix}\right\}v\begin{matrix}+256s^3\\-128q^2s^2\\+144r^2qs\\+16q^4s\\-27r^4\\-4r^2q^3\end{matrix}=0.$$

Sit $256s^3-128q^2s^2+144r^2qs+16q^4s-27r^4-4r^2q^3$ negativa quantitas; tum duas & non plures impoſſibiles radices habet data æquatio. Sit vero affirmativa quantitas, & ſigna haud continuo mutentur de + in —, & — in +; tum datæ æquationis quatuor radices erunt impoſſibiles: ſit nihilo æqualis, & ſigna haud continuo mutentur de + in —, & — in +; tum duæ vel quatuor radices datæ æquationis erunt impoſſibiles, prout ultimus ſolummodo vel duo ultimi datæ æquationis termini nihilo ſint æquales.

Paulo facilior reddi poteſt hæc ſolutio. Sit ultimus datæ æquationis terminus affirmativa, & vel q affirmativa, vel q^2-4s negativa quantitas; tum quatuor datæ æquationis radices erunt impoſſibiles; ſin aliter omnes ejus radices erunt poſſibiles.

Ex. 4. Sit æquatio $x^5+qx^3-rx^2+sx-t=0$, cujus radices ſint

$\alpha, \beta,$

$a, \beta, \gamma, \delta, \epsilon$: æquatio, cujus radices ſunt $(\overline{\alpha-\beta}^2, \overline{\alpha-\gamma}^2, \overline{\beta-\gamma}^2, \overline{\alpha-\delta}^2,$
$\overline{\beta-\delta}^2$, &c.) erit $w^{10} + 10qw^9 + 39q^2 + 10s \times w^8 + 80q^3 + 50qs + 25r^2$
$\times w^7 + 95q^4 + 124q^2s - 95s^2 + 92qr^2 + 200rt \times w^6 + 66q^5 - 360qs^2$
$+ 169q^3s + 118q^2r^2 + 260r^2s + 625t^2 + 400qrt \times w^5 + 25q^6 + 40s^3$
$- 53r^4 + 52q^3r^2 - 522q^2s^2 + 194q^4s + 708qr^2s + 240q^2rt +$
$1750qt^2 - 950srt \times w^4 + 4q^7 + 106q^5s - 80qs^3 - 308q^3s^2 -$
$102qr^4 - 7q^4r^2 + 570r^2s^2 + 612q^2r^2s + 700r^3t - 3750t^2s +$
$2500t^2q^2 + 80rtq^3 - 2150qrst \times w^3 + 400s^4 - 360q^2s^3 - 15q^4s^2$
$+ 24q^6s - 8q^5r^2 - 45q^2r^4 - 270r^4s + 140r^2sq^3 + 960r^2s^2q +$
$1875t^2r^2 + 1000trs^2 - 5000t^2qs + 1750t^2q^3 + 40trq^4 + 600tr^3q$
$- 1650trsq^2 \times w^2 + 36q^5s^2 - 224q^3s^3 + 320qs^4 + 4q^3r^4 + 27r^6$
$- 40r^2s^3 + 434r^2q^2s^2 - 24r^2sq^4 - 198r^4qs + 5000t^2s^2 - 450tr^3s$
$- 6250t^3r + 675t^2q^4 - 3750t^2q^2s + 3000t^2r^2q + 60tr^3q^2 + 200$
$trs^2q - 330trq^3s \times w + 3125t^4 - 3750qrt^3 + 2000s^2q + 2250r^2s$
$- 900sq^3 + 825r^2q^2 + 108q^5 \times t^2 - 1600s^3r - 560rq^2s^2 - 16r^3q^3$
$+ 630r^3qs + 72rsq^4 - 108r^5 \times t + 256s^5 - 128q^2s^4 + 144r^2qs^3 +$
$16q^4s^3 - 27r^4s^2 - 4r^2q^3s^2 = 0$: ſi continuo mutentur hujuſce æquationis ſigna de + in —, & — in +; nullas impoſſibiles radices habet data æquatio.

Si ſigna terminorum æquationis haud continuo mutentur de + in —, & — in +; duæ, vel quatuor datæ æquationis radices erunt impoſſibiles; prout ultimus ejus terminus ſit negativa, vel affirmativa quantitas.

Si ultimus terminus nihilo ſit æqualis, & ſigna terminorum æquationis haud continuo mutentur de + in —, & — in +; tum quatuor, vel duæ radices datæ æquationis erunt impoſſibiles, prout duo, & non plures ultimi datæ æquationis termini nihilo ſint æquales, necne.

Cor.

Cor. Eodem prorſus modo cum juſtâ terminorum comparatione inveniri poteſt numerus impoſſibilium radicum in quâcunque æquatione contentarum.

Cor. 2. Sit data æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = o$; transformetur hæc æquatio in alteram, cujus radix eſt $x - a$, & data & reſultans æquatio transformentur in æquationem methodo hoc problemate contentâ; ex utrâque transformatione eadem reſultabit æquatio.

Cor. 3. Si ultimus reſultantis æquationis terminus nihilo ſit æqualis, tum duæ radices datæ æquationis erunt inter ſe æquales; ſi duo ultimi reſultantis æquationis termini nihilo ſint æquales, tum duæ radices datæ æquationis bis fiunt æquales; ſi tres vel quatuor ultimi reſultantis æquationis termini nihilo ſint æquales, tum duæ radices ter vel quater fiunt inter ſe æquales; & ſic deinceps.

PROB. XIII.

Invenire numerum impoſſibilium radicum in datâ æquatione contentarum e diverſâ transformatione.

1. Sit æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \&c. = o$, & $ax^n - \overline{a+b}\,px^{n-1} + \overline{a+2b}\,qx^{n-2} - \overline{a+3b}\,rx^{n-3} + \&c. = v$; vel $ax^n - \overline{a+b}\,px^{n-1} + \overline{a+2b}\,qx^{n-2} - \overline{a+3b}\,rx^{n-3} + \&c. = o$ & $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = v$; ita reducantur duæ æquationes, ut exterminetur x, reſultabit æquatio $v^n - Av^{n-1} + Bv^{n-2} - Cv^{n-3} + \ldots \mp Pv^3 \pm Qv^2 \mp Rv \pm S = o$, cujus radix eſt v: cum duæ datæ æquationis radices ſint inter ſe æquales, tum erit in priori caſu $S = o$ & $R = o$; & in genere tot ultimi reſultantis æquationis termini nihilo erunt æquales, quot æquales inter ſe inveniuntur radices datæ æquationis: ex ultimo reſultantis æquationis termino dici poteſt; utrum duas, ſex, decem, &c.; vel nullas, quatuor, octo, &c. radices impoſſibiles habet data æquatio.

Ex. 1. Sit data æquatio $x^2 + q = o$, & ſupponatur $2x = v$, & æqua-

æquatio reſultans erit $v^2 + 4q = 0$: duæ vero radices per problema erunt impoſſibiles, necne; prout ultimus reſultantis æquationis terminus $(4q)$ ſit affirmativa vel negativa quantitas.

Ex. 2. Sit data æquatio $x^3 + qx - r = 0$, ſupponatur $3x^2 + q = v$; & erit $v^3 + 3qv^2 * - 4q^3 - 27r^2 = 0$.

Ex. 3. Sit data æquatio $x^4 + qx^2 - rx + s = 0$, & ſupponatur $4x^3 + 2qx - r = v$; & æquatio reſultans erit $v^4 - 8rv^3 + \overline{4q^3 - 16qs + 18r^2}\, v^2 * + 256s^3 - 128s^2q^2 + 16sq^4 + 144r^2sq - 4r^2q^3 - 27r^4 = 0$.

Cor. Sit data æquatio (A) $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0$, ſupponatur $nx^{n-1} - \overline{n-1}\,px^{n-2} + \overline{n-2}\,qx^{n-3} - \&c. = v$, tum erit æquatio $v^n - Av^{n-1} + Bv^{n-2} \ldots + Pv^3 + Qv^2 * + S = 0$: ſi omnes radices ſint poſſibiles, tum erit S negativa quantitas, cum n vel $= 4m + 2$ vel $4m + 1$, ubi m eſt integer numerus; ſin aliter S erit affirmativa quantitas.

Cor. 2. Sit æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. \pm \pi x^2 \mp \varrho x \pm \sigma = 0$, aſſumantur $nx^{n-1} - \overline{n-1}\,px^{n-2} + \overline{n-2}\,qx^{n-3} - \&c. = v$, & $nx^n - \overline{n-1}\,px^{n-1} + \overline{n-2}\,qx^{n-2} - \&c. = w$; ſint æquationes, quarum radices ſunt v & w, reſpective $v^n - Av^{n-1} + Bv^{n-2} - \ldots \pm Pv^3 \mp Qv^2 * \mp S = 0$ & $w^n - A'w^{n-1} + B'w^{n-2} - \ldots \pm P'w^3 \mp Q'w^2 \pm R'w \mp S' = 0$; tum erit $S : S' :: 1 : \sigma$ & $R' = \pm S$.

Sint radices datæ æquationis $(x^n - px^{n-1} + qx^{n-2} - \&c. = 0)$ reſpective $\alpha, \beta, \gamma, \delta$, &c.; tum erunt radices (v) reſpective $\overline{\alpha-\beta} \times \overline{\alpha-\gamma} \times \overline{\alpha-\delta} \times \&c.$, $\overline{\beta-\alpha} \times \overline{\beta-\gamma} \times \overline{\beta-\delta} \times \&c.$, $\overline{\gamma-\alpha} \times \overline{\gamma-\beta} \times \overline{\gamma-\delta} \times \&c.$, $\overline{\delta-\alpha} \times \overline{\delta-\beta} \times \overline{\delta-\gamma} \times \&c.$, &c.: radices vero (w) erunt $\overline{\alpha-\beta} \times \overline{\alpha-\gamma} \times \overline{\alpha-\delta} \times \&c. \times \alpha$, $\overline{\beta-\alpha} \times \overline{\beta-\gamma} \times \overline{\beta-\delta} \times \&c. \times \beta$, $\overline{\gamma-\alpha} \times \overline{\gamma-\beta} \times \overline{\gamma-\delta} \times \&c. \times \gamma$, &c.

Ex. 4. Sit data æquatio $x^2 - px + q = 0$; æquatio ad limites erit $2z - p = 0$, ſupponatur $z^2 - pz + q = v$, & reſultabit $v + \frac{1}{4}p^2 - q = 0$; & exinde duæ radices datæ æquationis erunt impoſſibiles, necne, prout $\frac{1}{4}p^2 - q$ ſit negativa vel affirmativa quantitas.

Ex.

Ex. 5. Sit $x^3 + qx - r = 0$; & æquatio ad limites erit $3z^2 + q = 0$; ſupponatur $z^3 + qz - r = v$, & reſultabit $v^2 + 2rv + r^2 + \frac{4q^3}{27} = 0$: duæ radices datæ æquationis erunt impoſſibiles, necne; prout $r^2 + \frac{4q^3}{27}$ ſit affirmativa vel negativa quantitas.

Ex. 6. Sit $x^4 + qx^2 - rx + s = 0$, & æquatio ad limites erit $4z^3 + 2qz - r = 0$, ſupponatur $z^4 + qz^2 - rz + s = v$, & reſultabit æquatio $v^3 + \overline{\frac{q^2}{2} - 3s}\, v^2 + \overline{3s^2 - q^2 s + \frac{q^4}{16} + \frac{9r^2q}{16}}\, v - s^3 + \frac{s^2q^2}{2} - \frac{sq^4}{16} - \frac{9r^2sq}{16} + \frac{r^2q^3}{64} + \frac{27r^4}{256} = 0$.

Si ultimus hujuſce æquationis terminus ſit affirmativa quantitas, tum duæ ſolummodo datæ æquationis radices erunt impoſſibiles.

Et ſic de pluribus exemplis.

2. Sit æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0$; ſingatur eodem modo, quo prius docetur

$$(P)\ z^{n-1} - \frac{n-1}{n} pz^{n-2} + \frac{n-2}{n} qz^{n-3} - \&c. = 0$$

$$\&\ z^n - pz^{n-1} + qz^{n-2} - rz^{n-3} + \&c. = v$$

transformentur hæ duæ æquationes in unam, ita ut exterminetur incognita quantitas (z), & æquatio reſultans ſit

$$v^{n-1} - Pv^{n-2} + Qv^{n-3} - Rv^{n-4} + Sv^{n-5} - \&c. = 0,$$

tum tot vel plures impoſſibiles radices habet æquatio $(A)\ x^n - px^{n-1} + qx^{n-2} - \&c. = 0$; 1. (ſi n ſit impar numerus) quot ſit differentia (τ) inter numerum progreſſuum ſignorum de + in +, & — in —, & numerum mutationum ſignorum de + in —, & — in +, in reſultante æquatione: 2. ſin vero (n) ſit par numerus; tum tot vel plures habet impoſſibiles radices, quot ſit (τ) prædicta differentia ab unitate diminuta, ſi numerus progreſſuum major ſit quam numerus mutationum; ſin aliter ab unitate aucta.

Ex.

Ex. 1. Sint duæ æquationes $3x^2 + 2px + q = 0$, & $x^3 + px^2 + qx + r = v$; inveniatur æquatio (cujus radix eſt v) $v^2 + Pv + Q = 0$; tum duæ radices æquationis $x^3 + px^2 + qx + r = 0$ erunt impoſſibiles, necne; prout Q ſit affirmativa vel negativa quantitas.

Ex. 2. Sint duæ æquationes $4x^3 + 3px^2 + 2qx + r = 0$ & $x^4 + px^3 + qx^2 + rx + s = v$; inveniatur æquatio (cujus radix eſt v) $v^3 + Pv^2 + Qv + R = 0$; tum, ſi R ſit affirmativa quantitas, duæ & haud plures radices æquationis $x^4 + px^3 + qx^2 + rx + s = 0$ erunt impoſſibiles: ſi vero omnes radices æquationis $4x^3 + 3px^2 + 2qx + r = 0$ ſint poſſibiles, & in æquatione $v^3 + Pv^2 + Qv + R = 0$ una & haud plures detur mutatio ſignorum de $+$ in $-$, vel $-$ in $+$; tum omnes radices prædictæ æquationis $x^4 + px^3 + qx^2 + rx + s = 0$ erunt poſſibiles: ſin aliter, nullæ.

Ex. 3. Sint duæ æquationes $5x^4 + 4px^3 + 3qx^2 + 2rx + s = 0$ & $x^5 + px^4 + qx^3 + rx^2 + sx + t = 0$; æquatio, cujus radix eſt v, ſit $v^4 + Pv^3 + Qv^2 + Rv + S = 0$: tum 1mo. ſi S ſit negativa quantitas, erunt duæ & haud plures radices æquationis $x^5 + px^4 + qx^3 + rx^2 + sx + t = 0$ impoſſibiles: 2do. ſi S ſit affirmativa quantitas, & omnes radices æquationis $5x^4 + 4px^3 + 3qx^2 + 2rx + s = 0$ poſſibiles, & duæ & haud plures in æquatione $v^4 + Pv^3 + Qv^2 + Rv + S = 0$ contineantur mutationes ſignorum de $+$ in $-$ & $-$ in $+$; tum erunt omnes radices æquationis $x^5 + px^4 + qx^3 + rx^2 + sx + t = 0$ poſſibiles: ſin aliter, quatuor ejus radices erunt impoſſibiles.

PROB. XIV.

Datâ æquatione $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0$; *invenire, quot ſint ejus radices poſſibiles, & quot impoſſibiles.*

Inveniantur quantitates, quæ nihilo fiant æquales, cum duæ radices fiant æquales; & cum duæ radices bis, ter, quater, &c. fiant æquales; & quæ nullum recipiant diviſorem; e quantitatibus reſultantibus erui poteſt numerus impoſſibilium radicum.

Quantitas, quæ nihilo ſit æqualis, cum duæ radices datæ æquationis fiant æquales, & quæ diviſores haud admittat; vel evadet negativa, cum 2, 6, 10, 14, 18, &c. radices fiant impoſſibiles; & affirmativa, cum 0, 4, 8, 12, 16, &c. fiant impoſſibiles: vel affirmativa, cum 2, 6, 10, 14, 18, &c.; & negativa, cum 0, 4, 8, 12, 16, &c. radices fiant impoſſibiles.

Dividatur data æquatio per $x^2 - 2ax + a^2$, & fiant duo reſidua ex hâc diviſione reſultantia nihilo æqualia; ita reducantur duæ æquationes reſultantes, ut exterminetur incognita quantitas (a); reſultat quantitas, quæ eſt ultimus terminus in quibuſdam æquationibus prius traditis; & quæ ſemper eſt nihilo æqualis, cum duæ radices datæ æquationis evadant inter ſe æquales. Dividatur data æquatio per $\overline{x^2 - 2ax + a^2} \times \overline{x^2 - 2bx + b^2}$; fiant quatuor reſidua exinde reſultantia nihilo æqualia; ita reducantur quatuor æquationes reſultantes, ut exterminentur duæ incognitæ quantitates a & b; reſultant duæ vel plures quantitates, quæ nihilo evadent æquales; cum duæ radices datæ æquationis bis evadant inter ſe æquales: deinde dividatur data æquatio per $\overline{x - a}^3$ & $\overline{x - a}^4$, & ita reducantur tria vel quatuor reſidua reſultantia, ut exterminetur incognita quantitas (a); reſultant duæ vel tres vel plures quantitates; quæ nihilo, cum tres vel quatuor radices datæ æquationis, evadent æquales: & ſic dividatur data æquatio per $\overline{x^2 - 2ax + a^2} \times \overline{x^2 - 2bx + b^2} \times \overline{x^2 - 2cx + c^2}$, & fiant ſex reſidua exinde reſultantia nihilo æqualia; ita reducantur ſex reſultantes æquationes, ut exterminentur tres incognitæ quantitates a, b & c; reſultant tres vel plures quantitates, quæ nihilo evadent æquales, cum duæ radices datæ æquationis ter fiant inter ſe æquales: & ſic deinceps: Ex quantitatibus ſic deductis, i. e. ex omnibus quantitatibus conjunctim perpenſis, quæ neceſſario evadant affirmativæ vel negativæ; cum duæ vel plures poſſibiles vel impoſſibiles datæ æquationis radices ſemel, bis, ter, &c. ſejunctim & conjunctim evadant inter ſe æquales; deduci poteſt impoſſibilium datæ æquationis radicum numerus: quantitates enim enim primum fiunt æquales, deinde

impoſſibiles; & conſequenter omnis regula, quæ invenit, cum radices fiant impoſſibiles; neceſſario etiam inveniet, cum fiant æquales: & vice versâ, omnis regulæ ope, quæ omnes diverſos caſus generaliter detegit; cum radices ſemel, bis, ter, &c. fiant æquales; inveniri poteſt, cum radices evadant impoſſibiles.

Ex. 1. Sit æquatio $x^4 + qx^2 - rx + s = 0$, dividatur hæc æquatio per $x^2 - 2ax + a^2$, reſultant duo reſidua $4a^3 + 2qa - r$ & $3a^4 + qa^2 - s$; fiant hæc reſidua nihilo æqualia, & ita reducantur hæ æquationes reſultantes in unam, ut exterminetur a; reſultat quantitas (A) $256s^3 - 128q^2s^2 + 144r^2qs + 16q^4s - 27r^4 - 4r^2q^3$, quæ nihilo erit æqualis, cum duæ radices datæ æquationis ſint æquales. 2do. Dividatur data æquatio per $(x^2 - 2ax + a^2) \times (x^2 - 2bx + b^2) = x^4 - 2a^2x^2 + a^4$, quoniam coefficiens ſecundi datæ æquationis termini nihilo eſt æqualis; duo reſidua erunt $q + 2a^2$ & $s - a^4$; fiant hæc reſidua nihilo reſpective æqualia, & reſultabunt $q + 2a^2 = 0$ & $s - a^4 = 0$; reducantur hæ duæ æquationes in unam, ita ut exterminetur a, & reſultabit $q^2 - 4s = 0$; inde e tribus quantitatibus A, q & $q^2 - 4s$ cognoſci poteſt numerus impoſſibilium radicum in datâ æquatione contentarum.

Ex. 2. Sit æquatio $x^n - Ax^m - B = 0$, & ſit 1mo. n par numerus & $-B$ negativa quantitas; tum duas radices poſſibiles habet data æquatio: ſit n par numerus, m vero impar & $-B$ affirmativa quantitas; tum duæ radices datæ æquationis erunt poſſibiles, ſi $\overline{\frac{m-n}{n^{n-m}}}^{\,n-m} A^n - \frac{n^m}{m^m} B^{n-m}$ ſit negativa quantitas; ſin aliter nullæ: ſint n & m pares numeri; & $-B$ & $-A$ affirmativæ quantitates, tum nullam habet poſſibilem radicem data æquatio; ſit $-B$ affirmativa & $-A$ negativa, & ſit $\overline{\frac{m-n}{n^{n-m}}}^{\,n-m} A^n - \frac{n^m}{m^m} B^{n-m}$ affirmativa quantitas, tum quatuor poſſibiles radices habet data æquatio, ſin aliter nullas. 2do. Sit n impar numerus & m impar numerus & $\overline{\frac{m-n}{n^{n-m}}}^{\,n-m} A^n - \frac{n^m}{m^m} B^{n-m}$ affir-

affirmativa quantitas, tum tres radices erunt poſſibiles; ſin aliter, una ſolummodo invenitur poſſibilis radix; ſit n impar & m par & A & B eadem habeant ſigna, tum unam ſolummodo poſſibilem radicem habet data æquatio; ſi vero diverſa habeant ſigna, & $\overline{\frac{m-n}{n^{n-m}}}^{\,n-m} A^n - \frac{n^m}{m^m} B^{n-m}$ idem habeat ſignum ac $-B$, tum tres habet poſſibiles radices data æquatio, ſin aliter unam.

THEOR. X.

1. Sit quæcunque æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0$, pro x ſcribatur $z + a$, & reſultet æquatio $z^n - Pz^{n-1} + Qz^{n-2} - Rz^{n-3} + \&c. = 0$: ſit v quæcunque functio differentiarum inter quaſcunque duas radices æquationis, cujus radix eſt x; & w eadem functio differentiarum inter quaſcunque duas radices æquationis, cujus radix eſt z; tum erit v eædem quantitas, ac w; & æquatio, cujus radix eſt v, eadem erit ac æquatio, cujus radix eſt w.

2. Sint duæ æquationes $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0$ & $z^m - az^{m-1} + bz^{m-2} - cz^{m-3} + \&c. = 0$; augeantur radices e ſingulis duabus æquationibus per a; i. e. ſcribantur $v - a$ & $w - a$ pro x & z reſpective, & ſint æquationes reſultantes $v^n - Pv^{n-1} + Qv^{n-2} - Rv^{n-3} + \&c. = 0$, & $w^m - Aw^{m-1} + Bw^{m-2} - Cw^{m-3} + \&c. = 0$; aſſumantur duæ æquationes $v^n - Pv^{n-1} + Qv^{n-2} - Rv^{n-3} + \&c. = 0$ & $v^m - Av^{m-1} + Bv^{m-2} - Cv^{m-3} + \&c. = u$; eadem erit æquatio, cujus radix eſt u, quicunque ſit valor quantitatis a.

Sint enim radices æquationis $v^n - Pv^{n-1} + Qv^{n-2} - Rv^{n-3} + \&c. = 0$ reſpective $a+\alpha$, $a+\beta$, $a+\gamma$, $a+\delta$, &c. & radices æquationis $w^m - Aw^{m-1} + Bw^{m-2} - Cw^{m-3} + \&c. = 0$ reſpective $a+\pi$, $a+\rho$, $a+\sigma$, $a+\tau$, &c.; tum erunt radices æquationis, cujus radix eſt u, reſpective $(a+\alpha-\overline{a+\pi}) \times (a+\alpha-\overline{a+\rho}) \times (a+\alpha-\overline{a+\sigma}) \times \&c.$, $(a+\beta-\overline{a+\pi}) \times (a+\beta-\overline{a+\rho}) \times (a+\beta-\overline{a+\sigma}) \times \&c.$, &c., i. e. $(\alpha-\pi) \times (\alpha-\rho) \times (\alpha-\sigma) \times \&c.$, $(\beta-\pi) \times (\beta-\rho) \times (\beta-\sigma) \times \&c.$, &c.: & conſequenter eædem erunt, quicunque ſit valor quantitatis a.

Ex. 1. Transformetur data æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0$ in alteram, cujus radices ſint majores vel minores quam radices

dices datæ æquationis per quantitatem $(\pm a)$, reſultat

$$\begin{array}{lllllll} z^n & +na & z^{n-1} & +n\times\frac{n-1}{2}a^2 & z^{n-2} & +n\times\frac{n-1}{2}\times\frac{n-2}{3}a^3 & z^{n-3}+\&c.=0 \\ & -p & & -\overline{n-1}\times pa & & -\overline{n-1}\times\frac{n-2}{2}pa^2 & \\ & & & +q & & +\overline{n-2}\times qa & \\ & & & & & -r & \end{array}$$

& æquatio, cujus radices ſunt limites inter reſultantis æquationis radices, fit

$$\begin{array}{lllllll} nz^{n-1} & +\overline{n-1}\times na & z^{n-2} & +\overline{n-2}\times n\times\frac{n-1}{2}a^2 & z^{n-3} & +\overline{n-3}\times n\times\frac{n-1}{2}\times\frac{n-2}{3}a^3 & z^{n-4}+\&c.=0 \\ & -\overline{n-1}\,p & & -\overline{n-2}\times\overline{n-1}\,pa & & -\overline{n-3}\times\overline{n-1}\times\frac{n-2}{2}pa^2 & \\ & & & +\overline{n-2}\,q & & +\overline{n-3}\times\overline{n-2}\,qa & \\ & & & & & -\overline{n-3}\,r & \end{array}$$

ſupponatur

$$\begin{array}{lllll} z^n & +na & z^{n-1} & +n\times\frac{n-1}{2}a^2 & z^{n-2}+\&c.=w \\ & -p & & -\overline{n-1}\,pa & \\ & & & +q & \end{array}$$

& æquatio, cujus radix eſt w, eadem erit; utcunque mutetur valor quantitatis (a).

Eadem accommodari poteſt methodus ad omnia duorum præcedentium caſuum exempla.

Eadem principia ulterius promovere liceat.

THEOR. 9.

Sint duæ æquationes $Hx^n - ax^{n-1} + bx^{n-2} - cx^{n-3} + dx^{n-4} + \&c. = 0$, cujus radices ſint $\alpha, \beta, \gamma, \delta, \varepsilon$, &c. & $Lx^m - px^{m-1} + qx^{m-2} - rx^{m-3} + \&c. = 0$, cujus radices ſint $\pi, \varrho, \sigma, \tau$, &c.; tum 1mo aſſumantur $Hx^n - ax^{n-1} + bx^{n-2} - cx^{n-3} + \&c. = 0$ & $Lx^m - px^{m-1} + qx^{m-2} - rx^{m-3} + \&c. = w$; unde ſcribendo $\alpha, \beta, \gamma, \delta, \varepsilon$, &c. radices æquationis

valoribus quantitatum v, w, y & z reſpective V, W, Y & Z; tum erit $V \times W \times Y \times Z =$ contento ſub ſingulis valoribus quantitatis u.

THEOR. 10.

Sint duæ æquationes $(H)\ lx^n - px^{n-1} + qx^{n-2} - rx^{n-3} +$ &c. $= o$ & $(K)\ Lx^m - Px^{m-1} + Qx^{m-2} - Rx^{m-3} +$ &c. $= o$, in quibus n major ſit quam m: fingatur $H = lx^n - px^{n-1} + qx^{n-2} -$ &c. $= o$ & $K = Lx^m - Px^{m-1} + Qx^{m-2} -$ &c. $= v$; etiamque $K = o$ & $H = w$; tum erit contentum ſub omnibus (n) valoribus quantitatis v ad contentum ſub omnibus (m) valoribus quantitatis w :: $\pm L^n : l^m$: deinde fingatur $H = o$ & $aH + bK = y$; tum per lem. & theor. præcedentia erit contentum ſub omnibus valoribus quantitatis v ad contentum ſub omnibus valoribus quantitatis y :: $1 : b^n$: fingatur $aH + bK = o$ & $H = Y$; tum erit contentum ſub omnibus valoribus quantitatis y ad contentum ſub omnibus valoribus quantitatis Y :: $\pm a^n l^n : l^n$ :: $\pm a^n : 1$; ultimo vero fingantur $aH + bK = o$ & $AH + BK = z$; (ubi a & b, A & B ſunt datæ quantitates) tum (quoniam $\frac{B}{b} \times \overline{aH + bK} + \frac{bA - Ba}{b} H = AH + BK$) erit contentum ſub omnibus valoribus quantitatis Y ad contentum ſub omnibus valoribus quantitatis z :: $1 : \frac{\overline{bA - Ba}^n}{b^n}$: & conſequenter, per compoſitionem rationum, contentum ſub omnibus valoribus quantitatis z ad contentum ſub omnibus valoribus quantitatis w :: $\frac{\overline{bA - aB}^n}{a^n} \times L^n : \pm l^m$; erit $-l^m$, ſi n ſit impar & m par numerus; ſin aliter, $+l^m$: etiamque contentum ſub omnibus valoribus quantitatis z ad contentum ſub omnibus valoribus quantitatis v :: $\frac{\overline{bA - aB}^n}{a^n} : \pm 1$; erit $+1$, ſi n ſit par; ſin aliter, -1.

Cor.

Cor. Sint $cH+dK=0$ & $CH+DK=V$; ubi c & d, C & D ſunt datæ quantitates; tum erit contentum ſub ſingulis valoribus quantitatis z ad contentum ſub ſingulis valoribus quantitatis V :: $\frac{\overline{bA-aB}^n}{a^n} : \frac{\overline{dC-cD}^n}{c^n}$.

Cor. 2. Sint duæ æquationes (H) $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} +$ &c. $=0$ & (K) $-px^{n-1} + 2qx^{n-2} - 3rx^{n-3} + 4sx^{n-4} -$ &c. $=0$; aſſumantur duæ æquationes $aH+bK = ax^n - \overline{a+b}\,px^{n-1} + \overline{a+2b}\,qx^{n-2} - \overline{a+3b}\,rx^{n-3} +$ &c. $=0$ & $AH+BK = Ax^n - \overline{A+B}\,px^{n-1} + \overline{A+2B}\,qx^{n-2} - \overline{A+3B}\,rx^{n-3} +$ &c. $=Z$; & ſimiliter aſſumantur duæ æquationes $cH+dK = cz^n - \overline{c+d}\,pz^{n-1} + \overline{c+2d}\,qz^{n-2} - \overline{c+3d}\,rz^{n-3} +$ &c. $=0$ & $CH+DK = Cz^n - \overline{C+D}\,pz^{n-1} + \overline{C+2D}\,qz^{n-2} - \overline{C+3D}\,rz^{n-3} +$ &c. $=V$; tum erit contentum ſub omnibus valoribus quantitatis Z ad contentum ſub omnibus valoribus quantitatis V :: $\frac{\overline{aB-bA}^n}{a^n} : \frac{\overline{cD-dC}^n}{c^n}$.

2. Sint n & m in præcedente caſu inter ſe æquales: aſſumantur (H) $lx^n - px^{n-1} + qx^{n-2} -$ &c. $=0$ & (K) $Lx^n - Px^{n-1} + Qx^{n-2} -$ &c. $=v$, etiamque $K=0$ & $H=w$; tum erit contentum ſub ſingulis valoribus quantitatis v ad contentum ſub ſingulis valoribus quantitatis w :: $\pm L^n : l^n$: fingetur $H=0$ & $aH+bK=y$, tum erit contentum ſub ſingulis valoribus quantitatis v ad contentum ſub ſingulis valoribus quantitatis y :: $L^n : b^n L^n$ :: $1 : b^n$: aſſumantur $aH+bK=0$ & $H=z$, tum erit contentum ſub ſingulis valoribus quantitatis z ad contentum ſub ſingulis valoribus quantitatis y :: $\pm l^n : \overline{al+bL}^n$: aſſumantur $aH+bK=0$ & $V = cH+dK = cH - \frac{da}{b}H$, quoniam $aH+bK=0$ & conſequenter $aH=-bK$; erit contentum ſub ſingulis valoribus quantitatis z ad contentum ſub ſingulis valoribus quantitatis V :: $1 : \frac{\overline{cb-da}^n}{b^n}$; unde e compoſitis rationi-

bus

bus reſultat contentum ſub ſingulis valoribus quantitatis V: ad contentum ſub ſingulis valoribus quantitatis $v :: \pm \overline{cb - ad}^n l^n : \overline{al + bL}^n$: ſi n ſit par numerus, ſignum affixum erit $+$; ſin aliter $-$.

Si vel $al + bL = o$ vel $cl + dL = o$, &c. vel $ad = cb$; tum e coefficientibus termini maximarum dimenſionum quantitatis x vel in æquatione $ab + bK = o$ vel in æquatione $cH + dK = o$ contenti, &c. conſimili methodo facile erui poteſt ratio, quam inter ſe habent prædicta contenta.

THEOR. XIII.

1. Sit cubica æquatio $x^3 + px^2 + qx + r = o$, cujus duæ radices ſint impoſſibiles; tertia erit negativa vel affirmativa quantitas, prout (r) ſit affirmativa vel negativa quantitas.

2. Sit æquatio biquadratica $x^4 + px^3 + qx^2 + rx + s = o$, cujus duæ radices ſint impoſſibiles & duæ poſſibiles: ſit s negativa quantitas; altera poſſibilis radix erit negativa, altera vero affirmativa: ſit s affirmativa quantitas, & termini continuo mutentur de $+$ in $-$, & $-$ in $+$; tum duæ poſſibiles radices erunt affirmativæ; ſin omnes termini ſint affirmativi, tum duæ poſſibiles radices erunt negativæ: ſin termini nec continuo mutentur de $+$ in $-$, & $-$ in $+$; nec ſint omnes affirmativi; tum duæ poſſibiles radices erunt negativæ vel affirmativæ, prout $p^3 - 4pq + 8r$ ſit affirmativa vel negativa quantitas. Aſſumatur enim æquatio $x^4 + px^3 + qx^2 + rx + s = (x^2 - 2ax + a^2 + c^2) \times (x^2 + 2bx + b^2 - d^2) = o$; ſi modo æquentur correſpondentes termini duarum æquationum inter ſe, & pro literis p, q & r in prædictâ quantitate $(p^3 - 4pq + 8r)$ ſcribantur earum valores in terminis quantitatum a, b, c & d; facile exinde deduci poſſunt caſus, in quibus litera b denotat affirmativam quantitatem, i. e. in quibus duæ poſſibiles radices ſunt negativæ.

Cor. Sit æquatio $x^4 + qx^2 + rx + s = o$, cujus duæ ſint poſſibiles radices; & ſit s affirmativa quantitas; duæ poſſibiles radices erunt negativæ vel affirmativæ, prout (r) ſit affirmativa vel negativa quantitas.

3. Sit

3. Sit $x^n + px^{n-1} + qx^{n-2} + rx^{n-3} + sx^{n-4} \ldots Rx + S = 0$, cujus ultimus terminus S ſit negativa quantitas, impar erit numerus affirmativarum radicum in datâ æquatione contentarum; ſit S affirmativa quantitas, & par erit numerus affirmativarum radicum. Si termini datæ æquationis continuo mutentur de + in —, & — in +; tum omnes poſſibiles radices erunt affirmativæ; ſi termini ſint omnes affirmativi, tum omnes poſſibiles radices erunt negativæ; & in genere haud plures erunt affirmativæ radices, quam ſignorum mutationes de + in — vel — in + in datâ æquatione continentur; nec plures negativæ, quam continui progreſſus de + in + vel — in —.

Ex. 1. Sit $x^{2n+1} + A = 0$; poſſibilis radix vel erit affirmativa vel negativa, prout A ſit negativa vel affirmativa quantitas: ſit $x^{2n} - A^2 = 0$; & una poſſibilis radix erit affirmativa, altera vero negativa quantitas.

Ex. 2. Sit $x^{2n} + Ax^m + B = 0$, ubi $2n$ major ſit quam m: ſi duæ ſolummodo radices ſint poſſibiles, & B negativa quantitas; una radix erit affirmativa, altera vero negativa quantitas: ſit B affirmativa quantitas, & duæ poſſibiles radices erunt negativæ vel affirmativæ, prout A ſit affirmativa vel negativa quantitas.

Sint quatuor poſſibiles; duæ erunt affirmativæ, & duæ negativæ radices.

Ex. 3. Sit $x^{2n+1} + Ax^m + B = 0$; ſi una radix ſolummodo ſit poſſibilis, ea erit affirmativa vel negativa quantitas, prout B ſit negativa vel affirmativa quantitas. Si vero tres ſint poſſibiles, & B affirmativa quantitas; tres poſſibiles radices erunt una negativa, duæ vero reliquæ affirmativæ.

Sit B negativa quantitas, & tres poſſibiles radices erunt una affirmativa, duæ vero reliquæ negativæ.

Hæc exempla ſatis manifeſto e caſu præcedente conſtabunt; ſi modo in datâ æquatione pro terminis deficientibus ſcribantur quantitates quam minimæ vel affirmativæ vel negativæ, prout exempli demonſtrationis gratiâ requiratur; quantitates quam minimæ vel affirmativæ vel negativæ nullam induœunt mutationem vel in numero

impoſſibilium vel affirmativarum vel negativarum radicum; ni duæ vel plures radices vel ſint inter ſe vel nihilo æquales.

4. Dato impoſſibilium radicum numero, invenire ſummam negativarum vel poſitivarum radicum in datâ æquatione $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} +$ &c. $=$ contentarum.

Sint quadratici datæ æquationis diviſores $x^2 - 2ax + a^2 \pm A^2 = 0$, $x^2 - 2bx + b^2 \pm B^2 = 0$, $x^2 - 2cx + c^2 \pm C^2 = 0$, $x^2 - 2dx + d^2 \pm D^2 = 0$, $x^2 - 2ex + e^2 \pm E^2 = 0$, &c.; inveniantur quantitates, quæ nihilo fiant æquales, cum A^2 vel B^2, C^2, D^2, E^2, &c. etiamque cum a, b, c, &c.; $a^2 \pm A^2$, $b^2 \pm B^2$, &c. nihilo fiant æquales, & quæ nullum recipiant diviſorem; e juſtâ comparatione quantitatum inter ſe reſultantium inveniri poteſt numerus affirmativarum & negativarum radicum.

Hinc conſtat, quod in æquationibus ſuperiorum dimenſionum calculus requiratur maxime operoſus.

PROB. XV.

Conceſsâ methodo detegendi numerum impoſſibilium cujuſcunque datæ æquationis radicum, in quibus continetur una ſolummodo incognita quantitas: Conceſsâ etiam methodo inveniendi limites, inter quos conſiſtant poſſibiles, affirmativi & negativi valores quantitatum exinde reſultantium: & datâ æquatione, quæ plures (x, v, w, z, *&c.) habet quam unam incognitam quantitatem; invenire, utrum hæ incognitæ quantitates admittant poſſibiles correſpondentes valores, necne.*

Aſſumantur omnes præter unam (x) incognitæ quantitates tanquam datæ; & inveniatur e conceſſis principiis, quando omnes valores hujus incognitæ quantitatis fiant impoſſibiles; reſultabunt quædam quantitates α, β, γ, &c. & vel π vel ρ, &c. e quibus exterminatur incognita quantitas x: hæ erunt affirmativæ vel negativæ, cum omnes radices datæ æquationis ſint impoſſibiles; fiant omnes hæ quantitates nihilo æquales, i. e. $\alpha = 0$, $\beta = 0$, $\gamma = 0$, &c. vel $\pi = 0$ vel $\rho = 0$, &c.; ex radicibus æquationum reſultantium dici poteſt, cum quantitates

tates α, β, γ, &c. vel $\pi = o$ vel $\varrho = o$, &c. evadant affirmativæ, & conſequenter cum radices datæ æquationis evadant impoſſibiles: ex eâdem methodo exterminari poſſunt reliquæ incognitæ quantitates (v, w, &c.), ita ut tandem invenietur, utrum poſſibiles correſpondentes valores e ſingulis incognitis quantitatibus admittat data æquatio, necne; etiamque detegentur limites ſingularum incognitarum quantitatum, inter quos conſiſtent prædicti correſpondentes poſſibiles valores.

Ferè per eandem methodum detegi poſſunt limites, inter quos omnis poſſibilis valor quantitatis (x, &c.) duobus, tribus, quatuor, &c. poſſibilibus quantitatis v vel w, &c. valoribus correſpondeat.

Haud refert in inveſtigandis prædictis limitibus quantitatis (z), utrum primo exterminetur incognita quantitas x vel v, &c. deinde, ſi modo quantitas x primo exterminetur, utrum ſecundo exterminetur quantitas v vel w, &c.; & ſic deinceps: iidem enim omnibus hiſce caſibus reſultabunt prædicti limites, inter quos conſiſtent omnes poſſibiles valores quantitatis (z), qui correſpondent poſſibilibus valoribus quantitatum (x, v, w, &c.) &c.

Ex. 1. Sit $y^2 + \overline{2z+8} \times y + 2z^2 + 10z + 20 = o$; invenire, num incognitæ quantitates z & y poſſibiles correſpondentes valores admittant, necne.

Aſſumatur z tanquam data quantitas; notum eſt quantitatem (y) nullos admittere poſſibiles valores, ſi modo $\overline{z+4}^2 - 2z^2 - 10z - 20$, i.e. $-z^2 - 2z - 4$ ſemper ſit negativa, vel $z^2 + 2z + 4$ ſemper ſit affirmativa quantitas. Radices autem æquationis $z^2 + 2z + 4 = o$ ſunt impoſſibiles; ergo $z^2 + 2z + 4$ ſemper erit affirmativa quantitas, & nullos poſſibiles correſpondentes valores admittent incognitæ quantitates z & y.

Ex. 2. Sit æquatio $y^2 + \overline{2x+4z+4}\,y + 2x^2 + 6z^2 + 5xz + 6 = o$; invenire, num incognitæ quantitates y, x & z poſſibiles correſpondentes valores recipiant, necne. Sint omnes valores quantitatis (y) impoſſibiles, tum $-\overline{x+2z+2}^2 + 2x^2 + 5zx + 6z^2 + 6 = x^2 +$

$z - 4$

$\overline{z-4}\,x+2z^2-8z+2$ ſemper erit affirmativa quantitas; ut ſemper fiat affirmativa quantitas, neceſſe eſt, ut radices æquationis $x^2+\overline{z-4}\,x+2z^2-8z+2=0$ ſint impoſſibiles, i. e. ut $\frac{\overline{z-4}^2}{4}-2z^2+8z-2$ ſemper ſit negativa quantitas, i. e. ut radices æquationis $7z^2-24z-8=0$ ſint impoſſibiles: ſed radices hujus æquationis haud erunt impoſſibiles; ſunt enim $\frac{12\pm10\sqrt{2}}{7}$: & exinde ſequitur, ſi modo radices quantitatis (x) ſint poſſibiles, tum radices quantitatis (z) inter $\frac{12+10\sqrt{2}}{7}$ & $\frac{12-10\sqrt{2}}{7}$ conſiſtent: aſſumatur igitur quæcunque quantitas inter $\frac{12+10\sqrt{2}}{7}$ & $\frac{12-10\sqrt{2}}{7}$ pro z; quâ pro ſuo valore in æquatione $x^2+\overline{z-4}\times x+2z^2-8z+2=0$ ſubſtitutâ, inveniantur duæ radices quantitatis (x) exinde reſultantes; aſſumatur quæcunque quantitas inter has duas radices, quâ pro x in primâ æquatione $y^2+\overline{4+2x+4z}\times y+2x^2+6z^2+5xz+6=0$ ſubſtitutâ, ex eâ reſultabunt duæ radices quantitatis (y) poſſibiles.

Eodem modo inveniri poſſunt $(4$ & $-\frac{12}{7})$ limites, inter quos conſiſtent poſſibiles valores quantitatis x, qui habent poſſibiles valores quantitatum y & z iis correſpondentes; & ſic $\frac{-46+5\sqrt{46}}{7}$ & $\frac{-46-5\sqrt{46}}{7}$ erunt limites, inter quos conſiſtent poſſibiles valores quantitatis y, qui habent poſſibiles valores quantitatum x & z iis correſpondentes.

Cor. Si ſinguli termini in datâ æquatione contenti haud plures reſpective habeant quam duas dimenſiones incognitarum quantitatum x, y, z, &c.; tum per methodum hiſce duobus exemplis traditam, ſemper reſolvi poteſt problema.

PROB.

PROB. XVI.

Cognito datæ æquationis impoſſibilium etiamque affirmativarum & negativarum, &c. radicum numero; & datâ methodo inveniendi, inter quos limites, &c. datæ æquationis radicum conſiſtant poſſibiles valores cujuſcunque functionis datæ æquationis radicum: invenire numerum impoſſibilium radicum, &c. quas habet æquatio, cujus radices ſint prædicta functio datæ æquationis radicum.

Primo e datâ functione perſcrutandum eſt, cujus generis ſint radices datæ æquationis, quæ dabunt impoſſibiles etiamque poſſibiles radices in reſultanti æquatione; & exinde e dato numero radicum prædicti generis in datâ æquatione contentarum, radicum impoſſibilium, quas habet reſultans æquatio, conſtabit numerus.

Hæc e terminis irrationalibus diverſis & rationalibus ſejunctim æſtimatis ſemper conſtabunt.

Conſimili autem methodo deduci poteſt numerus negativarum & affirmativarum radicum.

Hæc autem per exempla facilius doceri poſſunt.

Ex. 1. Sit æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = o$, cujus radices ſint $\alpha, \beta, \gamma, \delta$, &c.; transformetur hæc æquatio in aliam, cujus radices ſint $\alpha^2, \beta^2, \gamma^2, \delta^2$, &c. e numero impoſſibilium radicum in datâ æquatione dato, invenire earum numerum in transformatâ æquatione. Primo omnis poſſibilis radix in datâ æquatione alteram habet in reſultanti æquatione ſibi correſpondentem: 2do omnis impoſſibilis radix in datâ æquatione habet impoſſibilem radicem in reſultanti ſibi correſpondentem, ni radix datæ æquationis ſit hujuſce formulæ $\sqrt{-A^2}$; ergo per caput ſubſequens inveniendum eſt, quot radices hujuſce formulæ $\sqrt{-A^2}$ habet data æquatio, qui numerus ſit (m); in æquatione reſultante minor impoſſibilium radicum invenitur numerus, quam in datâ æquatione per numerum (m): numerus autem negativarum radicum erit m; tot autem habet affirmativas radices reſultans, quot poſſibiles habet data æquatio.

Ex. 2. Sit æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0$, cujus radices ſint α, β, γ, δ, &c. quarum ſint $2r$ impoſſibiles, & $n - 2r$ poſſibiles; transformetur hæc æquatio in alteram, cujus radices ſint $\overline{\alpha - \beta}^2$, $\overline{\alpha - \gamma}^2$, $\overline{\beta - \gamma}^2$, $\overline{\alpha - \delta}^2$, $\overline{\beta - \delta}^2$, $\overline{\gamma - \delta}^2$, &c. invenire numerum impoſſibilium radicum, &c. in transformatâ æquatione.

Inveſtigandum eſt, cujus generis ſint radices datæ æquationis, quæ habent impoſſibiles radices in reſultanti æquatione correſpondentes.

Sint 1mo α & β poſſibiles vel negativæ vel affirmativæ quantitates; $\overline{\alpha - \beta}^2$, quæ eſt radix reſultantis æquationis, erit poſſibilis & affirmativa quantitas; exinde, quoniam $n - 2r$ poſſibiles ſunt radices in datâ æquatione, $\overline{n - 2r} \times \frac{n - 2r - 1}{2}$ erunt affirmativæ in reſultante æquatione.

2do. Sint duæ correſpondentes datæ æquationis radices impoſſibiles $a + b\sqrt{-1}$ & $a - b\sqrt{-1}$; tum $\overline{a + b\sqrt{-1} - a + b\sqrt{-1}}^2 = \overline{2b\sqrt{-1}}^2 = -4b^2$, quæ erit radix reſultantis æquationis, erit poſſibilis & negativa quantitas; ſed in datâ continentur $(2r)$ impoſſibiles radices, ergo in reſultante æquatione erunt (r) negativæ radices.

Sint ergo nullæ radices inter ſe æquales, nec ullæ radices hujuſce generis $a + \sqrt{-b^2}$ & $a + \sqrt{-c^2}$, vel hujuſce $a + \sqrt{-b^2}$ & $c + \sqrt{-b^2}$: tum radices reſultantis æquationis erunt $\overline{n - 2r} \times \frac{n - 2r - 1}{2}$ (affirmativæ) $+ r$ negativæ $+ 2rn - 2r^2 - 2r$ impoſſibiles $= n \times \frac{n - 1}{2}$.

Si vero ullæ ſint radices datæ æquationis vel inter ſe æquales, vel harum formularûm $a + \sqrt{-b^2}$ & $a + \sqrt{-c^2}$; vel $a + \sqrt{-b^2}$ & $c + \sqrt{-b^2}$; inveniri poſſunt methodo in ſubſequenti capite traditâ radices harum formularum; quarum omnium cognito numero, exinde

inde facile inveniri poteſt numerus affirmativarum, negativarum & impoſſibilium radicum in æquatione reſultanti.

Ex. 3. Sit æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4}$ &c. $= 0$, cujus radices ſint $\alpha, \beta, \gamma, \delta,$ &c. quarum $2b$ ſint impoſſibiles, cæteræ vero poſſibiles.

Transformetur hæc æquatio in alteram, cujus radices ſint

$$nx^{n-1} - \overline{n-1}\,px^{n-2} + \overline{n-2}\,qx^{n-3} - \overline{n-3}\,rx^{n-4} + \overline{n-4}\,sx^{n-5} \text{ \&c.} = v;$$

invenire numerum impoſſibilium radicum in transformatâ æquatione. Omnis poſſibilis radix datæ æquationis habet poſſibilem transformatæ æquationis ſibi ipſi reſpondentem, & omnis impoſſibilis datæ æquationis radix habet impoſſibilem; ni quædam ſint radices duarum æquationum $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} +$ &c. $= 0$, &

$$nx^{n-1} - \overline{n-1}\,px^{n-2} + \overline{n-2}\,qx^{n-3} - \overline{n-3}\,rx^{n-4} + \text{\&c.} = 0,$$

inter ſe æquales; vel nonnullas radices impoſſibiles $(a + \sqrt{-b^2})$ hujuſmodi admittat data æquatio, ita ut $n \times \overline{n-1}\,a^{n-2} - n \times \overline{n-1} \times \frac{n-2}{2} \times \frac{n-3}{3} a^{n-4} b^2 + n \times \overline{n-1} \times \frac{n-2}{2} \times \frac{n-3}{3} \times \frac{n-4}{4} \times \frac{n-5}{5} a^{n-6} b^4 -$ &c. $- \overline{n-1} \times \overline{n-2}\,p a^{n-3} + \overline{n-1} \times \overline{n-2} \times \frac{n-3}{2} \times \frac{n-4}{3} p a^{n-5} b^2 - \overline{n-1} \times \overline{n-2} \times \frac{n-3}{2} \times \frac{n-4}{3} \times \frac{n-5}{4} \times \frac{n-6}{5} p a^{n-7} b^4 +$ &c. $+ \overline{n-2} \times \overline{n-3}\,q a^{n-4}$ &c. $= 0$: hæc æquatio deduci poteſt e ſcribendo $a + \sqrt{-b^2}$ pro x in quantitate $nx^{n-1} - \overline{n-1}\,px^{n-2} + \overline{n-2}\,qx^{n-3} -$ &c. $= 0$; & e ſupponendo impoſſibilem partem reſultantis quantitatis nihilo eſſe æqualem. Si nullas radices hujuſmodi habeat data æquatio; tum reſultans æquatio, cujus radix eſt v, habebit radices impoſſibiles $(2b)$; &, ſi n ſit par numerus, $\frac{n-2b}{2}$ & affirmativas & negativas radices; ſi vero (n) ſit impar numerus, tum $\frac{n-2b-1}{2}$ negativas, & $\frac{n-2b+1}{2}$ affirmativas, habebit reſultans æquatio.

Utrum vero ullæ prædictarum æquationum radices ſint inter ſe æquales, & exinde quædam radices reſultantis æquationis nihilo fiant æquales; vel ulla quantitas prædictæ formulæ nihilo ſit æqualis, & exinde quædam ſint impoſſibiles radices datæ æquationis, quæ poſſibilibus radicibus æquationis reſultantis reſpondent, necne; e ſubſequenti capite colligi poteſt.

2. Et vice verſâ dato impoſſibilium radicum, &c. in reſultanti æquatione contentarum numero, inveniri poteſt impoſſibilium datæ æquationis radicum numerus.

Eadem erit methodus ratiocinandi in hoc ac in priori caſu: aliter: ex aſſumpto numero radicum poſſibilium & impoſſibilium datæ æquationis radicum tanquam cognito detegatur numerus radicum poſſibilium & impoſſibilium reſultantis æquationis ei reſpondens; deinde regredi liceat e numero poſſibilium & impoſſibilium reſultantis æquationis radicum ad numerum poſſibilium & impoſſibilium datæ æquationis radicum.

E.G. Sit data æquatio $x^3 + qx - r = 0$, cujus radices ſint α, β, $-\alpha-\beta$; data æquatio erit $x^3 - \overline{\alpha^2+\alpha\beta+\beta^2}\,x + \alpha\beta \times \overline{\alpha+\beta} = 0$: transformetur hæc æquatio in alteram, cujus radix ſit z; ubi relatio inter x & z per æquationem $z^2 - zx = \frac{q}{3} = -\frac{\alpha^2+\alpha\beta+\beta^2}{3}$ exprimitur, reſultat æquatio $z^6 - rz^3 = \frac{q^3}{27}$. Sint primum omnes tres radices datæ æquationis poſſibiles, & ſubſtituantur pro (x) in æquatione $z^2 - xz = -\frac{\alpha^2+\alpha\beta+\beta^2}{3}$ ſucceſſive α, β & $-\alpha-\beta$; reſultant tres æquationes $z^2 - \alpha z = -\frac{\alpha^2+\alpha\beta+\beta^2}{3}$, $z^2 - \beta z = -\frac{\alpha^2+\alpha\beta+\beta^2}{3}$, $z^2 + \overline{\alpha+\beta}\,z = -\frac{\alpha^2+\alpha\beta+\beta^2}{3}$; & exinde $z = \frac{\alpha}{2} + \frac{\overline{\alpha+2\beta}\times\sqrt{-3}}{6}$ & $z = \frac{\beta}{2} + \frac{\overline{\beta+2\alpha}\times\sqrt{-3}}{6}$, & $z = -\frac{\alpha+\beta}{2}$

$+\alpha-\beta$

$+\frac{\overline{\alpha-\beta\sqrt{-3}}}{6}$; ergo, ſi radices datæ æquationis ſint omnes poſſibiles, radices reſultantis æquationis erunt omnes impoſſibiles: ſint duæ radices (x) datæ æquationis impoſſibiles $a+\sqrt{-A^2}$ & $a-\sqrt{-A^2}$; tertia autem erit $-2a$, & data æquatio $x^3-\overline{3a^2-A^2}\,x+2a^3+2A^2a=0$; ſupponatur $z^2-zx=\frac{q}{3}=-a^2+\frac{A^2}{3}$; ſubſtituantur pro ($x$) ejus tres reſpectivi valores ſucceſſive, reſultabunt tres quadraticæ æquationes, quarum radix eſt z, viz. $z^2-\overline{a+\sqrt{-A^2}}\,z=-a^2+\frac{A^2}{3}$, & $z^2-\overline{a-\sqrt{-A^2}}\,z=-a^2+\frac{A^2}{3}$, & $z^2+2ax=-a^2+\frac{A^2}{3}$: duarum priorum æquationum omnes radices erunt impoſſibiles, ultimæ autem æquationis radices (z) erunt $z=\frac{A}{\sqrt{3}}-a$: ergo quatuor impoſſibiles & duas poſſibiles radices habet reſultans æquatio $z^6-rz^3=\frac{q^3}{27}$, ſi modo duas impoſſibiles habeat data æquatio $x^3+qx-r=0$: ſin aliter, nullas poſſibiles habet: & vice verſâ, ſi modo reſultans æquatio $z^6-rz^3=\frac{q^3}{27}$ duas habeat poſſibiles radices, duas impoſſibiles habebit data æquatio: ſin aliter nullas.

Ex. 2. Sit æquatio $x^4+qx^2-rx+s=0$, cujus radices ſint $\alpha, \beta, \gamma, \delta$: æquatio, cujus radices ſint ſummæ quarumcunque duarum reſpective, i. e. ſint $\alpha+\beta$, $\alpha+\gamma$, $\beta+\gamma$, $\alpha+\delta$, $\beta+\delta$, $\gamma+\delta$, erit

$$z^6+2qz^4\left.\begin{matrix}+q^2\\-4s\end{matrix}\right\}z^2-r^2=0.$$

Si omnes datæ æquationis radices ($\alpha, \beta, \gamma, \delta$) ſint poſſibiles, tum omnes reſultantis æquationis radices erunt poſſibiles: ſi vero duæ vel quatuor datæ æquationis radices ſint impoſſibiles, tum quatuor reſultantis æquationis radices erunt impoſſibiles, & duæ poſſibiles; ni

duæ radices impoſſibiles bis fiant æquales, i. e. æquatio ſit $x^4 + 2A^2x^2 + A^4 = 0$, in quo caſu omnes radices datæ æquationis erunt impoſſibiles, duæ vero reſultantis æquationis erunt impoſſibiles & quatuor nihilo æquales.

Sint enim quatuor radices datæ æquationis impoſſibiles & reſpective per $a \pm A\sqrt{-1}$, $-a \pm B\sqrt{-1}$ deſignatæ; quatuor reſultantis æquationis impoſſibiles radices erunt reſpective $\overline{B+A}\sqrt{-1}$, $-\overline{B+A}\sqrt{-1}$, $\overline{B-A}\sqrt{-1}$, $\overline{A-B}\sqrt{-1}$, duæ vero poſſibiles radices erunt $2a$ & $-2a$; & earum quadrata erunt quatuor negativæ quantitates, duæ vero affirmativæ: & quoniam negativæ & affirmativæ reſultantis æquationis radices ſunt inter ſe æquales; æquatio $v^3 + 2qv^2 + \overline{q^2 - 4s}\,v - r^2 = 0$, cujus radices ſunt quadrata radicum reſultantis æquationis, i. e. $v = z^2$, habebit omnes ejus radices poſſibiles, duas vero negativas & unam affirmativam; ni $a = 0$ & $A = B$, in quo caſu data æquatio erit $x^4 + 2A^2x^2 + A^4 = 0$; & reſultantis æquationis quatuor radices nihilo erunt æquales, duæ vero reliquæ invenientur impoſſibiles.

Sint duæ radices datæ æquationis poſſibiles, duæ vero impoſſibiles; & per α, β, $-\frac{\alpha+\beta}{2} + \sqrt{-A^2}$, $-\frac{\alpha+\beta}{2} - \sqrt{-A^2}$ reſpective deſignatæ; erunt quatuor reſultantis æquationis radices impoſſibiles, viz. $\frac{\alpha-\beta}{2} \pm \sqrt{-A^2}$, $\frac{\beta-\alpha}{2} \pm \sqrt{-A^2}$, duæ vero poſſibiles $\alpha+\beta$ & $-\alpha-\beta$; impoſſibilium radicum quadrata etiam erunt impoſſibiles quantitates; ni $\alpha = \beta$, in quo caſu duæ radices cubicæ æquationis $v^3 + 2qv + \overline{q^2 - 4s}\,v - r^2 = 0$, cujus radices ſunt quadrata radicum reſultantis æquationis, erunt inter ſe æquales; erunt enim $-A^2$, & $-A^2$.

Si igitur cubicæ æquationis $v^3 + 2qv^2 + \overline{q^2 - 4s} \times v - r^2 = 0$ omnes radices ſint poſſibiles & affirmativæ, omnes radices datæ biquadraticæ etiam erunt poſſibiles: ſi omnes radices æquationis cubicæ ſint poſſibiles; duæ vero radices negativæ, & una affirmativa; quatuor

quatuor radices datæ æquationis erunt impoſſibiles: ſi vero duæ radices cubicæ ſint impoſſibiles, tum duæ radices biquadraticæ etiam erunt impoſſibiles, duæ vero poſſibiles.

Cor. Datâ generali reductione æquationum, hinc conſtabit methodus inveniendi, quot impoſſibiles radices habet data æquatio; ſi modo enim reducatur, neceſſe eſt, ut transformetur in æquationem quoad formulam minores dimenſiones habentem, data æquatio: data autem relatio inter radices datæ & transformatæ æquationis ſemper exprimitur per æquationes quoad formulas earum minores quam datæ æquationis dimenſiones habentes, ergo e numero earum impoſſibilium radicum conſtat numerus impoſſibilium radicum in datâ æquatione contentarum.

SCHOL.

Modus demonſtrandi maxime accommodatus hiſce rebus, ubi quæritur, quot radices impoſſibiles continentur in datâ æquatione? eſt vel ſubſtituere quantitates ipſas pro ſuis valoribus; vel multiplicare datam æquationem 1mo per $x - a = 0$; deinde per $x^2 - 2ax + a^2 + b^2 = 0$; cum enim prior æquatio unicam radicem habeat, eamque poſſibilem; poſterior autem duas impoſſibiles; manifeſtum eſt eundem futurum eſſe numerum radicum impoſſibilium in æquatione, quæ oritur e priori multiplicatione, ac in datâ æquatione: æquatio autem, quæ oritur e ſecundâ multiplicatione duas habebit radices impoſſibiles, quæ in datâ æquatione non continentur.

Ex. Sit data æquatio $x^2 - 2x + 2 = 0$, quæ ſecundum regulam in prob. 9. exemp. 1. traditam duas habet radices impoſſibiles; multiplicetur per $x + 2 = 0$, & fit $x^3 - 2x + 4 = 0$, quæ ſecundum eandem regulam nullas habet radices impoſſibiles, debet autem habere duas; regula igitur non ſemper accurata eſt.

Modus autem demonſtrationis, de quo loquor, non huic quæſtioni ſolummodo proprius eſt; ſed commode adhiberi poteſt, ubicunque quæritur, quot quantitates dati cujuſvis generis in datâ æquatione continentur?

Ex. 1.

Ex. 1. Sit notiſſima Carteſii regula, "tot eſſe radices affirmativas, "quot ſignorum in continuâ ſerie mutationes de + in — & — in "+; ſi modo nullæ æquationis radices ſint impoſſibiles."

Si enim data quævis æquatio multiplicetur per $x - a = o$, ubi a ſit affirmativa quantitas, mutationum ſignorum numerus ſaltem augebitur per unitatem; ſi autem eadem æquatio multiplicetur per $x + a = o$, iidem ſaltem erunt ſignorum progreſſus de + in + vel — in — in novâ, ac in datâ æquatione; vel magis generaliter hìc primum aſſeritur, ſi data quævis æquatio multiplicetur per $x - a$, tum ſingula mutatio ſignorum de + in — & — in + in datâ æquatione habet unam vel tres vel quinque, &c. correſpondentes mutationes ſignorum de + in — & — in + in reſultante æquatione: ſi vero multiplicetur per $x + a$, tum ſingulus continuus progreſſus de + in + & — in — in datâ æquatione habet unum vel tres vel quinque, &c. continuos progreſſus de + in + & — in — in reſultante æquatione.

Sit enim æquatio $x^n + px^{n-1} + qx^{n-2} + \ldots\ldots + Px^{n-m} - Qx^{n-m-1} \pm$ &c. $= o$, ubi p, q, &c.; P & Q ſunt affirmativæ quantitates, & $Px^{n-m} - Qx^{n-m-1}$ eſt prima mutatio ſignorum de + in —; ducatur æquatio $x^n + px^{n-1} + qx^{n-2} \ldots\ldots + Px^{n-m} - Qx^{n-m-1} \pm$ &c. $= o$ in $x - a$

reſultat $x^{n+1} + \overline{p - a}\,x^n + \overline{q - ap}\,x^{n-1} \ldots\ldots - \overline{Q + aP}\,x^{n-m} \pm$ &c. $= o$; ſed $-Q - aP$ eſt negativa quantitas; ergo in æquatione reſultante eſt una vel tres vel quinque, &c. mutationes ſignorum de + in — & — in +, quæ correſpondent primæ mutationi ſignorum in datâ æquatione de + in —.

Sit eadem æquatio $x^n + px^{n-1} + qx^{n-2} \ldots\ldots + \alpha x^{n-r} - \beta x^{n-r-1} \ldots\ldots - \pi x^{n-s} + \varrho x^{n-s-1} \ldots\ldots + \phi x^{n-t} - \chi x^{n-t-1} \pm$ &c. $= o$, ubi α, β; π, ϱ; ϕ & χ reſpective denotant affirmativas quantitates, & ſit $\alpha x^{n-r} - \beta x^{n-r-1}$ quæcunque mutatio ſignorum de + in — in datâ æquatione contenta, & $-\pi x^{n-s} + \varrho x^{n-s-1}$ ejus proxime ſubſequens mutatio ſignorum de — in +; & ſic ſit $-\pi x^{n-s} + \varrho x^{n-s-1}$ quæcunque

mutatio ſignorum de — in + in datâ æquatione, & ejus proxime ſubſequens mutatio de + in — ſit $+\phi x^{n-t} - \chi x^{n-t-1}$: ducatur data æquatio

$$x^n + px^{n-1} + qx^{n-2} \ldots + \alpha x^{n-r} - \beta x^{n-r-1} \ldots - \pi x^{n-s} + \varrho x^{n-s-1} \ldots + \phi x^{n-t} - \chi x^{n-t-1} \pm \&c. = 0$$

in $x - a$

resultat $x^{n+1} + \overline{p-a}\, x^n + \overline{q-ap}\, x^{n-1} \ldots - \overline{\beta + a\alpha}\, x^{n-r} \ldots + \overline{\varrho + a\pi}\, x^{n-s} \ldots - \overline{\chi + a\phi}\, x^{n-t}$ &c. $= 0$.

In hâc æquatione coefficiens $-\beta - a\alpha$ eſt negativa, $\varrho + a\pi$ eſt affirmativa, & $-\chi - a\phi$ iterum negativa; ergo de termino negativo $-\overline{\beta + \alpha a}\, x^{n-r}$ ad affirmativum $\overline{\varrho + a\pi}\, x^{n-s}$ continetur una vel plures (tres, quinque, &c.) mutationes ſignorum, quæ correſpondent uni mutationi ſignorum de termino $-\beta x^{n-r-1}$ ad terminum $+\varrho x^{n-s-1}$ in datâ æquatione contentæ; & ſimiliter de affirmativo termino $+\overline{\varrho + a\pi}\, x^{n-s}$ ad terminum negativum $-\overline{\chi + a\phi}\, x^{n-t}$ in reſultante æquatione eſt una vel tres, vel quinque, &c. mutationes ſignorum de + in — & — in +, quæ correſpondent uni mutationi ſignorum de termino $+\varrho x^{n-s-1}$ ad terminum $-\chi x^{n-t-1}$ in datâ æquatione contentæ; ergo ſingula mutatio de + in — vel — in + in datâ æquatione habet unam vel plures (tres, quinque, &c.) correſpondentes mutationes ſignorum de + in — & — in + in reſultante æquatione: conſequenter tot vel plures erunt mutationes de + in — & — in + in reſultante æquatione $x^{n+1} + \overline{p - a}\, x^n + \overline{q - ap}\, x^{n-1} \ldots + \overline{M - aL}\, x - aM = 0$ a primo ad ejus penultimum terminum $\overline{M - aL} \times x$, quot inveniuntur in datâ æquatione $x^n + px^{n-1} + qx^{n-2} \ldots Lx + M = 0$: ſi vero tot ſolummodo contineantur mutationes ſignorum de + in — & — in + a primo termino reſultantis æquationis ad ejus penultimum, quot mutationes ſignorum de + in — & — in + habet data æquatio; tum penultimus terminus $(M - aL)\, x$ reſultantis æquationis idem habet ſignum ac ultimus terminus M datæ æquationis; & ſi penultimus terminus $(M - aL)\, x$ reſultantis æquationis idem habeat ſignum ac M; tum de penultimo termino $\overline{M - aL}\, x$ ad ultimum $-Ma$ una datur mutatio ſignorum de + in — vel — in + (viz: $+\overline{M - aL}\, x - aM$); & conſequenter reſultans

tans æquatio habet plures ab unâ vel tribus vel quinque, &c. mutationes ſignorum de + in — & — in +, quam habet data æquatio.

2. Sit æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} \ldots - Px^{n-m+1} + Qx^{n-m} + Rx^{n-m-1} \pm$ &c. $= 0$, ubi literæ p, q, r, &c.; P, Q, R, &c. ſunt affirmativæ quantitates, & $+ Qx^{n-m} + Rx^{n-m-1}$ eſt primus progreſſus de + in + vel — in — in datâ æquatione contentus; e. g. ſit de + in +; ducatur hæc æquatio

$$\begin{array}{ll} & x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \ldots - Px^{n-m+1} + Qx^{n-m} + Rx^{n-m-1} \pm \text{\&c.} = 0; \\ \text{in} & x + a \\ \hline \text{reſultat} & x^{n+1} - \overline{p-a}\,x^n + \overline{q-ap}\,x^{n-1} - \overline{r-aq}\,x^{n-2} \ldots + \overline{Q-aP}\,x^{n-m+1} + \overline{R+aQ}\,x^{n-m} \pm \text{\&c.} = 0; \end{array}$$

ſi modo nullus in reſultante æquatione uſque ad terminum $\overline{Q-aP}\,x^{n-m+1}$ detur progreſſus de + in + vel — in —; erunt p, q, r, &c. & Q reſpective majores quam a, ap, aq, &c.; & aP; & conſequenter $Q - aP$ idem habet ſignum ac Q, i. e. erit affirmativa quantitas & $\overline{Q-aP}\,x^{n-m+1} + \overline{R+aQ}\,x^{n-m}$ erit progreſſus de + in +; ergo in æquatione reſultante erit unus vel tres vel quinque, &c. ſignorum progreſſus de + in + vel — in —, qui correſpondent primo ſignorum progreſſui de + in + in datâ æquatione, i. e. in iiſdem primis terminis contenti.

Et ſic mutatis mutandis eâdem methodo ratiocinari liceat de æquatione $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} \ldots + Px^{n-m+1} - Qx^{n-m} - Rx^{n-m-1} \pm$ &c. $= 0$, in quâ primus continuus progreſſus eſt de — in —; i. e. unus vel tres vel quinque, &c. ſignorum progreſſus de + in + & — in — in reſultante æquatione continentur, qui correſpondent primo progreſſui $- Qx^{n-m} - Rx^{n-m-1}$ de — in — in datâ æquatione.

Sit æquatio $x^n - px^{n-1} + qx^{n-2} - \ldots + \alpha x^{n-m+1} + \beta x^{n-m} - \gamma x^{n-m-1} + \delta x^{n-m-2} \ldots - \pi x^{n-r} + \varrho x^{n-r-1} + \sigma x^{n-r-2} \pm$ &c. $= 0$, in quâ ſit $\alpha x^{n-m+1} + \beta x^{n-m}$ progreſſus ſignorum de + in + vel — in —; & ſit $+ \varrho x^{n-r-1} + \sigma x^{n-r-2}$ ejus proxime ſubſequens progreſſus de + in + vel — in —; ducatur æquatio

$$\begin{array}{ll} & x^n - px^{n-1} + qx^{n-2} \ldots + \alpha x^{n-m+1} + \beta x^{n-m} - \gamma x^{n-m-1} + \delta x^{n-m-2} \ldots - \pi x^{n-r} + \varrho x^{n-r-1} + \sigma x^{n-r-2} \pm \text{\&c.} \ldots + Lx + M \\ \text{in} & x + a \\ \hline \text{reſultat} & x^{n+1} - \overline{p-q}\,x^n + + \overline{q-ap}\,x^{n-1} \ldots + \overline{\beta + a\alpha}\,x^{n-m+1} - \overline{\gamma - a\beta}\,x^{n-m} + \overline{\delta - a\gamma}\,x^{n-m-1} \ldots + \overline{\varrho - a\pi}\,x^{n-r} + \overline{\sigma + a\varrho}\,x^{n-r-1} \pm \text{\&c.} \end{array}$$

In hâc æquatione quoniam, α & β idem habent ſignum, $\beta + a\alpha$ habebit

habebit idem ſignum ac β: ſigna terminorum $\beta x^{n-m} - \gamma x^{n-m-1} + \delta x^{n-m-2}$ — &c. .... $- \pi x^{n-r} + \varrho x^{n-r-1}$ in datâ æquatione continuo mutantur de + in —, & — in +; nam per hypotheſin $+ \varrho x^{n-r-1} + \sigma x^{n-r-2}$ eſt proxime ſubſequens progreſſus de + in + & — in — in datâ æquatione contentus: ſi modo nullus detur progreſſus de + in + vel — in — in æquatione reſultante a termino $\overline{\beta + a\alpha}\, x^{n-m+1}$ uſque ad terminum $\overline{\varrho - a\pi}\, x^{n-r}$; tum erunt γ, δ, &c., & ϱ reſpective majores quam $a\beta$, $a\gamma$, &c., & $a\pi$; conſequenter $+ \overline{\varrho - a\pi}$ habet idem ſignum ac ϱ; & quoniam ϱ & σ eadem habent ſigna, ergo $+ \varrho - a\pi$ & $\overline{+ \sigma + a\varrho}$ eadem habent ſigna; & conſequenter datur progreſſus $(\overline{\varrho - a\pi} \times x^{n-r} + \overline{\sigma + a\varrho}\, x^{n-r-1})$ ſignorum in reſultante æquatione de + in + vel — in —: unde datur unus vel plures (tres, quinque, &c.) progreſſus ſignorum de + in + vel — in —, qui correſpondent cuicunque progreſſui $(\varrho x^{n-r-1} + \sigma x^{n-r-2})$ ſignorum de + in + vel — in — in datâ æquatione; & conſequenter a primo termino ad penultimum reſultantis æquationis tot vel plures per duo, quatuor, &c. continentur progreſſus ſignorum de + in + & — in —, quot continentur in datâ æquatione: ſi vero tot ſolummodo contineantur prædicti progreſſus a primo termino x^{n+1} ad penultimum $\overline{M + aL}\, x$ reſultantis æquationis, tum tot erunt etiam mutationes ſignorum de + in — & — in +, quot in datâ æquatione continentur; & conſequenter penultimus terminus $\overline{M + aL}\, x$ reſultantis æquationis idem habebit ſignum ac ultimus terminus (M) datæ æquationis; & exinde in reſultante æquatione $x^{n+1} - \overline{p - a}\, x^n + \overline{q - ap}\, x^{n-1} \ldots \overline{M + aL}\, x + aM = 0$ adjicitur progreſſus $(\overline{M + aLx + aM})$ ſignorum de + in + vel — in —, qui in datâ æquatione haud invenitur; ergo e prædictis ſequitur reſultantem æquationem plures per unum, tres, &c. habere continuos progreſſus de + in + vel — in —, quam eos, quos habet data æquatio.

Cor. Sit data æquatio $x^n + px^{n-1} + qx^{n-2} \ldots - Px^{n-m} + Qx^{n-m-1} + Rx^{n-m-2} \ldots \pm fx^{n-m-r} \pm gx^{n-m-r-1} \pm hx^{n-m-r-2} \pm kx^{n-m-r-3} \ldots$

$\pm \alpha x$

$\pm \alpha x^{n-m-r-s} \pm \beta x^{n-m-r-s-1} \pm \gamma x^{n-m-r-s-2} \ldots + \pi x^{n-m-r-s-t} - \varrho x^{n-m-r-s-t-1} - \sigma x^{n-m-r-s-t-2} + \&c. = 0$, ubi P, Q, R, &c.; f, g, h, &c.; α, β, &c.; π, ϱ & σ vel ſint omnes affirmativæ vel omnes negativæ quantitates: ſit $-Px^{n-m} + Qx^{n-m-1}$ quæcunque mutatio ſignorum de + in — vel — in +; & $+\pi x^{n-m-r-s-t} - \varrho x^{n-m-r-s-t-1}$ ejus proxime ſubſequens mutatio de — in + vel + in —; ſit g major quam af; at ag major quam h, & ah quam k, & ſic deinceps uſque ad β; at γ ſit major quam $a\beta$: tum, ſi data æquatio ducatur in $x - a$, duæ evadunt mutationes ſignorum de + in — & — in + in reſultante æquatione $x^{n+1} + \overline{p-a}\,x^{n} + \overline{q-ap}\,x^{n-1} \ldots + \overline{Q+aP}\,x^{n-m} + \overline{R-aQ}\,x^{n-m-1} \ldots + \overline{g-af}\,x^{n-m-r} - \overline{ag-h}\,x^{n-m-r-1} - \overline{ah-k}\,x^{n-m-r-2} \ldots - \overline{a\alpha-\beta}\,x^{n-m-r-s} + \overline{\gamma-a\beta}\,x^{n-m-r-s-1} \pm \&c. = 0$, viz. $(+\overline{g-af}\,x^{n-m-r} - \overline{ag-h}\,x^{n-m-r-1}$ & $-\overline{a\alpha-\beta}\,x^{n-m-r-s} + \overline{\gamma-a\beta}\,x^{n-m-r-s-1})$ quæ in datâ haud inveniuntur: & pro omni conſimili caſu in datâ æquatione contento conſtat duas alias in reſultante æquatione evadere mutationes de + in — & — in +, quæ in datâ haud continentur.

Cor. 2. Sit $(+Qx^{n-m-1} + Rx^{n-m-2})$ quicunque progreſſus ſignorum de + in + vel — in —, & $(-\varrho x^{n-m-r-s-t-1} - \sigma x^{n-m-r-s-t-2})$ ejus proxime ſubſequens progreſſus de + in + vel — in —; ergo quantitates f, g, h, k, &c., α, β, γ, &c. habent ſigna + & — alternatim; tum, ſi g major ſit quam af; at ag quam h, & ah quam k, & ſic deinceps uſque ad coefficientem β; ſit autem γ major quam $a\beta$: tum, ſi data æquatio ducatur in $x + a$, duo evadunt progreſſus de + in + & — in — in reſultante æquatione $(x^{n+1} + \overline{p+a}\,x^{n} + \overline{q+ap}\,x^{n-1} \ldots + \overline{Q-aP}\,x^{n-m} + \overline{R+aQ}\,x^{n-m-1} \ldots \pm \overline{g-af}\,x^{n-m-r} \pm \overline{ag-h}\,x^{n-m-r-1} \pm \overline{ah-k}\,x^{n-m-r-2} \ldots \mp \overline{a\alpha-\beta}\,x^{n-m-r-s} \mp \overline{\gamma-a\beta}\,x^{n-m-r-s-1} \ldots - \overline{\varrho-a\pi}\,x^{n-m-r-s-t} - \overline{\sigma+a\varrho}\,x^{n-m-r-s-t-1} \pm \&c. = 0)$ viz. $(\pm \overline{g-af}\,x^{n-m-r} \pm \overline{ag-h}\,x^{n-m-r-1}$ & $\mp \overline{a\alpha-\beta}\,x^{n-m-r-s} \mp \overline{\gamma-a\beta}\,x^{n-m-r-s-1})$, qui in datâ æquatione haud inveniuntur; & pro

O omni

omni conſimili caſu in datâ æquatione contento conſtat duos alios in reſultante æquatione evadere progreſſus de + in + & — in —, qui in datâ haud continentur.

Cor. Hinc facile patet, ſi modo ducatur data æquatio in $x - a$ vel $x + a$; & in priori caſu numerus mutationum ſignorum de + in — & — in + augeatur, in poſteriori vero caſu diminuatur per unitatem a duobus, quatuor, ſex, &c. auctam; tum ſaltem duas, quatuor, ſex, &c. reſpective impoſſibiles radices habet data æquatio.

CAP. III.

DE REDUCTIONE ET RESOLUTIONE ÆQUATIONUM.

PROB. XVII.

DATIS duabus æquationibus $x^n - p x^{n-1} + q x^{n-2} - r x^{n-3} + \&c. = 0$, & $x^m - P x^{m-1} + Q x^{m-2} - R x^{m-3} + \&c. = 0$; *prioris æquationis* (r) *radices ſint etiam* (r) *radices poſterioris æquationis.*

Reducere priorem æquationem in duas, quarum una habet dimenſiones (r), *altera vero* (n — r); *poſteriorem vero in duas alias, quarum una habet dimenſiones* (r), *cujus radices eædem erunt ac* (r) *radices prædictæ æquationis; altera vero* (m — r).

Duarum datarum æquationum inveniatur maximus communis diviſor, cujus dimenſiones erunt *(r)*, & nihilo æqualis ſupponatur; erit una æquatio quæſita, quæ utriſque datis æquationibus communis eſt: dividantur datæ æquationes per hunc communem diviſorem, & reſpectivæ quotientes nihilo æquales ſupponantur; erunt duæ reliquæ æquationes, quarum dimenſiones ſunt $n - r$ & $m - r$ reſpective.

Ex. 1. Sint duæ æquationes $x^5 - 2x^4 + x^3 + 20x^2 - 68x + 48 = 0$ & $x^3 + 2x^2 - 13x + 10 = 0$.

Harum æquationum inveniatur maximus communis diviſor $x^2 - 3x + 2$; dividantur datæ æquationes $x^3 + 2x^2 - 13x + 10 = 0$, & $x^5 - 2x^4 + x^3 + 20x^2 - 68x + 48 = 0$ per hunc communem diviſorem; reducitur æquatio $x^3 + 2x^2 - 13x + 10 = 0$ in quadraticam $x^2 - 3x + 2 = 0$, & ſimplicem $x + 5 = 0$; & æquatio $x^5 - 2x^4 + x^3 + 20x^2 - 68x + 48 = 0$ in cubicam $x^3 + x^2 + 2x + 24 = 0$ & communem quadraticam $x^2 - 3x + 2 = 0$.

Ex. 2.

Ex. 2. Datâ æquatione $x^n - px^{n-1} + qx^{n-2} - \&c. = o$, cujus radices ſint $\alpha, \beta, \gamma, \delta$, &c. i. e. ſit $x^n - px^{n-1} + qx^{n-2} - \&c. = \overline{x-\alpha} \times \overline{x-\beta} \times \overline{x-\gamma} \times \overline{x-\delta} \times \&c.$; datis etiam quibuſdam ex ejus radicibus α, β, γ; vel data æquatione (m) dimenſionum $\overline{x-\alpha} \times \overline{x-\beta} \times \overline{x-\gamma} \times \&c. = o$, cujus ($m$) radices ſint α, β, γ, &c. radices datæ æquationis; dividatur data æquatio $x^n - px^{n-1} + qx^{n-2} - \&c. = o$ per hanc æquationem; deprimetur data æquatio in duas alias, quarum una habet (m) dimenſiones; altera vero ($n-m$).

PROB. XVIII.

Datâ æquatione $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \&c. = o$, *cujus quædam radices* ($\alpha, \beta, \gamma, \delta, \varepsilon$, &c.) *datam habeant inter ſe relationem.*

Invenire illas datæ æquationis radices ($\alpha, \beta, \gamma, \delta, \varepsilon$, &c.), *quæ datam inter ſe habent relationem.*

Scribantur in datâ æquatione pro radice (x) reſpectivæ radices ($\alpha, \beta, \gamma, \delta, \varepsilon$, &c.) omnes inter ſe diverſæ, & datam habentes relationem; quantitatum reſultantium communes diviſores erunt radices quæſitæ.

Si numerus incognitarum quantitatum minor ſit quam æquationum reſultantium: methodus inveniendi communes diviſores ipſas quantitates deducit, quæ profecto non ſolummodo duarum quantitatum maximum communem diviſorem invenit, ſed quam plurimarum; ſi omnes diviſores per minimum reſiduum continuo dividantur.

Ex. Sint radices datam inter ſe habentes relationem termini arithmeticæ ſeriei a, $a+b$, $a+2b$, $a+3b$, &c.

Scribantur in datâ æquatione pro radice (x) termini prædicti a, $a+b$, $a+2b$, $a+3b$, &c. reſultant æquationes

$$a^n - pa^{n-1} + qa^{n-2} - ra^{n-3} + \&c. = o,$$

$$\overline{a+b}^n - p \times \overline{a+b}^{n-1} + q \times \overline{a+b}^{n-2} - r \times \overline{a+b}^{n-3} + \&c. = o,$$

$$\overline{a+2b}^n - p \times \overline{a+2b}^{n-1} + q \times \overline{a+2b}^{n-2} - r \times \overline{a+2b}^{n-3} + \&c. = o,$$

&c. &c. &c.

harum æquationum reſultantium quicunque communis diviſor radicem quæſitam præbebit.

2. Si

2. Si data relatio inter (m) radices ($\alpha, \beta, \gamma, \delta, \epsilon$, &c.) datæ æquationis ($x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \&c. = 0$) ſolummodo exprimi poſſit per æquationem hujuſmodi $x^m - Px^{m-1} + Qx^{m-2} - Rx^{m-3} + Sx^{m-4} - \&c. = 0$. Aſſumatur data æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \&c. = w$ & $x^m - Px^{m-1} + Qx^{m-2} - Rx^{m-3} + Sx^{m-4} - \&c. = 0$; problematis quinti ope inveniatur æquatio, cujus radix eſt (w), quæ ſit $w^m - aw^{m-1} + bw^{m-2} - cw^{m-3} + dw^{m-4} - \&c. = 0$; fiant coefficientes a, b, c, d, e, &c. nihilo reſpective æquales, æquationum reſultantium quicunque communis diviſor radicem quæſitam præbet.

3. Eſt etiam alia methodus hoc problema reſolvendi.

Sit data æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0$, cujus ſint $\alpha, \beta, \gamma, \delta$, &c. radices datam inter ſe habentes relationem.

Scribatur $x - \alpha = 0$, $x - \beta = 0$, $x - \gamma = 0$, $x - \delta = 0$, &c. & per æquationem ex horum factorum continuâ multiplicatione $\overline{x - \alpha} \times \overline{x - \beta} \times \overline{x - \gamma} \times \overline{x - \delta} \times \&c. = 0$ generatam dividatur data æquatio.

Singulus reſidui terminus nihilo æqualis fiat; æquationum reſultantium quicunque communis diviſor erit radix quæſita.

Ex. 1. Sint α & β duæ datæ æquationis $x^n - px^{n-1} + \&c. = 0$ radices, quæ datam inter ſe habeant relationem, æquales; i. e. $\alpha = a$, & $\beta = a$: ſcribatur $x - a = 0$ & $x - a = 0$; rectangulum $\overline{x - a} \times \overline{x - a} = x^2 - 2ax + a^2 = 0$; per hoc rectangulum data dividatur æquatio, operatio eſt hujuſmodi

$$
\begin{array}{l}
x^2 - 2ax + a^2 \,\big)\, x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0 \,\big(\, x^{n-2} + \overline{2a - p}\,x^{n-3} + \&c. \\
\qquad\qquad\qquad\;\; x^n - 2ax^{n-1} + a^2x^{n-2} \\
\qquad\qquad\qquad\;\; \overline{2a - p}\,x^{n-1} + \overline{q - a^2}\,x^{n-2} \\
\qquad\qquad\qquad\;\; \overline{2a - p}\,x^{n-1} - \overline{4a^2 - 2ap}\,x^{n-2} + \overline{2a^3 - a^2p}\,x^{n-3} \\
\qquad\qquad\qquad\qquad\qquad\quad \overline{3a^2 - 2ap + q}\,x^{n-2} - \overline{2a^3 - a^2p + r}\,x^{n-3}, \&c.
\end{array}
$$

patet prioris reſidui termini coefficientem eſſe

$$na^{n-1} - \overline{n-1}\,pa^{n-2} + \overline{n-2}\,qa^{n-3} - \overline{n-3}\,ra^{n-4} + \&c.$$

poſteriorem vero reſidui terminum eſſe

$$\overline{n-1}\,a^n - \overline{n-2}\,pa^{n-1} + \overline{n-3}\,qa^{n-2} - \overline{n-4}\,ra^{n-3} + \&c.$$

Fiant

Fiant hæ duæ quantitates nih
æquationum reſultantium quicun
quæſita.

Ex. 2. Sint (m) radices α, β, γ
æquales.

Pro radicibus (α, β, γ, δ, &c.),
$x - a = 0$, $x - a = 0$, &c. ducantu
& fit contentum

$$\overline{x-a}^{m} = x^{m} - max^{m-1} + m.\frac{m-1}{2}a^{2}x^{m-}$$

per hoc contentum data dividatur

$$x^{m} - max^{m-1} + m.\frac{m-1}{2}a^{2}x^{m-2} - \&c.$$

Patet coefficientem primi reſidui

$$m\times\frac{m+1}{2}\times\frac{m+2}{3}\cdots\frac{n}{n-m+1}a^{n-m+1} - m\times\frac{m+}{2}$$

Secundi autem reſidui termini co

$$\overline{n-m+1}\times\overline{m-1}\times\frac{m}{2}\times\frac{m+1}{3}\cdots\frac{n}{n-m+2}a^{n}$$

Tertii reſidui termini coefficiente

$$\overline{n-m+1}\times\frac{n-m+2}{2}\times\overline{m-2}\times\frac{m-1}{2}\times\frac{m}{3}\times\frac{m+1-2}{4-m+1}qa^{n-m+1} - \&c.$$

Quarti reſidui termini coefficient

$$\overline{n-m+1}\times\frac{n-m+2}{2}\times\frac{n-m+3}{3}\times\overline{m-3}\times\frac{m-}{2}\frac{-m+1}{3}\times\overline{m-3}\cdots\frac{n-2}{n-m+2}qa^{n-m+1}\&c.$$

& ſic deinceps.

Fiant hæ coefficientes nihilo reſ
ſultantium communis diviſor erit r

Ex. 3. Sint α, β, γ, δ, ε, &c. termini arithmeticæ ſeriei a, $a+b$, $a+2b$, .. $a+\overline{m-1}\,b$.

Scribatur pro ſummâ numerorum $(1, 2, 3, 4, .. m-1)$, pro ſumma rectangulorum ſub ſingulis binis $(1\times 2,\ 1\times 3,\ 2\times 3,\ 1\times 4,\ 2\times 4,\ 3\times 4,\ 1\times 5,\ 2\times 5$, &c.), contentorum ſub ſingulis ternis, quatuor, quinque, &c. P, Q, R, S, T, &c. reſpective; multiplicentur factores $\overline{x-a}\times\overline{x-a-b}\times\overline{x-a-2b}\times\overline{x-a-3b}\ ..\ \overline{x-a-\overline{m-1}\,b}$ continuo in ſeſe; contentum erit

$$\begin{array}{llll} x^m - max^{m-1} + m\times\dfrac{m-1}{2}a^2x^{m-2} & - m\times\dfrac{m-1}{2}\times\dfrac{m-2}{3} & a^3\times x^{m-3} & + \text{\&c.} = 0 \\ \quad -Pb \qquad +\overline{m-1}\times Pba & -\overline{m-1}\times\dfrac{m-2}{2} & Pba^2 & + \\ \qquad\qquad\quad + Qb^2 & -\overline{m-2} & Qb^2a & + \\ & - & Rb^3 & + \end{array}$$

Per hoc contentum data dividatur æquatio; reſidui terminorum quicunque communis diviſor erit radix quæſita.

Ex. 4. Sint radices α, β, γ, δ, &c. geometricæ progreſſionis termini $(r, ra, r^2a, r^3a, .. r^{m-1}a)$: multiplicentur factores $\overline{x-a}\times\overline{x-ra}\times\overline{x-r^2a}\ ...\ \overline{x-r^{m-1}a}$ continuo in ſeſe; contentum erit

$$x^m - \frac{1-r^m}{1-r}\times ax^{m-1} + \frac{\overline{1-r^m}\times\overline{1-r^{m-1}}}{\overline{1-r}\times\overline{1-r^2}}\times ra^2x^{m-2}$$

$$-\frac{\overline{1-r^m}\times\overline{1-r^{m-1}}\times\overline{1-r^{m-2}}}{\overline{1-r}\times\overline{1-r^2}\times\overline{1-r^3}}\times r^3\times a^3x^{m-3} + \text{\&c.} = 0.$$

Coefficiens termini (x^{m-s}) hujus æquationis erit æqualis fractioni, cujus numerator eſt

$$a^s\times r^{s\times\frac{s-1}{2}}\times\overline{1-r^{m-s+1}}\times\overline{1-r^{m-s+2}}\times\overline{1-r^{m-s+3}}\times\ ....\ \overline{1-r^m}.$$

Denominator vero $\overline{1-r}\times\overline{1-r^2}\times\overline{1-r^3}\ ....\ \overline{1-r^s}$.

Per hoc contentum data dividatur æquatio, terminorum reſidui communis diviſor erit radix quæſita.

Forſan haud multum abs re erit obſervare hinc oriri methodum

inveniendi

inveniendi infinitam æquationem, quæ infinitas habet radices & poſſibiles & cognitas.

Sit enim æquatio

$$1 - \frac{a}{1-r}x + \frac{ra^2}{1-r \times 1-r^2} \times x^2 - \frac{r^3a^3}{1-r \times 1-r^2 \times 1-r^3} x^3 + \frac{r^6a^4}{1-r\times 1-r^2 \times 1-r^3 \times 1-r^4} x^4 - \frac{r^{10}a^5}{1-r\times 1-r^2\times 1-r^3\times 1-r^4\times 1-r^5} x^5 +$$

&c. in infinitum $= o$: & æquatio habet infinitas radices $\frac{1}{a}$, $\frac{1}{ra}$, $\frac{1}{r^2a}$, $\frac{1}{r^3a}$, $\frac{1}{r^4a}$, &c. in infinitum.

Et ſic aſſumi poſſunt infinitæ conſimiles æquationes.

Hinc facile conſtabit, ut in pleriſque caſibus hæc methodus difficiliorem præbet ſolutionem quam præcedens.

2. 2. Si modo relatio inter (m) radices datæ æquationis exprimatur per æquationem $x^m - ax^{m-1} + bx^{m-2} - cx^{m-3} +$ &c. $= o$, methodus omnino eadem erit ac præcedens; i. e. dividatur data æquatio ($x^n - px^{n-1} + qx^{n-2} -$ &c. $= o$) per æquationem $x^m - ax^{m-1} + bx^{m-2} -$ &c. $= o$, quæ exprimit relationem inter (m) radices datæ æquationis; terminorum reſidui quicunque communis diviſor (m) radices datæ æquationis quæſitas oſtendet.

Ex. 1. Sit data æquatio $x^4 - x^3 - 24x^2 + 4x + 80 = o$, & æquatio exprimens relationem inter duas datæ æquationis radices $x^2 - 7x + \mathcal{Q} = o$; dividatur prior æquatio per poſteriorem, i. e.

$$x^2 - 7x + \mathcal{Q}\,(x^4 - x^3 - 24x^2 + 4x + 80)\,x^2 + 6x + 18 - \mathcal{Q}$$

erit reſiduum $\overline{130 - 13\mathcal{Q}} \times x + 80 - 18\mathcal{Q} + \mathcal{Q}^2$

Duorum reſultantium reſidui terminorum $130 - 13\mathcal{Q}$ & $80 - 18\mathcal{Q} + \mathcal{Q}^2$ inveniatur communis diviſor; operatio erit $10 - \mathcal{Q}\,(80 - 18\mathcal{Q} + \mathcal{Q}^2)\,8 - \mathcal{Q}$; & exinde $\mathcal{Q} - 10 = o$, & $\mathcal{Q} = 10$; & conſequenter æquatio quæſita erit $x^2 - 7x + 10 = o$, cujus radices datam habent relationem, i. e. cujus ſumma radicum erit 7.

Paulo

Paulo aliter reſolvi poteſt hoc exemplum. Quoniam per hypotheſin ſumma duarum radicum eſt 7, pro (x) ſcribantur reſpective v & $7-v$, reſultant duæ æquationes $v^4-v^3-24v^2+4v+80=0$, & $v^4-27v^3+249v^2-893v+990=0$. Harum duarum æquationum communis diviſor erit radix quæſita.

Ex. 2. Sit data æquatio $x^6+qx^4-rx^3+sx^2-tx+v=0$, & æquatio exprimens relationem inter tres datæ æquationis radices $x^3+ax-b=0$: dividatur prior æquatio per poſteriorem, i. e.

$$x^3+ax-b)\,x^6+qx^4-rx^3+sx^2-tx+v\,)\,x^3+\overline{q-a}\,x+b-r$$

reſiduum erit $\overline{s-qa+a^2}\,x^2+\overline{bq-t+ra-2ba}\,x+b^2-br+v$.

E methodo inveniendi communem diviſorem ſequitur nullum eſſe communem diviſorem horum trium reſidui terminorum $s-qa+a^2=0$, & $bq-t+ra-2ba=0$, & $b^2-br+v=0$; & conſequenter datam æquationem nullas habere dati generis radices, ni $q^2v-qrt+r^2s-4sv+t^2=0$: ſi vero $q^2v-qrt+r^2s-4sv+t^2=0$, tum æquationes quæſitæ erunt $x^3+\overline{\sqrt{\tfrac{1}{4}q^2-s}+\tfrac{1}{2}q}\,x-\tfrac{1}{2}r+\sqrt{\tfrac{1}{4}r^2-v}=0$, & $x^3-\overline{\sqrt{\tfrac{1}{4}q^2-s}-\tfrac{1}{2}q}\,x-\tfrac{1}{2}r-\sqrt{\tfrac{1}{4}r^2-v}=0$.

Si vero $v=0$; tum æquationes quæſitæ erunt

$x^3+\overline{\sqrt{\tfrac{1}{4}q^2-s}+\tfrac{1}{2}q}\,x-r=0$, & $x^3-\overline{\sqrt{\tfrac{1}{4}q^2-s}-\tfrac{1}{2}q}\,x=0$.

Cor. 1. Sint plures (n) æquationes quam incognitæ (m) quantitates, e prædictis (n) æquationibus inveniantur $(n-m)$ independentes æquationes $A=0$, $B=0$, $C=0$, &c. quæ ſolummodo unam incognitam quantitatem (x) involvant; e quibuſque (m) prædictis æquationibus inveniri poteſt æquatio, cujus radix eſt x; tum inveniatur maximus communis diviſor quantitatum A, B, C, &c.; qui ſit P; tot (b) poſſunt eſſe datæ formulæ æquationes, quot diverſos valores (α, β, γ, δ, &c.) habet incognita quantitas (x), quæ ſit radix communis diviſoris $P=0$; ni duas, vel tres, vel plures radices æquales habeat prædicta incognita quantitas (x), in quo caſu forſan poſſunt eſſe $b+1$, $b+2$, vel plures datæ formulæ æquationes. Omnis valor (α) cujuſcunque incognitæ quantitatis (x) habet unum ſolummodo valorem e

ſingulis incognitis quantitatibus ſibimetipſi (α) correſpondentem; ni duo vel plures valores incognitæ quantitatis prædictæ (x) ſint (α) inter ſe æquales; in quo caſu tot poſſunt eſſe diverſi valores ex unâ vel pluribus incognitis quantitatibus valori (α) radicis (x) reſpondentes, quot valores incognitæ quantitatis (x) ſint α.

Scribantur enim $\alpha, \beta, \gamma, \delta$, &c. pro x in datis æquationibus, ex æquationibus reſultantibus facile conſtabit coroll.

Cor. 2. Sit numerus æquationum major quam numerus incognitarum quantitatum, & exterminentur omnes incognitæ quantitates; ſi modo numerus æquationum major ſit quam numerus incognitarum quantitatum per unitatem, reſultat una ſolummodo æquatio cognitas quantitates involvens: ſi per duo, tres, &c. major ſit; tum duæ, tres, &c. reſultant æquationes cognitas quantitates ſolummodo involventes, quæ detegunt, quibus caſibus poſſunt eſſe æquationes hujuſmodi.

Hæc duo corollaria facile conſtabunt ex operatione ad communes diviſores detegendos, uſitatâ.

E duabus, tribus, &c. æquationibus cognitas quantitates ſolummodo involventibus, facile formari poſſunt ex earum additione, ſubductione, multiplicatione in datas quantitas, &c. infinitæ aliæ.

Cor. 3. Datâ quâcunque quantitate e diverſis irrationalibus quantitatibus $(A+B+C+D+E+$ &c.) conflatâ; invenire, utrum ullos admittat poſſibiles valores data quantitas, necne.

1. Aſſumantur pro radice irrationalis quantitatis A quantitas $a+p\sqrt{-1}$; pro radice quantitatis B quantitas $b+q\sqrt{-1}$; pro radicibus quantitatum irrationalium C, D, E, &c. reſpective $c+r\sqrt{-1}$; $d+s\sqrt{-1}$; $e+t\sqrt{-1}$; &c. & ſupponatur $p+q+r+s+t+$ &c. $=o$.

Per problema inveniatur, utrum æquationes reſultantes ex aſſumptis valoribus hanc habeant relationem $(p+q+r+s+t+$&c. $=o)$ inter ſe, & perficitur corollarium.

2. Hoc etiam inveſtigari poteſt, ſupponendo datam quantitatem $A+B+C+$ &c. $=x$, & exterminando irrationales quantitates, & exinde

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$$\left.\begin{matrix} \\ \\ \end{matrix}\right\} \alpha^{n-5}$$

$$\left.\begin{matrix} \\ z \end{matrix}\right.$$

$$\left.\begin{matrix} \\ \\ \\ \frac{n-7}{3}\, b \\ \frac{5}{} \times \frac{n-7}{4} \end{matrix}\right\} \left.\begin{matrix} \alpha^{n-8} - k \\ + \overline{n-7} \times g \\ - \overline{n-5} \times \frac{n-8}{2}\, e \\ + \overline{n-3} \times \frac{n-7}{2} \times \frac{n-8}{3}\, c \\ - \overline{n-1} \times \frac{n-6}{2} \times \frac{n-7}{3} \times \frac{n-8}{4}\, a \end{matrix}\right\} \alpha^{n-9} + \&c. = 0.$$

exinde inveniendo, utrum reſultans æquatio, cujus radix eſt x, poſſibiles admittat valores, necne.

Ex. 1. Sit quantitas data $\sqrt[2]{-3+4\sqrt{-1}} + \sqrt[3]{-16-16\sqrt{-1}}$: invenire utrum poſſibilem admittat valorem, necne.

Sint $\sqrt[2]{-3+4\sqrt{-1}} = a+p\sqrt{-1}$, & $\sqrt[3]{-16-16\sqrt{-1}} = b+q\sqrt{-1}$: exinde reſultant quatuor æquationes $a^2-p^2=-3$; $2ap=4$; $b^3-3bq^2=-16$, $3b^2q-q^3=-16$; ſupponatur etiam $p+q=0$, i. e. $q=-p$; per methodum communes diviſores detegendi prædictam inveniantur $a=1$, $p=2$, $b=2$, $q=-2$; ergo recipiet poſſibilem valorem data quantitas.

Ex. 2. Sit quantitas irrationalis $\sqrt[n]{a+\sqrt{-b^2}} + \sqrt[n]{-a\sqrt{-b^2}}$; tum facile conſtat, ſi modo $r+p.\sqrt{-1}$ ſit radix quantitatis $\sqrt[n]{a+\sqrt{-b^2}}$, $a-p\sqrt{-1}$ etiam eſſe radicem quantitatis $\sqrt[n]{a-\sqrt{-b^2}}$, & poſſibiles eſſe valores.

Cor. 1. In pleriſque hujuſmodi caſibus e datâ relatione inter radices deduci poteſt maximus radicum numerus, quas habere poteſt quæcunque (n) dimenſionum æquatio: ſed minime generaliter dici poteſt, ni quis prædictâ utatur methodo communes diviſores inveniendi, utrum ullas radices dati generis habeat data æquatio, necne.

Cor. 2. Quædam æquationes datæ formulæ neceſſario habent radices, quæ aſſignabilem inter ſe habeant relationem: in his autem caſibus haud plures reſultant diverſæ æquationes, quam incognitæ quantitates: ergo ex æquationibus reſultantibus dici poteſt numerus radicum aſſignabilem inter ſe relationem habentium, quas data recipiat æquatio.

Ex. 1. Sit æquatio

$x^{nm} + Px^{\overline{n-1}\times m} + Qx^{\overline{n-2}\times m} + Rx^{\overline{n-3}\times m} + Sx^{\overline{n-4}\times m} + \&c. = 0$: Hæc æquatio neceſſario habet radices, quæ eandem relationem inter ſe habeant, quam habent radices æquationis $x^m+p=0$, $x^m+q=0$, x^m+

Q 2 $r=0$,

Coefficiens cujuſcunque literæ, quæ ſit coefficiens termini $x^{2n-m+2r}$ datæ æquationis, erit $\overline{n-m+2r} \times \frac{n-m+1}{2} \times \frac{n-m+2}{3} \times \frac{n-m+3}{4} \times \frac{n-m+r-1}{r}$: ſi vero r ſit o, tum coefficiens erit 1; ſi r ſit 1, tum coefficiens erit $n-m+2$.

Signum coefficienti literæ (quæ ſit coefficiens termini $x^{2n-m+2r}$) affixum erit + vel —, prout $m+r$ ſit par vel impar numerus.

Cor. 1. Sit æquatio prædicta $x^{2n} \pm x^{2n-1} + x^{2n-2} \pm x^{2n-3} + x^{2n-4} \pm$ &c. $= o$: ea in quadraticam $x^2 + vx + 1 = o$ reduci poteſt ope æquationis $v^n \mp v^{n-1} - \overline{n-1}\, v^{n-2} \pm \overline{n-2}\, v^{n-3} + \overline{n-2} \times \frac{n-3}{2} v^{n-4} \mp \overline{n-3} \times \frac{n-4}{2} v^{n-5} - \overline{n-3} \times \frac{n-4}{2} \times \frac{n-5}{3} v^{n-6} \pm \overline{n-4} \times \frac{n-5}{2} \times \frac{n-6}{3} v^{n-7} + \overline{n-4} \times \frac{n-5}{2} \times \frac{n-6}{3} \times \frac{n-7}{4} v^{n-8} \mp$ &c. $= o$.

Cor. 2. Tres radices æquationis $x^3 + 1 = o$ erunt reſpective -1, $\frac{+1 \pm \sqrt{-3}}{2}$; quatuor radices æquationis $x^4 + 1 = o$ erunt $\pm \sqrt{\tfrac{1}{2}}$ $\pm \sqrt{-\tfrac{1}{2}}$: quinque radices æquationis $x^5 + 1 = o$ erunt -1, $\frac{\mp \sqrt{1\frac{1}{4}} + \frac{1}{2}}{2} \pm \frac{\sqrt{-2\frac{1}{2} \pm \sqrt{1\frac{1}{4}}}}{2}$: ſex radices æquationis ($x^6 + 1 = o$ erunt $\pm \sqrt{-1}$, $\frac{\pm \sqrt{3} \pm \sqrt{-1}}{2}$: ſeptem vero radices æquationis $x^7 + 1 = o$ haud exprimi poſſunt per quadraticas radices.

Ex. 3. Sit æquatio $x^{2n+1} + ax^{2n} + bx^{2n-1} + cx^{2n-2} + dx^{2n-3} + ex^{2n-4} + fx^{2n-5} + gx^{2n-6} + hx^{2n-7} + kx^{2n-8} \ldots + px^{n+1} + px^n \ldots kx^9 + hx^8 + gx^7 + fx^6 + ex^5 + dx^4 + cx^3 + bx^2 + ax + 1 = o$. Hujus æquationis una radix erit -1, & (n) etiam erunt quadraticæ æquationes hujuſce formulæ $x^2 + \alpha x + 1 = o$, $x^2 + \beta x + 1 = o$, $x^2 + \gamma x + 1 = o$; &c. æquatio, cujus radix eſt α, erit

$\alpha^n - a$

$$\alpha^n - \overline{a-1}\times\alpha^{n-1} \left.\begin{array}{l} +b \\ -a \\ -\overline{n-1} \end{array}\right\} \alpha^{n-2} \left.\begin{array}{l} -c \\ +b \\ +\overline{n-2}\,a \\ -\overline{n-2} \end{array}\right\} \alpha^{n-3} \left.\begin{array}{l} +d \\ -c \\ -\overline{n-3}\,b \\ +\overline{n-3}\,a \\ +\overline{n-2}\times\frac{n-3}{2} \end{array}\right\} \alpha^{n-4} \left.\begin{array}{l} -e \\ +d \\ +\overline{n-4}\,c \\ -\overline{n-4}\,b \\ -\overline{n-3}\times\frac{n-4}{2}\,a \\ +\overline{n-3}\times\frac{n-4}{2} \end{array}\right\} \alpha^{n-5} \left.\begin{array}{l} +f \\ -e \\ -\overline{n-5}\,d \\ +\overline{n-5}\,c \\ +\overline{n-4}\times\frac{n-5}{2}\,b \\ -\overline{n-4}\times\frac{n-5}{2}\,a \\ -\overline{n-3}\times\frac{n-4}{2}\times\frac{n-5}{3} \end{array}\right\} \alpha^{n-6} - \&c. = 0.$$

Lex, quam obſervat ſeries, quæ exprimit coefficientem termini (α^{n-m}) hujus ſeriei; vel literas, vel earum coefficientes, reſpicit.

Coefficiens cujuſcunque literæ, quæ ſit coefficiens termini $x^{2n-m+2r+1}$ vel termini $x^{2n-m+2r+2}$ datæ æquationis, erit $\overline{n-m+1}\times\frac{n-m+2}{2}\times\frac{n-m+3}{3}\ldots\frac{n-m+r}{r}$. ſi r ſit o, coefficiens erit 1.

Signum coefficienti literæ (quæ ſit coefficiens termini $x^{2n-m+2r+1}$) affixum, erit + vel —; prout $m+r$ ſit par vel impar numerus.

Hæc exempla facile erui & demonſtrari poſſunt, ſi modo in datâ æquatione pro x ſcribatur ejus valor $-\frac{1}{2}\alpha+\sqrt{\frac{1}{4}\alpha^2-1}$.

Cor. Omnis recurrens æquatio parium dimenſionum $x^{2n}+ax^{2n-1}+bx^{2n-2}\ldots+bx^2+ax+1=(x^n+\alpha x^{n-1}+\beta x^{n-2}+\gamma x^{n-3}\ldots+\pi x^3+\rho x^2+\sigma x+\tau)\ (x^n+\frac{\sigma}{\tau}x^{n-1}+\frac{\rho}{\tau}x^{n-2}+\frac{\pi}{\tau}x^{n-3}\ldots+\frac{\gamma}{\tau}x^3+\frac{\beta}{\tau}x^2+\frac{\alpha}{\tau}x+\frac{1}{\tau})=o$.

Cor. Quadratum, cubus, & omnis poteſtas recurrentis quantitatis erit recurrens quantitas.

PROB. XIX.

1. *Invenire, utrum duæ datæ irrationales quantitates* A & B *unum valorem, vel duos, vel plures habeant inter ſe æquales, necne.*

Fingatur $x=A$ & $x=B$; ita reducantur datæ æquationes, ut exterminentur irrationales quantitates; deinde inveniatur communis diviſor

diviſor duarum reſultantium æquationum, ſi modo quidam detur; & fingatur communis diviſor inventus nihilo æqualis; erunt radices æquationis exinde reſultantis valores quæſiti.

2. *Ex hâc methodo etiam deduci poteſt, annon radix vel radices unius vel duarum vel plurium irrationalium quantitatum* A, B, C, D, *&c. datam habeant relationem ad radicem vel radices earundem irrationalium quantitatum* A, B, C, *&c. vel diverſarum* a, b, c, *&c.*

Aſſumantur $x = A$, $y = B$, $z = C$, &c.; $u = a$, $v = b$, $w = c$, &c. ita reducantur omnes hæ æquationes, ut exterminentur irrationales quantitates, & reſultent æquationes $P = 0$, $Q = 0$, &c. $p = 0$, $q = 0$, &c. deinde ita reducantur æquationes reſultantes, & æquationes inter quas exprimitur data relatio, ut exterminentur incognitæ quantitates x, y, z, &c. u, v, w, &c., &c.; reſultabunt æquationes, quæ manifeſto oſtendunt, utrum ullæ ſint radices dati generis, necne.

PROB. XX*.

Transformare irrationales quantitates in alias, quæ datis irrationalibus quantitatibus æquales ſint, & tamen ſub diverſâ irrationalitatis formulâ lateant.

Inveniatur quæcunque functio datarum irrationalium quantitatum, cujus reductio cognoſcitur; deinde ex cognitâ reductione inveniri poteſt valor datarum irrationalium quantitatum ſub diverſâ irrationalitatis formulâ latens.

Ex. 1. Sint datæ irrationales quantitates $\sqrt{a + \sqrt{a^2 - b^2}} + \sqrt{a - \sqrt{a^2 - b^2}} = x$; quadratum eſt functio, cujus reſolutio datur; ſed erit quadratum $(x^2) = 2a \pm 2b$; unde valor (x) datarum irrationalium quantitatum $= \sqrt{a + \sqrt{a^2 - b^2}} + \sqrt{a - \sqrt{a^2 - b^2}} = \sqrt{2a \pm 2b}$..

Ex. 2. Sint datæ irrationales quantitates $x = \sqrt[3]{a + \sqrt{a^2 - b^3}} + \sqrt{a -}$

$\sqrt[3]{a-\sqrt{a^2-b^3}}$; hujus quantitatis cubus erit $x^3 = 2a + 3$ (1 vel $\frac{-1\pm\sqrt{-3}}{2}$) $b\left(\sqrt[3]{a+\sqrt{a^2-b^3}} + \sqrt[3]{a-\sqrt{a^2-b^3}}\right)$, & conſequenter erit $x = \sqrt[3]{2a + \left(1 \text{ vel } \frac{-1\pm\sqrt{-3}}{2}\right) 3b \left(\sqrt[3]{a+\sqrt{a^2-b^3}} + \sqrt[3]{a-\sqrt{a^2-b^3}}\right)} = \sqrt[3]{a+\sqrt{a^2-b^3}} + \sqrt[3]{a-\sqrt{a^2-b^3}}$.

Ex. 3. Sit $x = \sqrt{a} + \sqrt{b}$, unde $x^2 - 2px = a + b + 2\sqrt{ab} - 2p\sqrt{a} - 2p\sqrt{b}$; & exinde erit $x = p + \sqrt{a + b + p^2 + 2\sqrt{ab} - 2p\sqrt{a} - 2p\sqrt{b}} = \sqrt{a} + \sqrt{b}$.

Hæ quantitates reſultantes facile reduci poſſunt in præcedentes, ex quibus deductæ fuerunt.

THEOR. XIV*.

1. Sint b, i, k, l, &c. integri affirmativi numeri; & α, β, γ, δ, ε, &c. radices æquationis $x^n - 1 = 0$; tum erit contentum $\alpha^b \times \beta^i \times \gamma^k \times \delta^l \times$ &c. etiam radix prædictæ æquationis $x^n - 1 = 0$.

Scribatur $\alpha^b\, \beta^i\, \gamma^k\, \delta^l$ &c. pro x in æquatione $x^n - 1 = 0$, reſultat $x^n = \alpha^{n\times b}(1) \times \beta^{n\times i}(1) \times \gamma^{n\times k}(1) \times$ &c. $= 1$, unde $x^n - 1 = 1 - 1 = 0$.

2. Sit $x^n + 1 = 0$, cujus radices ſint α, β, γ, δ, &c.; erit $\alpha^b \times \beta^i \times \gamma^k \times \delta^l \times$ &c. radix æquationis $x^n + 1 = 0$, vel $x^n - 1 = 0$; prout $b + i + k + l +$ &c. ſit impar vel par numerus.

3. Sit n primus numerus, & α radix æquationis $x^n - 1 = 0$; tum erunt α, α^2, α^3, $\alpha^4 \ldots \alpha^{n-1}$, 1, omnes diverſæ radices prædictæ æquationis $x^n - 1 = 0$: etiamque ſit α radix æquationis $x^n + 1 = 0$, tum α, $-\alpha^2$, α^3, $-\alpha^4$, α^5, $-\alpha^6$, ... -1 erunt omnes radices æquationis $x^n + 1 = 0$.

4. 1. Sit $n = m \times r \times s \times t \times$ &c.; ubi m, r, s, t, &c. ſint primi numeri; tum, ſi modo radices æquationum $x^m - 1 = 0$, $x^r - 1 = 0$, $x^s - 1 = 0$, $x^t - 1 = 0$, &c. ſint reſpective a, b, c, &c.; e, f, g, &c.;

b, *i*, *k*, &c., &c.; erunt *aeb* &c., *aei* &c., *aek* &c., &c., *afb* &c., *afi* &c., *afk* &c., &c., *agb* &c., *agi* &c., *agk* &c., &c., *beb* &c., *bei* &c., *bek* &c., &c.; & ſic deinceps omnes diverſæ radices (n) æquationis $x^n - 1 = 0$.

4. 2. Sint prædictæ quantitates radices æquationum $x^m + 1 = 0$, $x^r + 1 = 0$, $x^s + 1 = 0$, &c.; tum, ſi numerus diviſorum m, r, s, t, &c. ſit impar, prædicta contenta erunt omnes diverſæ radices æquationis $x^n + 1 = 0$; ſi vero ſit par, tum prædicta contenta reſpective multiplicata in quemcunque valorem quantitatis $\sqrt[n]{-1} = \pi$ erunt omnes diverſæ radices æquationis $x^n + 1 = 0$.

4. 3. Iiſdem poſitis ſit α radix æquationis $x^n - 1 = 0$, quæ haud ſit radix vel æquationum $x^m - 1 = 0$, vel $x^r - 1 = 0$, vel $x^s - 1 = 0$, &c.; tum $\alpha, \alpha^2, \alpha^3, \alpha^4, \ldots \alpha^{n-1}, 1$, erunt univerſæ radices æquationis $x^n - 1 = 0$: ſit α radix æquationis $x^n + 1 = 0$, quæ haud ſit radix æquationum $x^m + 1 = 0$, $x^r + 1 = 0$, $x^s + 1 = 0$, &c.; tum erunt α, $\pi \times \alpha^2$, α^3, $\pi \times \alpha^4$, α^5, $\pi\alpha^6$, &c.; univerſæ radices æquationis $x^n + 1 = 0$.

5. Sit n impar numerus, & ſi $\alpha, \beta, \gamma, \delta$, &c. ſint radices æquationis $x^n - 1 = 0$, tum $-\alpha$, $-\beta$, $-\gamma$, $-\delta$, &c. erunt reſpective radices æquationis $x^n + 1 = 0$.

6. Sit $\frac{n}{r} = m$; tum radices æquationis $x^r - 1 = 0$ erunt (m) poteſtates e ſingulis radicibus æquationis $x^n - 1 = 0$: erunt radices æquationis $x^r + 1 = 0$ (m) poteſtates e ſingulis radicibus æquationis $x^n + 1 = 0$. Conſtant hæc corollaria e ſubſtitutione.

Cor. Sint α & β duæ correſpondentes radices $(a + \sqrt{-b^2}$ & $a - \sqrt{-b^2})$ æquationis $x^n \mp 1 = 0$, conſequenter earum productum $\overline{x-\alpha} \times \overline{x-\beta} = x^2 - \overline{\alpha + \beta}x + \alpha\beta$ nullam continet impoſſibilem quantitatem; tum erit $a^2 + b^2 = \overline{a + \sqrt{-b^2}} \times \overline{a - \sqrt{-b^2}} = 1$.

THEOR.

THEOR. XV*.

Sint $a+\sqrt{-b^2}$ & $a-\sqrt{-b^2}$; $c+\sqrt{-d^2}$ & $c-\sqrt{-d^2}$; $e+\sqrt{-f^2}$ & $e-\sqrt{-f^2}$; &c. correſpondentes impoſſibiles quantitates: ſit $\mathfrak{Q}$ rationalis functio harum quantitatum vel fractionalis, necne; in quâ ſimiliter involvuntur correſpondentes quantitates $a+\sqrt{-b^2}$ & $a-\sqrt{-b^2}$; etiamque $c+\sqrt{-d^2}$ & $c-\sqrt{-d^2}$; & ſimiliter $e+\sqrt{-f^2}$ & $e-\sqrt{-f^2}$; &c.: tum erit functio $\mathfrak{Q}$ poſſibilis quantitas.

Facile conſtat ex principiis in theor. 1. datis.

2. Quæcunque rationalis functio ($\mathfrak{Q}$) correſpondentium impoſſibilium radicum etiam erunt correſpondentes impoſſibiles radices; e. g. ſit $\mathfrak{Q}$ functio impoſſibilis radicis $(a+\sqrt{-b^2}) = A+\sqrt{-B^2}$, tum erit $\mathfrak{Q}$ functio ejus correſpondentis radicis $(a-\sqrt{-b^2}) = A-\sqrt{-B^2}$, ubi $A+\sqrt{-B^2}$ & $A-\sqrt{-B^2}$ ſunt correſpondentes radices.

THEOR. XVI.

1. Sit reſolutio æquationis $W = ap^{\frac{1}{n}} + bp^{\frac{2}{n}} + cp^{\frac{3}{n}} + \&c.$, ubi a, b, c, d, &c., & p ſint poſſibiles quantitates: ſcribatur $p^{\frac{1}{n}} = q\sqrt[n]{\pm 1}$, ubi q eſt poſſibilis quantitas; unde reſolutio $W = aq\sqrt[n]{\pm 1} + bq^2\sqrt[n]{\pm 1}^{\,2} + cq^3\sqrt[n]{\pm 1}^{\,3} + \&c.$: ſint α & $\frac{1}{\alpha}$ radices correſpondentes æquationis $x^n \mp 1 = 0$; ſcribantur $aq\alpha + bq^2\alpha^2 + cq^3\alpha^3 + dq^4\alpha^4 + \&c. = P$, & $aq \times \frac{1}{\alpha} + bq^2 \times \frac{1}{\alpha^2} + cq^3 \times \frac{1}{\alpha^3} + dq^4 \times \frac{1}{\alpha^4} + \&c. = \mathfrak{Q}$; unde P & $\mathfrak{Q}$ erunt duo correſpondentes impoſſibiles valores prædictæ radicis W; & conſequenter $\overline{W-P} \times \overline{W-\mathfrak{Q}} = W^2 - (P+\mathfrak{Q})W + P\mathfrak{Q}$ erit poſſibile productum, i. e. productum in quo haud continetur impoſſibilis quantitas.

4. Datâ reſolutione ad modum præcedentem præparatâ, & literis $\alpha, \delta, \theta, \lambda, \pi,$ &c. reſpective denotantibus radicem e ſingulis æquationibus $x^n \mp 1 = 0$, $x^m \mp 1 = 0$, $x^l \mp 1 = 0$, &c.; ſi in datâ reſolutione contineantur radices $(n, m, l,$ &c.) poteſtatum e quibuſcunque quantitatibus $a, b, c, d,$ &c.; tum in datâ reſolutione pro $\sqrt[n]{a}, \sqrt[m]{b}, \sqrt[l]{c},$ &c., ſcribantur reſpective $\alpha\sqrt[n]{a}, \delta\sqrt[m]{b}, \theta\sqrt[l]{c},$ &c., & reſultet quantitas H: deinde in prædictâ reſolutione pro $\sqrt[n]{a}, \sqrt[m]{b}, \sqrt[l]{c},$ &c. ſcribantur reſpective $\frac{1}{\alpha} \times \sqrt[n]{a}, \frac{1}{\delta} \times \sqrt[m]{b}, \frac{1}{\theta} \times \sqrt[l]{c},$ &c., & reſultet quantitas K; tum erit $\overline{W-H} \times \overline{W-K} = W^2 - \overline{H+K}W + H \times K$ poſſibile productum, in quo haud continetur impoſſibilis quantitas.

Hæc facile conſtat e coroll. . theor. 1. eadem principia in omnes reſolutiones, quæ ſint functiones algebraicarum quantitatum, e. g. in reſolutiones hujuſce generis $\sqrt[n]{a + \sqrt[m]{b + \sqrt[l]{c + \&c.}} + \sqrt[l]{d} + \&c.}$, &c.; etiamque in reſolutiones, quæ ſint functiones exponentialium quantitatum, applicari poſſunt.

Cor. Sit reſolutio $x = A$, cujus æquatio e ſurdis libera ſit $R = 0$; tum fractio $\frac{1}{R}$ dividi poteſt per hanc methodum in ſimplices & quadraticas poſſibiles fractiones; nam per hanc methodum dantur correſpondentes poſſibiles valores datæ æquationis, & conſequenter correſpondentes poſſibiles radices æquationis $R = 0$, unde facile conſtat corollarium.

THEOR. XVII*.

Erit $\dfrac{A + B\sqrt{-1}}{C + D\sqrt{-1}} = \dfrac{BD + CA + \overline{CB - DA}\sqrt{-1}}{D^2 + C^2}$.

Facile probari poteſt ex ducendo numeratorem unius fractionis in denominatorem alterius, evadent enim duo rectangula inter ſe æqualia.

Cor.

&c. totidem enim diverſos valores habent coefficientes vel a, vel b, vel c, &c. & conſequenter eædem erunt æquationum dimenſiones, quarum radices ſunt vel a, vel b, vel c, &c. reſpective.

Datâ autem unâ coefficiente, quâ ſubſtitutâ pro ſuo valore, e præcedenti inveniendi methodo, utrum datæ æquationis radices datam inter ſe habeant relationem, i. e. e methodo communes diviſores inveniendi, erui poſſunt reliquæ incognitæ quantitates. Si vero duo valores datæ coefficientis ſint æquales, tum reliquæ incognitæ quantitates prædictâ methodo & quadraticarum æquationum reſolutionum ope invenientur; ſi tres valores ſint æquales, tum reliquæ incognitæ quantitates prædictâ methodo & cubicarum æquationum reſolutionum ope inveſtigari poſſunt; & ſic deinceps.

Hoc in loco obſervetur, quod poſteà generaliter docebitur, omnes incognitas quantitates a, b, c, &c. eandem habere irrationalitatem; ni duo vel plures valores unius incognitæ quantitatis ſint inter ſe æquales; ſi duo uniuſcujuſque quantitatis valores (α) ſint inter ſe æquales, tum reſultabunt quadraticæ æquationes, quarum radices ſint reliquæ incognitæ quantitates, & quæ ſolummodo habebunt irrationalitatem, quam habet valor (α) prædictæ incognitæ quantitatis; ſi tres valores (α) ſint inter ſe æquales, tum reſultant conſimiles cubicæ æquationes; & ſic deinceps.

Hæc facile conſtabunt, ex eo quod per vulgarem communes diviſores detegendi operationem, haud novæ irrationales quantitates exorientur.

Ex. Biquadratica $x^4 - px^3 + qx^2 - rx + s = o$ reduci poteſt in quadraticam $x^2 - Px + Q = o$, æquationis $(4 \times \frac{4-1}{2} = 6)$ dimenſionum ope.

1. Æquatio, cujus radix eſt P, erit $\overline{P - \frac{p}{2}}|^6 + \overline{2q - \frac{3}{4}p^2} \times \overline{P - \frac{p}{2}}|^4 + \overline{q^2 - 4s - qp^2 + rp + \frac{3}{16}p^4} \times \overline{P - \frac{p}{2}}|^2 - \overline{\frac{p^3}{8} - \frac{qp}{2} + r}|^2 = o.$

Sit

2. Sit $p = o$, i. e. ſumma quatuor radicum (α, β, γ, δ) datæ æquationis ſit $= o$; ſumma autem duarum ($\alpha + \beta$) eſt P, ergo ſumma duarum reliquarum ($\gamma + \delta$) erit $-P = -\alpha - \beta$; & tres radices æquationis, cujus radix eſt P, erunt affirmativæ; tres vero negativæ & affirmativis æquales; & conſequenter ejus formula quoad reſolutionem erit cubica.

Æquatio, cujus radix eſt P, erit

$$P^6 + 2qP^4 + \overline{q^2 - 4s}\,P^2 - r^2 = o.$$

Hujuſce æquationis radices ($\alpha + \beta$, $\alpha + \gamma$, $\alpha + \delta$, $\beta + \gamma$, $\beta + \delta$, $\gamma + \delta$) ſint π, ϱ, σ, τ, υ, ϕ reſpective; radices datæ æquationis erunt $\frac{\pi + \varrho - \tau}{2}$, $\frac{\pi + \tau - \varrho}{2}$, $\frac{\tau + \varrho - \pi}{2}$, $\frac{\sigma + \upsilon - \pi}{2}$ reſpective, etiamque erunt $\frac{\pi + \sigma - \upsilon}{2}$, $\frac{\pi + \upsilon - \sigma}{2}$, $\frac{\varrho + \phi - \sigma}{2}$, $\frac{\sigma + \phi - \varrho}{2}$; etiamque $\frac{\varrho + \sigma - \phi}{2}$, $\frac{\tau + \upsilon - \phi}{2}$, $\frac{\tau + \phi - \upsilon}{2}$, $\frac{\upsilon + \phi - \tau}{2}$. Facile conſtat e ſubſtituendo pro literis π, ϱ, σ, τ, υ, ϕ earum valores prius aſſignatos.

3. Data unâ radice (π) quantitatis P, tum quatuor datæ æquationis radices erunt $\pm\sqrt{\frac{-\pi^3 - 2q\pi - 2r}{4\pi}} + \frac{1}{2}\pi$, $\pm\sqrt{\frac{-\pi^3 - 2q\pi + 2r}{4\pi}} - \frac{1}{2}\pi$.

DEMONSTRATIO.

Sit π valor quantitatis P inventus, tum erit $\overline{x^2 - \pi x + Q} \times \overline{x^2 + \pi x + R} = x^4 + \overline{R + Q - \pi^2}\,x^2 - (R - Q)\,\pi x + RQ = x^4 + qx^2 - rx + s = o$; ex æquatis coefficientibus correſpondentium terminorum reſultant tres æquationes $R + Q - \pi^2 = q$, $\overline{R - Q} \times \pi = r$, $QR = s$; ſed hæ tres æquationes duas ſolummodo involvunt incognitas quantitates (R & Q); hæ vero duæ quantitates e methodo communes diviſores inveniendi plurimis modis exprimi poſſunt in rationalibus terminis literarum q, r, s & π, qui revera omnes iidem ſunt; ni duo vel plures valores hujus (Q) vel illius (R) incognitæ

incognitæ quantitatis ſint inter ſe æquales; e. g. e duabus prioribus æquationibus $R + \mathfrak{Q} - \pi^2 = q$ & $(R - \mathfrak{Q})\pi = r$ deduci poteſt $R = \frac{1}{2}\left(q + \pi^2 + \frac{r}{\pi}\right)$ & $\mathfrak{Q} = \frac{1}{2}\left(q + \pi^2 - \frac{r}{\pi}\right)$; ſubſtituantur hi valores reſpective pro R & $\mathfrak{Q}$ in quadraticis æquationibus $x^2 - \pi x + \mathfrak{Q} = 0$ & $x^2 + \pi x + R = 0$; quatuor radices duarum reſultantium quadraticarum erunt eæ, quæ pro quatuor datæ æquationis radicibus prius aſſignatæ fuere.

Cor. 1. Hæc reſolutio ſolummodo generalis erit, cum duæ radices datæ æquationis ſint poſſibiles, duæ vero impoſſibiles: hoc enim in caſu cubicæ æquationis $P^3 + 2qP^2 + \overline{q^2 - 4s}\,P - r^2 = 0$ una radix erit poſſibilis, duæ vero impoſſibiles; in quo caſu ſolummodo inveniri poſſunt ejus radices per algebraicam methodum cubicas æquationes reſolvendi.

Æquatio, cujus radix eſt $\mathfrak{Q}$ rectangulum ſub quibuſque duabus biquadraticæ æquationis $x^4 + qx^2 - rx + s = 0$ radicibus, erit $\mathfrak{Q}^6 - q\mathfrak{Q}^5 - s\mathfrak{Q}^4 + \overline{2sq - r^2}\,\mathfrak{Q}^3 - s^2\mathfrak{Q}^2 - qs^2\mathfrak{Q} + s^3 = 0$.

Cor. 2. Hâc vero methodo æquatio reducens plures habet dimenſiones quam data æquatio, & haud generaliter admittit diviſores, nec ejus formula ea eſt, cujus reſolutio cognoſcitur: fruſtra igitur per talem reductionem æquationum reſolutio generalis quæritur.

LEMMA.

Omnis biquadratica æquatio hujuſce formulæ $x^2 - 2x\,\overline{a + b\sqrt{-1}}\;x - c - d\sqrt{-1} = 0$ ſemper diſtingui poteſt in duas quadraticas æquationes poſſibiles quantitates ſolummodo habentes.

Sint enim $\alpha^2 = \frac{a^2 - b^2 + c + \sqrt{\overline{a^2 - b^2 + c}^2 + \overline{2ab + d}^2}}{2}$ &

$$\beta^2 = \frac{+\sqrt{\overline{a^2 - b^2 + c}^2 + \overline{2ab + d}^2} - a^2 + b^2 - c}{2},$$

ubi α & β poſſibiles quantitates ſemper denotant.

Poſſibiles

Poſſibiles quadraticæ æquationes erunt reſpective
$x^2 - \overline{2a+\alpha}\,x + \overline{a+\alpha}^2 + \overline{\beta+b}^2 = 0$, & $x^2 - \overline{2a-\alpha}\,x + \overline{a-\alpha}^2 + \overline{\beta-b}^2 = 0$.

Hoc lemma facile erui poteſt, ſi modo inveniatur radix datæ quadraticæ æquationis $x^2 - \overline{2a + b\sqrt{-1}}\,x - c - d\sqrt{-1} = 0$, quæ erit $a + b\sqrt{-1} \pm \sqrt{a^2 - b^2 + c + (2ab + d)\sqrt{-1}}$; ejus vero correſpondens radix, quæ præbet poſſibilem quadraticam, erit $a - b\sqrt{-1} \pm \sqrt{a^2 - b^2 + c - (2ab + d)\sqrt{-1}}$: fingatur enim $\sqrt{a^2 - b^2 + c + (2ab + d)\sqrt{-1}} = \alpha + \beta\sqrt{-1}$, tum erit $\sqrt{a^2 - b^2 + c - (2ab + d)\sqrt{-1}} = \alpha - \beta\sqrt{-1}$; exinde erui poſſunt valores quantitatum α & β ſupra traditi.

THEOR. XVIII.

Omnis æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0$ vel componitur e ſimplicibus diviſoribus in ſeſe ductis ($\overline{x-\alpha} \times \overline{x-\beta} \times \overline{x-\gamma} \times$ &c.), quæ ſolummodo poſſibiles quantitates ($\alpha, \beta, \gamma, \delta$, &c.) continent, vel e ſimplicibus & quadraticis diviſoribus $x-\alpha$, $x-\beta$, $x-\gamma$, &c. $x^2 - \pi x + P$, $x^2 - \varrho x + R$, $x^2 - \sigma x + S$, &c. in ſeſe ductis, qui ſolummodo poſſibiles quantitates continent.

DEMONSTRATIO.

Omnis æquatio $x^{2n+1} - px^{2n} + qx^{2n+1} - \&c. = 0$, cujus dimenſiones $(2n+1)$ ſunt impares, unum ſaltem habet poſſibilem diviſorem $(x-\alpha)$, cujus α ſit poſſibilis quantitas; & hujus ope deprimi poteſt data æquatio in alteram parium $(2n)$ dimenſionum.

Omnis æquatio $x^{4n+2} - px^{4n+1} + \&c. = 0$ dimenſionum $(4n+2)$ deprimi poteſt in quadraticam æquationis $\overline{4n+2} \times \frac{4n+1}{2} = \overline{2n+1} \times \overline{4n+1}$ dimenſionum ope: una autem radix æquationis reducentis, cujus dimenſiones ſint $\overline{2n+1} \times \overline{4n+1}$, erit poſſibilis quantitas; ergo

o una

una quadratica æquatio, quæ ſolummodo poſſibiles habet quantitates, erit diviſor datæ æquationis.

Omnis æquatio $x^{8n+4} - px^{8n+3} + qx^{8n+2} - \&c. = o$ dimenſionum $(8n+4)$ deprimi poteſt in quadraticam æquationis $\overline{8n+4} \times \frac{8n+3}{2} = \overline{4n+2} \times \overline{8n+3}$ dimenſionum ope: hæc autem æquatio unam habet quadraticam æquationem diviſorem, cujus duæ coefficientes ſunt poſſibiles quantitates, quæ ſit $x^2 - ax + b = o$; cujus radices erunt $\frac{a \pm \sqrt{a^2 - 4b}}{2}$: ſed æquatio $(8n+4)$ dimenſionum deprimi poteſt in quadraticam utrarum harum radicum ope; quibus ſubſtitutis, conſtat æquationem ſaltem habere biquadraticam æquationem hujuſce formulæ $x^2 + \overline{A + c\sqrt{a^2 - 4b}}\,x + B + d\sqrt{a^2 - 4b} = o$ diviſorem, ubi A, B, c & d poſſibiles denotant quantitates: ſed per lemma præcedens hæc biquadratica conficitur e duabus poſſibilibus quadraticis; ergo æquatio $(8n+4)$ dimenſionum ſaltem habet duas poſſibiles quadraticas diviſores.

Eodem prorſus modo probari poteſt æquationem $x^{16n+8} - px^{16n+7} + qx^{16n+6} - \&c. = o$ ſaltem habere quatuor poſſibiles quadraticas æquationes diviſores: & eodem modo conſtat æquationem $x^{32n+16} - px^{32n+15} + \&c. = o$ ſaltem habere octo poſſibiles quadraticas æquationes diviſores; & ſic deinceps: ergo conſtat omnes æquationes habere vel ſimplicem, vel quadraticas poſſibiles æquationes diviſores: dividatur data æquatio per ſimplicem vel quadraticas poſſibiles æquationes, deprimetur ea in alteram pauciores dimenſiones habentem: æquationis autem reſultantis poſſibiles erunt ſimplices vel quadraticæ æquationes diviſores, ergo adhuc deprimi poteſt reſultans æquatio; & ſic deinceps; unde conſtat theorema.

2. Sit $2^r = n$, ubi r eſt integer numerus; tum æquatio (n) dimenſionum reduci poteſt ad æquationem 2^{r-1} dimenſionum, æquationis $n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \ldots \frac{\frac{1}{2}n+1}{\frac{1}{2}n}$ dimenſionum, ope; formulæ vero $4m+2$ dimenſionum, ubi m eſt integer numerus.

Sit 2^r minor quam n, at 2^{r+1} major quam n; tum æquatio (n) dimenſionum reduci poteſt ad æquationem 2^r dimenſionum, æquationis $n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \ldots \frac{n-2^r+1}{2^r}$ dimenſionum, quæ erunt impares, ope.

THEOR. XIX.

Æquationes reduci poſſint, ſi ita in duas æquales partes dividantur, ut ex utrâque parte eadem radix extrahi poſſit.

PROB. XXII.

Biquadraticam $x^4 + 2px^3 = qx^2 + rx + s$ *in quadraticam ope cubicæ æquationis reducere.*

Addatur utrique æquationis parti $\overline{p^2+2n}\,x^2 + 2pnx + n^2$, & prodit

$$x^4 + 2px^3 + \overline{p^2+2n}\,x^2 + 2pnx + n^2 = \overline{p^2+2n+q}\,x^2 + \overline{2pn+r}\,x + s + n^2.$$

Supponatur $4 \times \overline{s+n^2} \times \overline{p^2+2n+q} = \overline{2pn+r}^{\,2}$: reducatur hæc æquatio, & tranſponantur termini; fit cubica

$$8n^3 + 4qn^2 + \overline{8s-4rp}\,n + 4qs + 4p^2s - r^2 = 0.$$

Hinc inveniatur incognita quantitas (n), quâ pro ſuo valore (n) in æquatione $x^4 + 2px^3 + \overline{p^2+2n}\,x^2 + 2pnx + n^2 = \overline{p^2+2n+q}\,x^2 + \overline{2pn+r}\,x + s + n^2$ ſubſtitutâ, & extractâ utrobique radice quadraticâ; reſultat æquatio quadratica $x^2 + px + n = \sqrt{p^2+2n+q}\;x + \sqrt{s+n^2}$.

Cor. 1. Biquadraticæ æquationis quatuor radices (ſi modo π ſit radix cubicæ æquationis $8n^3 + 4qn^2 + \overline{8s-4rp} \times n + 4qs + 4p^2s - r^2 = 0$) erunt reſpective radices duarum quadraticarum $x^2 + \left(p + \sqrt{p^2+2\pi+q}\right)x + \pi - \sqrt{s+\pi^2} = 0$, & $x^2 + \left(p - \sqrt{p^2+2\pi+q}\right)x + \pi + \sqrt{s+\pi^2} = 0$,

duæ

duæ etiam aliæ inveniuntur quadraticæ æquationes ex hâc reſolutione $\left(x^2+p+\sqrt{p^2+2\pi+q}\right)x+\pi+\sqrt{s+\pi^2}=0$, & $\left(x^2+p-\sqrt{p^2+2\pi+q}\right)x+\pi-\sqrt{s+\pi^2}=0$.

i. e. ſi modo quatuor radices datæ æquationis ſint $\alpha, \beta, \gamma, \delta$: & duæ priores quadraticæ æquationes ſint $x^2-\overline{\alpha+\beta}\,x+\alpha\beta=0$, & $x^2-\overline{\gamma+\delta}\,x+\gamma\delta=0$; duæ poſteriores erunt $x^2-\overline{\alpha+\beta}\,x+\gamma\delta=0$, & $x^2-\overline{\gamma+\delta}\,x+\alpha\beta=0$.

Si vero $p=0$; tum hæ quatuor quadraticæ æquationes erunt diviſores duarum biquadraticarum æquationum $x^4-qx^2-rx-s=0$, & $x^4-qx^2+rx-s=0$: i. e. reſolutio ex hâc methodo deducta præbet omnes datæ æquationis radices, & omnes earum negativas: e. g. ſit biquadratica æquatio $x^4-25x^2+60x-36=0$; pro p, q, r, s ſcribantur 0, 25, —60, 36 reſpective in reducenti æquatione $8n^3+4qn^2+\overline{8s-4rp}\times n+4qs+4p^2s-r^2=0$; fit $8n^3+100n^2+288n=0$; cujus radices ſunt reſpective $0, -8, -\frac{9}{2}$; quibus pro ſuo valore (n) in æquatione $x^2+px+n=\sqrt{q+p^2+2n}\,x+\sqrt{s+n^2}$ ſubſtitutis, reſultant tres æquationes $x^2=\sqrt{25}\,x+\sqrt{36}$, & $x^2-8=\sqrt{9}\,x+\sqrt{100}$, & $x^2-\frac{9}{2}=\sqrt{16}\,x+\frac{\sqrt{225}}{2}$. Ex ſingulis his tribus æquationibus radices vel ſunt 1, 2, 3, —6; qui numeri ſunt radices datæ æquationis $x^4-25x^2+60x-36=0$; vel earum negativi —1, —2, —3, +6.

Cor. 2. Sint $\alpha, \beta, \gamma, \delta$ reſpective datæ æquationis $x^4+2px-qx^2-rx-s=0$ radices, tres radices æquationis reducentis $8n^3+4qn^2+\overline{8s-4pr}\times n+4qs+4p^2s-r^2=0$ erunt reſpective $\frac{\alpha\beta+\gamma\delta}{2}, \frac{\alpha\gamma+\beta\delta}{2}, \frac{\alpha\delta+\beta\gamma}{2}$; & quantitatis $\sqrt{q+p^2+2n}$ ſex valores erunt reſpective $\frac{\alpha+\beta-\gamma-\delta}{2}, \frac{\gamma+\delta-\alpha-\beta}{2}, \frac{\alpha+\gamma-\beta-\delta}{2}$,

 $\beta+\delta$

$\frac{\beta+\delta-\alpha-\gamma}{2}$, $\frac{\alpha+\delta-\beta-\gamma}{2}$, $\frac{\beta+\gamma-\alpha-\delta}{2}$; ſex vero valores quantitatis $\sqrt{s+n^2}$ erunt reſpective $\frac{\alpha\beta-\gamma\delta}{2}$, $\frac{\gamma\delta-\alpha\beta}{2}$, $\frac{\alpha\gamma-\beta\delta}{2}$, $\frac{\beta\delta-\alpha\gamma}{2}$, $\frac{\alpha\delta-\beta\gamma}{2}$, $\frac{\beta\gamma-\alpha\delta}{2}$.

Hæc facile conſtant; ſi modo pro p, q, r & s ſcribantur earum valores, a functionibus radicum α, β, γ, δ deſignati, in reſultante æquatione $x^2 + \left.\begin{matrix}2p \\ -\sqrt{q+p^2+2n}\end{matrix}\right) x + \begin{matrix}n \\ +\sqrt{s+n^2}\end{matrix} = 0$, cujus æquationis radices erunt etiam datæ æquationis radices; erunt enim vel α & β, vel α & γ, vel α & δ, vel β & γ, vel β & δ, vel γ & δ.

Cor. 3. Inventis vero omnibus diverſis valoribus quantitatis $\sqrt{p^2+2n+q}$, ſine ulteriori radicum extractione inveſtigari poſſunt datæ æquationis radices: ſint enim $a = \frac{\alpha+\beta-\gamma-\delta}{2}$, $b = \frac{\alpha+\gamma-\beta-\delta}{2}$, $c = \frac{\alpha+\delta-\beta-\gamma}{2}$; erit $\alpha = \frac{+a+b+c-p}{2}$, $\beta = \frac{+a-b-c-p}{2}$, $\gamma = \frac{+b-a-c-p}{2}$, $\delta = \frac{+c-a-b-p}{2}$: & ſic de cæteris quantitatibus.

Cor. 4. 1. Sit data æquatio $x^4 - px^3 + qx^2 - rx + s = 0$; tum æquatio, cujus radix eſt $z = \alpha+\beta-\gamma-\delta$, $\gamma+\delta-\alpha-\beta$, $\alpha+\gamma-\beta-\delta$, $\beta+\delta-\alpha-\gamma$, $\alpha+\delta-\beta-\gamma$, $\beta+\gamma-\alpha-\delta$; erit $z^6 + \overline{8q-3p^2}\, z^4 + (3p^4 - 16p^2q + 16q^2 - 64s + 16pr)\, z^2 - p^6 - 16p^2q^2 + 8p^4q - 64r^2 - 16p^3r + 64pqr = 0$.

2. Sit data æquatio $x^4 + qx^2 - rx + s = 0$; æquatio, cujus radix eſt $\alpha\beta-\gamma\delta$, $\alpha\gamma-\beta\delta$, $\alpha\delta-\beta\gamma$, $\gamma\delta-\alpha\beta$, $\beta\delta-\alpha\gamma$, $\beta\gamma-\alpha\delta$; erit $x^6 - \overline{q^2-4s}\, x^4 - 2r^2qx^2 - r^4 = 0$.

Cor. 4. Ex hâc regulâ haud inveniri poſſunt radices datæ biquadraticæ; ni duæ ejus radices ſint poſſibiles, duæ vero impoſſibiles: ſi enim nullæ vel quatuor radices datæ æquationis ſint impoſſibiles; tum reducens

ducens æquatio, quæ ſit cubica, tres poſſibiles habet radices; & exinde conſtat nullam eſſe regulam, e quâ generaliter inveniri poſſunt ejus radices.

Cor. 5. Cum ſupponatur in reſolutione problematis $2\times\sqrt{p^2+2n+q}\times\sqrt{s+n^2}=2pn+r$, tum recte concluditur $4\times\overline{p^2+2n+q}\times\overline{s+n^2}=\overline{2pn+r}^2$: ſuppoſito autem $4\times\overline{p^2+2n+q}\times\overline{s+n^2}=\overline{2pn+r}^2$, minime ſequitur omnem valorem rectanguli $2\sqrt{p^2+2n+q}\times\sqrt{s+n^2}$ æqualem eſſe $2pn+r$: quot enim valores rectanguli prædicti ſint $2pn+r$, tot erunt $-2pn-r$; & conſequenter exinde in reſolutionem irrepſerunt e ſingulo valore quantitatis (n) duo valores præter quæſitos in quantitatem $\sqrt{p^2+2n+q}\times\sqrt{s+n^2}$; & exinde quatuor radices in reſolutionem: unde tribus valoribus quantitatis (n) ſubſtitutis in datam reſolutionem irrepſerunt duodecim radices, & reſolutio erit reſolutio æquationis (16) dimenſionum, cujus quatuor radices erunt quatuor datæ æquationis radices.

Ex. 2. Æquationem quadrato-cubicam $(x^5+qx^3+rx^2+sx+t=0)$ in cubicam reducere.

Multiplicetur data æquatio per (x), reſultat $x^6+qx^4+rx^3+sx^2+tx=0$; addatur utrique æquationis parti $\overline{kx^3+dx+b}^2$, & prodit $\overline{kx^3+dx+b}^2=\overline{k^2+1}\,x^6+\overline{2kd+q}\,x^4+\overline{2kb+r}\,x^3+\overline{d^2+s}\,x^2+\overline{2db+t}\,x+b^2$. Supponatur $2\sqrt{k^2+1}\times b=2kb+r$ $(A=0)$; $2\sqrt{k^2+1}\times\sqrt{d^2+s}=2kd+q$ $(B=0)$; $2\sqrt{d^2+s}\times b=2db+t$ $(C=0)$; quibus reductis, reſultant tres æquationes $4b^2=4rkb+r^2$; $4sk^2+4d^2+4s=4qkd+q^2$; $4sb^2=4tdb+t^2$; e primâ æquatione ſequitur $k=\frac{4b^2-r^2}{4rb}$, e tertiâ vero $d=\frac{4sb^2-t^2}{4tb}$; quibus quantitatibus pro ſuis valoribus in ſecundâ æquatione ſubſtitutis, reſultat æquatio biquadratica formulæ vero quadraticæ $4\times\frac{\overline{4sb^2-t^2}^2}{16t^2b^2}+4s\times\frac{\overline{4b^2-r^2}^2}{16r^2b^2}+4s=4q\times\frac{4b^2-r^2}{4rb}\times\frac{4sb^2-t^2}{4tb}+q^2$:

reducatur

reducatur hæc æquatio, & fit $\overline{sr^2+t^2-qtr}\times\overline{16sb^4+4qrtb^2+r^2t^2}=0$.

Cor. 1. Hæc reſolutio ſolummodo inveniet radices datæ æquationis, cum $sr^2+t^2-qtr=0$: ſi vero $sr^2+t^2-qtr=0$, aſſumatur pro literâ (b) quæcunque quantitas, $k=\frac{4b^2-r^2}{4rb}$, & $d=\frac{4sb^2-t^2}{4tb}$; tum data æquatio erit (differentia inter duo quadrata) $\overline{\frac{4b^2+r^2}{4rb}x^3+\frac{4sb^2+t^2}{4tb}x+b}\Big|^2-\overline{\frac{4b^2-r^2}{4rb}x^3+\frac{4sb^2-t^2}{4tb}x+b}\Big|^2=x^6+qx^4+rx^3+sx^2+tx=0$; quæ quidem quadrata infinitis modis variari poſſunt, quoniam pro literâ (b) ſcribatur quæcunque quantitas: e. g. pro $2b$ ſcribatur r; evaneſcet terminus $\frac{4b^2-r^2}{4rb}x^3$, & $\frac{4b^2+r^2}{4rb}$ fiet 1: æquationes autem depreſſæ erunt $\frac{4b^2+r^2}{4rb}x^3+\frac{4sb^2+t^2}{4tb}x+b=\pm\left(\frac{4b^2-r^2}{4rb}x^3+\frac{4sb^2-t^2}{4tb}x+b\right)$; unde $x^3+\frac{rs}{t}x+r=0$ & $x^3+\frac{t}{r}x=0$.

Hinc æquatio $16sb^4+4qrtb^2+r^2t^2=0$ haud generalis erit æquatio, quam exigat problema: quantitas enim (b) ad libitum aſſumi poteſt: etiamque omnis valor quantitatis (b) præbet æquationem $sr^2+t^2-qtr=0$.

Quoniam in prædictis tribus æquationibus $A=0$, $B=0$ & $C=0$ continetur æquatio $sr^2+t^2-qtr=0$, & hæ tres æquationes tres incognitas quantitates k, d & b involvunt; ergo una ex iis ad libitum aſſumi poteſt.

Cor. 2. Æquatio autem $16sb^4+4qrtb^2+r^2t^2=0$ hinc oritur: ſuppoſito $4\overline{k^2+1}\times b^2=\overline{2kb+r}^2$, concluditur $2b\sqrt{k^2+1}=2kb+r$; duo autem ſunt diverſi valores quantitatum $\sqrt{k^2+1}$ & $\sqrt{d^2+s}$, qui bis in prædictis tribus æquationibus occurrunt, & forſan haud

iidem valores harum quantitatum in utriſque caſibus applicati fuere.

Ex. 3. Æquationem cubo-cubicam ($x^6 + qx^4 + rx^3 + sx^2 + tx + v = 0$) in cubicam reducere.

Addatur utrique æquationis parti $\overline{kx^3 + dx + b}^2$, prodit $\overline{kx^3 + dx + b}^2 = \overline{k^2 + 1}\, x^6 + \overline{2kd + q}\, x^4 + \overline{2kb + r}\, x^3 + \overline{d^2 + s}\, x^2 + \overline{2db + t}\, x + b^2 + v$: ſupponantur $2\sqrt{k^2 + 1} \times \sqrt{s + d^2} = q + 2kd$; $2\sqrt{k^2 + 1} \times \sqrt{v + b^2} = r + 2kb$; & $2\sqrt{s + d^2} \times \sqrt{v + b^2} = t + 2db$; quibus reductis, reſultant tres æquationes $4vk^2 + 4b^2 + 4v = 4rkb + r^2$; $4sk^2 + 4d^2 + 4s = 4qkd + q^2$; & $4vs + 4vd^2 + 4sb^2 = 4tdb + t^2$; e primâ æquatione ſequitur $k = \frac{\sqrt{r^2 - 4v} \times \sqrt{v + b^2} + rb}{2v}$, e tertiâ $d = \frac{\sqrt{t^2 - 4vs} \times \sqrt{v + b^2} + tb}{2v}$; quibus in ſecundâ æquatione pro ſuis valoribus ſubſtitutis, reſultat æquatio, quâ reductâ, ita ut extermimetur irrationalis quantitas $\sqrt{v + b^2}$, prodit biquadratica formulæ vero quadraticæ: ſi vero exterminentur omnes irrationales quantitates; prodit æquatio octo dimenſionum, formulæ vero biquadraticæ. Hæ autem æquationes reſultantes multiplicabuntur in $q^2v - qrt + r^2s - 4sv + t^2$.

In his & conſimilibus caſibus, ubi haud datur quantitas b, ni $\mathfrak{Q} = 0$; tum, ſi ex vulgari reductione inveniatur æquatio, cujus radix eſt b; æquatio ſic inventa in æquationem $\mathfrak{Q} = 0$ ſemper ducetur.

Cor. Hæc reſolutio ſolummodo inveniet radices datæ æquationis, cum quantitas $q^2v - qrt + r^2s - 4sv + t^2 = 0$. Hoc enim in caſu trium radicum reſultantis cubicæ æquationis ſumma nihilo erit æqualis, quod in primâ ſubſtitutione revera hujus & præcedentis exempli pro conceſſo aſſumitur: pro literâ b aſſumi poteſt quæcunque quantitas; & exinde conſequuntur valores quantitatum k & d correſpondentes: reſultantis autem æquationis radices, quæ ſint b, ex eodem fonte ac in præcedenti exemplo ſuam originem duxere.

Ex. 4. Sit æquatio

$$x^{2n} + qx^{2n-2} - rx^{2n-3} + sx^{2n-4} - tx^{2n-5} + vx^{2n-6} \&c. = 0,$$

aſſumatur $\overline{x^n + ax^{n-2} + bx^{n-3} + cx^{n-4} + \&c.}^2 = \overline{\alpha x^{n-1} + \beta x^{n-2} + \gamma x^{n-3} + \&c.}^2$ & exinde $x^{2n} + \overline{2a - \alpha^2}\, x^{2n-2} + \overline{2b - 2\alpha\beta}\, x^{2n-3} + \overline{2c + a^2 - 2\alpha\gamma - \beta^2}\, x^{2n-4} + \&c. = 0.$

Fiant correſpondentes hujus & datæ æquationis coefficientes inter ſe æquales, i. e. $2a - \alpha^2 = q$, $2b - 2\alpha\beta = -r$, $2c + a^2 - 2\alpha\gamma - \beta^2 = s$, &c. & reducantur æquationes, ita ut exterminentur omnes præter unam incognitam quantitatem; reſultat æquatio reducens; ſed hæc æquatio reducens plures habebit dimenſiones quam data æquatio: e. g. ſi incognita quantitas, quæ ſit radix reſultantis æquationis, ſit α; tum $2n \times \frac{2n-1}{2} \times \frac{2n-2}{3} \ldots \frac{n+1}{n}$ habet dimenſiones: ſi modo ſit a, eundem habet dimenſionum numerum; & ſic deinceps. Sint enim radices datæ æquationis $\pi, \varrho, \sigma, \tau$, &c. & α erit ſumma e quibuſque n radicibus datæ æquationis; & ſic deinceps.

Ex. 5. Sit data æquatio $x^6 + qx^4 - rx^3 + sx^2 - tx + v = 0$, aſſumatur $x^2 + b = ax + c$; exinde reſultat

$$\begin{aligned} x^6 + 3bx^4 + 3b^2x^2 + b^3 &= a^3x^3 + 3a^2cx^2 + 3ac^2x + c^3 \\ Ax^4 + 2Abx^2 + Ab^2 &= Aa^2x^2 + 2Acx + Ac^2 \\ Bx^2 + Bb &= Bax + Bc. \end{aligned}$$

Fiant coefficientes reſultantis æquationis eædem ac datæ æquationis coefficientes, i. e. $3b + A = q$, & $a^3 = r$, &c. reſultabunt quinque æquationes quinque incognitas quantitates habentes; ſed hæ æquationes ſunt dependentes; ſint enim radices datæ æquationis $\alpha, \beta, \gamma, \delta, \epsilon, \zeta$; erit $a = \alpha + \beta = \sqrt[3]{r}$: in quo caſu hoc exemp. reſolutionem ex hâc methodo accipit.

Si generalis quæratur ſolutio; cavendum eſt, ne ex æquatis datæ & reſultantis æquationis terminis ullæ oriantur æquationes ex aliis pendentes; ſemper quidem tot vel plures eſſe debent incognitæ quantitates, quot reſultantes independentes æquationes.

Si datæ æquationis coefficientes r, s, t, &c. ex terminis radicum $\alpha, \beta,$ $\gamma, \delta,$

γ, δ, &c. exprimantur, & obſerventur etiam aſſumptæ & reducentis æquationis coefficientes; inveniri poſſunt, per functiones radicum datæ æquationis, coefficientes æquationis aſſumptæ.

Hæc methodus igitur plerumque facile converti poteſt in transformationem æquationum.

PROB. XXIII.

Invenire reſolutionem æquationum ex earum transformatione in alias, quarum radices datam habeant relationem ad radices datæ æquationis.

1. Transformentur datæ æquationes in alias, quarum radices per notas regulas inveniri poſſunt; & talis ſit relatio inter radices datæ & transformatæ æquationis, ut e radicibus transformatæ æquationis datis inveniri poſſint radices datæ æquationis.

Ex. 1. Sit æquatio cubica $x^3 + qx - r = 0$, fingatur $z - \frac{q}{3z} = x$; & transformetur cubica æquatio in alteram, cujus radix ſit z; æquatio reſultans erit $z^6 - rz^3 - \frac{q^3}{27} = 0$.

Hujus æquationis radices erunt $\sqrt[3]{\frac{1}{2}r \pm \sqrt{\frac{1}{4}r^2 + \frac{q^3}{27}}}$; & exinde radices datæ æquationis erunt $x = \sqrt[3]{\frac{1}{2}r \pm \sqrt{\frac{q^3}{27} + \frac{r^2}{4}}} +$ $\frac{\frac{1}{3}q}{\sqrt[3]{\frac{1}{2}r \pm \sqrt{\frac{q^3}{27} + \frac{r^2}{4}}}} = \sqrt[3]{\frac{1}{2}r + \sqrt{\frac{q^3}{27} + \frac{r^2}{4}}} + \sqrt[3]{\frac{1}{2}r - \sqrt{\frac{q^3}{27} + \frac{r^2}{4}}}$.

Cor. 1. Sit $a + b\sqrt{-3} = \sqrt[3]{\frac{1}{2}r + \sqrt{\frac{q^3}{27} + \frac{r^2}{4}}}$; tres radices cubicæ æquationis $x^3 + qx - r = 0$ erunt reſpective $a + b\sqrt{-3} + \frac{\frac{1}{3}q}{a + b\sqrt{-3}} = 2a$, $\frac{-1 + \sqrt{-3}}{2} \times \overline{a + b\sqrt{-3}} +$

$$\frac{\frac{1}{3}q}{\frac{-1+\sqrt{-3}}{2}\,\overline{a+b\sqrt{-3}}}\left(\frac{-1-\sqrt{-3}}{2}\,\overline{a-b\sqrt{-3}}\right)=-a-3b,$$

$$\&\ \frac{-1-\sqrt{-3}}{2}\times\overline{a+b\sqrt{-3}}+\frac{\frac{1}{3}q}{\frac{-1-\sqrt{-3}}{2}\,\overline{a+b\sqrt{-3}}}$$

$$\left(\frac{-1+\sqrt{-3}}{2}\times\overline{a-b\sqrt{-3}}\right)=-a+3b.$$

Sit $a+b=\sqrt[3]{\tfrac{1}{2}r+\sqrt{\tfrac{1}{4}r^2+\frac{q^3}{27}}}$, & $a-b=\sqrt[3]{\tfrac{1}{2}r-\sqrt{\tfrac{1}{4}r^2+\frac{q^3}{27}}}$, & tres radices datæ æquationis inveniuntur præcedenti methodo reſpective $2a$, $-a+b\sqrt{-3}$, & $-a-b\sqrt{-3}$.

Cor. 2. Sit æquatio prædicta $x^3+qx-r=0$, cujus radices ſint α, β, γ; & ſex radices æquationis reducentis, cujus radix eſt z, erunt reſpective $\overline{\tfrac{1}{2}\alpha+\beta}\sqrt{-\tfrac{1}{3}}+\tfrac{1}{2}\alpha$, $-\overline{\tfrac{1}{2}\alpha+\beta}\sqrt{-\tfrac{1}{3}}+\tfrac{1}{2}\alpha$; $\overline{\tfrac{1}{2}\beta+\alpha}\sqrt{-\tfrac{1}{3}}+\tfrac{1}{2}\beta$, $-\overline{\tfrac{1}{2}\beta+\alpha}\sqrt{-\tfrac{1}{3}}+\tfrac{1}{2}\beta$; $\overline{\tfrac{1}{2}\alpha-\tfrac{1}{2}\beta}\sqrt{-\tfrac{1}{3}}-\tfrac{1}{2}\alpha-\tfrac{1}{2}\beta$, $\overline{\tfrac{1}{2}\beta-\tfrac{1}{2}\alpha}\sqrt{-\tfrac{1}{3}}-\tfrac{1}{2}\alpha-\tfrac{1}{2}\beta$: ex his radicibus inter ſe comparatis facile conſtat æquationem eſſe hujuſce formulæ $z^6+\pi z^3+\rho=0$.

Cor. 3. Sint $\pi, \rho, \sigma, \tau, \upsilon, \phi$ ſex prædictæ radices æquationis reducentis $z^6-rz^3-\frac{q^3}{27}=0$ reſpective; tres radices datæ æquationis $x^3+qx-r=0$ erunt reſpective $\pi+\rho$, $\sigma+\tau$, $\upsilon+\phi$.

Omnia hæc etiam applicari poſſunt ad æquationem cubicam $x^3-px^2+qx-r=0$.

2. Ita transformare æquationem $x^n-px^{n-1}+qx^{n-2}-rx^{n-3}+$ &c. $=0$ in alteram, ut e reſultanti æquatione deſtruantur (i. e. nihilo fiant æquales) quicunque *(m)* termini.

Aſſumatur æquatio exprimens relationem inter radices datæ & reducentis æquationis, & quæ ſaltem tot involvat incognitas quantitates, quot *(m)* termini æquationis reducentis deſtruendi ſunt; fiant

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m prædicti termini reſultantis æquationis nihilo æquales; reſultant tot æquationes, quot termini deſtruendi ſunt.

Ex his æquationibus capite ſubſequenti inveniri poſſunt dimenſiones æquationis reſultantis; ſi modo ita transformentur hæ (m) æquationes in unam, ut exterminentur $(m-1)$ incognitæ quantitates.

Ex. 1. Sit æquatio $x^n + qx^{n-2} - rx^{n-3} + sx^{n-4} -$ &c. $= 0$, hanc æquationem in alteram n dimenſiones habentem transformare, viz. $v^n ** + Rv^{n-3} +$ &c. $= 0$; cujus duo termini, v^{n-1}, v^{n-2} deficiant.

Aſſumatur quantitas $(x^2 + ax + b = v)$ duas incognitas quantitates $(a$ & $b)$ habens: inveniatur per caput. 1. æquatio, cujus radix eſt v; fiant coefficientes terminorum v^{n-1} & v^{n-2} nihilo æquales; & invenientur $b = \frac{2q}{n}$ & $qa^2 - 3ra - \frac{\overline{n-2} \times q^2}{n} + 2s = 0$: e quibus æquationibus inveniri poſſunt b & a, & exinde per problema quintum æquatio, cujus radix eſt v.

1.2. Sit æquatio $x^3 + qx - r = 0$, aſſumo æquationem $x^2 + ax + \frac{2q}{3} = v$, ubi $qa^2 - 3ra - \frac{q^2}{3} = 0$, & reſultat æquatio $v^3 ** - ra^3 - \frac{2}{3} q^2 a^2 + rqa - r^2 - \frac{2}{27} q^3 = 0$.

Sint α, β, $-\alpha - \beta$ tres radices datæ æquationis; erit $a = \frac{\overline{2\alpha^3 - 2\beta^3 + 3\alpha^2\beta - 3\beta^2\alpha} \times \sqrt{-1} + 3\sqrt{3} \times \alpha\beta \times \overline{\alpha + \beta}}{\overline{2\alpha^2 + \alpha\beta + \beta^2} \times \sqrt{3}}$; quâ pro ſuo valore in æquatione $x^2 + ax + b = v$ ſubſtitutâ, reſultant diverſæ radices quantitatis (v).

Ex. 2. Sit æquatio $x^n + qx^{n-2} - rx^{n-3} + sx^{n-4} - tx^{n-5} + ux^{n-6} -$ &c. $= 0$, transformare hanc æquationem in alteram n dimenſiones habentem $v^n * - Qv^{n-2} * + Sv^{n-4} +$ &c. $= 0$, ubi termini v^{n-1} & v^{n-3} deficiunt.

 Aſſumo

Aſſumo æquationem $(x^2+ax+\frac{2q}{n}=v)$, ubi $ra^3+\overline{2\times\frac{n-2}{n}q^2-4s}\times a^2+\overline{5t-\frac{5n-12}{n}qr}\times a-\frac{\overline{n-4}\times\overline{n-2}}{3.n.n}\times 2q^3+\frac{n-4}{n}\times 2qs+r^2-2u=0$; inveniatur æquatio, cujus radix eſt v, & erit æquatio quæſita.

Sit æquatio $x^4+qx^2-rx+s=0$, & aſſumatur $x^2+ax+\frac{q}{2}=v$, ubi $ra^3+\overline{q^2-4s}a^2-2qra+r^2=0$; inveniatur æquatio, cujus radices ſit v; & æquatio reſultans erit biquadratica, formulæ vero quadraticæ $v^4 * +\overline{qa^2-3ra+2s-\frac{1}{2}q^2}\times v^2 * +sa^4+\frac{qr}{2}a^3+\overline{\frac{q^3}{4}-sq}\times a^2+\overline{sr-\frac{rq^2}{4}}\times a+s^2+\frac{r^2q}{2}-\frac{q^2s}{2}+\frac{q^4}{16}=0$, cujus ope inveniatur v; & conſequenter ex aſſumptâ æquatione $x^2+ax+\frac{q}{2}=v$ inveniri poteſt (x) radix datæ æquationis quæſita.

Quæ autem functiones radicum datæ æquationis ſint quantitates a & v; facile conſtabit pro literis q, r, s ſubſtituendo earum valores in terminis radicum datæ æquationis.

Cor. 5. Transformare per hanc methodum æquationem $x^n-px^{n-1}+qx^{n-2}-rx^{n-3}+sx^{n-4}-tx^{n-5}+$ &c. $=0$, ita ut reſultet æquatio hujuſce formulæ $v^n+Q=0$, exigere poteſt æquationem reducentem $(1\times 2\times 3\times 4\ldots\overline{n-1})$ dimenſionum.

Cor. 6. Sint incognitæ in aſſumptâ quantitate coefficientes plures quam termini tollendi.

Inveniantur termini tollendi, nihilo reſpective fiant æquales, & transformentur reſultantes æquationes in unam, ita ut exterminentur quædam incognitæ coefficientes; reſultat æquatio duas vel plures incognitas quantitates habens; inveniantur ex ſingulis his

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quæcunque radices inter ſe correſpondentes; quibus ſubſtitutis, tolluntur ut oportet termini.

Eadem eſt transformandi methodus, cum requiratur, ut termini reſpectivi reſultantis æquationis quamcunque habeant inter ſe relationem.

Hoc in loco animadvertendum eſt æquationem, cujus radix eſt rationalis functio radicis datæ æquationis, haud plures habere radices, & conſequenter haud aſcendere ad majores dimenſiones, quam eas, quas habet data æquatio.

PROB. XXIV.

Invenire æquationes, quas deprimere licet.

Aſſumatur æquatio (n) dimenſionum, & per problema quintum inveniantur aliæ, quarum radices ſunt quæcunque datæ functiones aſſumptæ æquationis radicum; ita vero ut ex radicibus aſſumptæ æquationis inveniri poſſunt per cognitas methodos radices reſultantium æquationum; æquationes reſultantes, ſi modo plures habeant dimenſiones quam aſſumpta æquatio, deprimi poſſunt in aſſumptam æquationem.

Ex. Aſſumatur æquatio $x^n - px^{n-1} + qx^{n-2} - \&c. = o$, cujus radices ſint $\alpha, \beta, \gamma, \delta$, &c. inveniatur æquatio $v^{n \times \frac{n-1}{2}} - Pv^{n \times \frac{n-1}{2} - 1} + \&c. = o$, cujus radices ſint $\alpha + \beta, \alpha + \gamma, \beta + \gamma, \alpha + \delta, \beta + \delta, \gamma + \delta$, &c. & hæc æquatio $\left(n \times \frac{n-1}{2}\right)$ dimenſionum facile deprimi poteſt in prædictam aſſumptam æquationem (n) dimenſionum.

Æquatio enim reſultans $\left(n \times \frac{n-1}{2}\right)$ dimenſionum habet ſolummodo (n) incognitas quantitates, quæ ſunt coefficientes aſſumptæ æquationis; ex eâ vero reſultant $n \times \frac{n-1}{2}$ æquationes (n) prædictas incognitas quantitates ſolummodo involventes; ergo quoniam plures

reſultant æquationes quam incognitæ quantitates, e methodo inveniendi communes diviſores, conſtabunt prædictæ incognitæ quantitates.

Si vero radices reſultantis æquationis ſint $\alpha+\beta+\gamma$, $\alpha+\beta+\delta$, $\alpha+\gamma+\delta$, $\beta+\gamma+\delta$, &c. vel $\alpha+\beta+\gamma+\delta$, &c. tum reſultantes æquationes $(n \times \frac{n-1}{2} \times \frac{n-2}{3}$; $n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, &c.) dimenſionum facile deprimi poſſunt in æquationes n dimenſionum.

Cor. Datis radicibus datæ æquationis (n) dimenſionum $x^n - px^{n-1} + qx^{n-2} - \&c. = 0$, exinde inveniri poſſunt radices infinitarum aliarum æquationum.

Transformetur enim hæc æquatio in quaſlibet alias, quarum radices ſint quæcunque datæ functiones radicum datæ æquationis, & conſtant radices reſultantium æquationum e radicibus datæ æquationis.

Ex. 1. Sit functio $\frac{ax^m + bx^{m-1} + cx^{m-2} + \&c.}{Ax^r + Bx^{r-1} + Cx^{r-2} + \&c.} = z$ radici reſultantis æquationis; ubi m & r ſint integri numeri; & a, b, c, &c. A, B, C, &c. ſint rationales quantitates; reſultans æquatio haud plures habet dimenſiones quam data æquatio.

Ex.g. Sit æquatio $y^n + b = 0$, aſſumatur $\overline{x+b} \times y = x + a$, exterminetur (y), & reſultat $\overline{1+b} \times x^n + n \times \overline{a+bb}\, x^{n-1} + n \times \frac{n-1}{2} \overline{a^2+bb^2}\, x^{n-2} + n \times \frac{n-1}{2} \times \frac{n-2}{3} \overline{a^3+bb^3}\, x^{n-3} + \&c. + a^n + bb^n = 0$: ſit $bb + a = 0$, & reſultat (A) $x^n - n \times \frac{n-1}{2} ab x^{n-2} - n \cdot \frac{n-1}{a} \cdot \frac{n-2}{3} ab \times \overline{a+b}\, x^{n-3} - n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} ab \times \overline{a^2+ab+b^2}\, x^{n-4} - \&c. = 0$, cujus æquationis radices erunt $\sqrt[n]{a^{n-1}b} + \sqrt[n]{a^{n-2}b^2} + \sqrt[n]{a^{n-3}b^3} \ldots \sqrt[n]{ab^{n-1}}$.

Sit data æquatio $x^n + qx^{n-2} - rx^{n-3} + \&c. = 0$; & tres primi termini

mini datæ & æquationis (A) inter ſe æquales eſſe ſupponantur, reſultat æquatio $a^2 + \frac{3ra}{\overline{n-2}.q} - \frac{2q}{n.\overline{n-1}} = 0$.

2. Aſſumantur æquationes, quarum ſingulæ duæ vel tres, &c. correſpondentes radices datam habeant inter ſe relationem; ex numero correſpondentium hujuſmodi radicum inveniri poſſunt dimenſiones æquationis reducentis; i. e. aſſumantur (m) quadraticæ, cubicæ, &c. æquationes, quarum duæ vel tres, &c. radices datam inter ſe habeant relationem; ducantur hæ quadraticæ, cubicæ, &c. æquationes in ſeſe, & æquatio reſultans deprimi poteſt in quadraticam, cubicam, &c. æquationis m dimenſionum ope.

Ex. Aſſumantur $(2m)$ radices $\alpha, n\alpha; \beta, n\beta; \gamma, n\gamma$; &c.; quæ habent inter ſe relationem $1:n$: i. e. aſſumantur (m) quadraticæ æquationes hujuſce formulæ $x^2 - (n+1)\alpha x + n\alpha^2 = 0$, $x^2 - (n+1)\beta x + n\beta^2 = 0$, $x^2 - (n+1)\gamma x + n\gamma^2 = 0$, &c.; ducantur hæ æquationes in ſeſe; reſultat $x^{2m} - \overline{n+1}\,p x^{2m-1} + (\overline{n^2+1}\,q + np^2) x^{2m-2} - (\overline{n^3+1}\,r + \overline{n^2+n}\,pq) x^{2m-3} + (\overline{n^4+1}\,s + \overline{n^3+n}\,pr + n^2q^2) x^{2m-4} - (\overline{n^5+1}\,t + \overline{n^4+n}\,ps + \overline{n^3+n^2}\,qr) x^{2m-5} + (\overline{n^6+1}\,v + \overline{n^5+n}\,pt + \overline{n^4+n^2}\,qs + n^3r^2) x^{2m-6} - (\overline{n^7+1}\,w + \overline{n^6+n}\,pv + \overline{n^5+n^2}\,qt + \overline{n^4+n^3}\,rs) x^{2m-7} + \&c. = 0$. In hac æquatione continentur (m) diverſæ quantitates p, q, r, s, t, v, w, &c. Hæc æquatio facile deprimi poteſt in duas æquationes $x^m - px^{m-1} + qx^{m-2} - rx^{m-3} + \&c. = 0$, & $x^m - npx^{m-1} + n^2qx^{m-2} - n^3rx^{m-3} + \&c. = 0$; vel in quadraticam $x^2 - \overline{n+1}\,\pi x + n\pi^2 = 0$ æquationis (m) dimenſionum ope.

Et ſimiliter facile dari poſſunt infinita hujuſmodi exempla.

PROB. XXV.

Datâ æquatione $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0$, *& alterâ* $z^m - Pz^{m-1} + Qz^{m-2} - Rz^{m-3} + Sz^{m-4} - \&c. = 0$; *ubi* P, Q, R, S, *&c. ſint quæcunque functiones coefficientium* p, q, r, s, *&c. invenire quæ functio radicis* (x) *ſit quantitas* (z).

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Sint $\alpha, \beta, \gamma, \delta$, &c. radices æquationis $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} -$ &c. $= 0$; pro p, q, r, s, &c. ſcribantur in coefficientibus alterius æquationis earum valores in terminis radicum $\alpha, \beta, \gamma, \delta$, &c. i. e. pro p ejus valor $\alpha + \beta + \gamma + \delta +$ &c., &c. exinde invenitur quæ functiones radicum $\alpha, \beta, \gamma, \delta$, &c. ſint coefficientes P, Q, R, S, &c. inveniatur radix æquationis reſultantis, & fit problema.

Hujus problematis exempla antea tradidi.

PROB. XXVI.

Datâ quantitate utcunque irrationali; invenire alteram, quæ in datam multiplicata, rationalem faciat quantitatem.

Datæ irrationalis quantitatis inveniantur ſinguli diverſi valores, quibus in ſeſe continuo ductis, contentum erit rationalis quantitas; dividatur hoc contentum per datam quantitatem, & quotiens erit irrationalis quantitas quæſita.

Ex. 1. Invenire irrationalem quantitatem, quæ in datam $\sqrt[2]{a+bx} + \sqrt[3]{c+dx+ex^2} + f + gx$ multiplicata, rationalem quantitatem facit.

Diverſi ſex valores datæ irrationalis quantitatis $\sqrt[2]{a+bx} + \sqrt[3]{c+dx+ex^2} + f + gx$ erunt reſpective

$$\sqrt{a+bx} + \sqrt[3]{c+dx+ex^2} + f + gx,$$

$$\& - \sqrt{a+bx} + \sqrt[3]{c+dx+ex^2} + f + gx,$$

$$\& + \sqrt{a+bx} - \frac{1+\sqrt{-3}}{2}\sqrt[3]{c+dx+ex^2} + f + gx,$$

$$\& - \sqrt{a+bx} - \frac{1+\sqrt{-3}}{2}\sqrt[3]{c+dx+ex^2} + f + gx,$$

$$\& + \sqrt{a+bx} - \frac{1-\sqrt{-3}}{2}\sqrt[3]{c+dx+ex^2} + f + gx,$$

$$\& - \sqrt{a+bx} - \frac{1-\sqrt{-3}}{2}\sqrt[3]{c+dx+ex^2} + f + gx;$$ conten-

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tum ex his ſex valoribus continuo in ſeſe ductis erit rationalis quantitas $\overline{f+gx}^6 - 3\overline{f+gx}^4 \times \overline{a+bx} + 2\overline{f+gx}^3 \times \overline{c+dx+ex^2} + 3\overline{f+gx}^2 \times \overline{a+bx}^2 + 6\overline{f\pm gx} \times \overline{a+bx} \times \overline{c+dx+ex^2} + \overline{c+dx+ex^2}^2 - \overline{a+bx}^3$: & contentum ex omnibus his præter primam irrationalem quantitatem (quæ ſit data), erit irrationalis quantitas quæſita.

Cor. 1. Hoc problemate irrationalis quantitas (A) a denominatore in numeratorem, & vice versâ a numeratore in denominatorem transferri poteſt.

Inveniatur rationalis quantitas B, cujus data irrationalis eſt diviſor; ſit $C = \frac{B}{A}$, & pro A ſubſtitatur $\frac{B}{C}$; id, quod requirebatur, perficietur.

Cor. 2. Datâ æquatione irrationales & incognitam quantitatem x involvente; facile inveniri poſſunt dimenſiones æquationis reſultantis; ſi modo ita reducatur æquatio data, ut exterminentur irrationales quantitates.

Ducantur maximæ dimenſiones incognitæ quantitatis (x) in datâ æquatione contentæ in contentum e ſingulis dimenſionibus (quarum radices in datâ æquatione continentur) in ſe invicem multiplicatis, & rectangulum exinde reſultans ſit A.

Ducantur minimæ dimenſiones incognitæ quantitatis (x) in datâ æquatione contentæ in prædictum contentum e ſingulis dimenſionibus (quarum radices in datâ æquatione continentur) in ſe invicem multiplicatis, & dicatur reſultans rectangulum B; dimenſiones quæſitæ erunt $A+B$, ſi modo B ſit negativa quantitas: ſi vero B ſit affirmativa quantitas; $A-B$; erunt enim hoc in caſu valores (B) quantitatis x nihilo æquales.

Cum vero terminorum, in quibus inveniuntur maximæ vel minimæ dimenſiones incognitæ quantitatis (x), quidam valores evaneſcant; tum evaneſcent etiam quædam dimenſiones incognitæ quantitatis (x) in æquatione reſultanti: quot autem evaneſcent dimenſiones incognitæ quantitatis (x) in reſultanti æquatione, e terminis, in

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quibus maximæ vel minimæ dimenſiones incognitæ quantitatis (x) inveniuntur, & eorum proximis, &c. facile detegi poſſunt.

Ex. 1. Sit data æquatio $\sqrt[2]{a+bx+cx^2} + \sqrt[3]{f+gx+hx^2+kx^3} + \sqrt[\frac{5}{2}]{p+qx+rx^2+sx^3} + P + Qx = 0$. In hâc æquatione dimenſio maxima incognitæ quantitatis (x) in primâ irrationali quantitate $\sqrt{a+bx+cx^2}$ erit $\frac{2}{2} = 1$; in ſecundâ irrationali quantitate $\overline{f+gx+hx+kx^3}\,|^{\frac{1}{3}}$ maxima dimenſio, etiam erit $\frac{3}{3} = 1$; in tertiâ $\sqrt[\frac{5}{2}]{p+qx+rx+sx^3}$ erunt ejus dimenſiones $3 \times \frac{2}{5} = \frac{6}{5}$; in rationali quantitate $P + Qx$ dimenſio erit 1; dimenſiones igitur termini, in quo dimenſiones incognitæ quantitatis inveniuntur maximæ datâ æquatione, erunt $\frac{6}{5}$: ducantur hæ dimenſiones $\left(\frac{6}{5}\right)$ in contentum ex omnibus dimenſionibus $(2 \cdot 3 \cdot 5)$, quarum radices continentur in datâ æquatione, erit rectangulum $2 \cdot 3 \cdot 5 \cdot \frac{6}{5} = 36$; ergo dimenſiones incognitæ quantitatis (x) in æquatione reſultanti erunt 36; ni terminorum, in quibus maximæ vel minimæ inveniuntur dimenſiones incognitæ quantitatis (x), quidam valor vel valores evaneſcant; in hoc exemplo unus ſolummodo terminus $\sqrt[\frac{5}{2}]{sx^3 + \&c.}$ involvitur in datâ æquatione, in quo maximæ $\left(\frac{6}{5}\right)$ continentur dimenſiones: ergo nullus datur evaneſcens valor incognitæ quantitatis (x) e terminis, in quibus maximæ continentur dimenſiones: termini autem, in quibus minimæ vel potius nullæ dimenſiones inveniuntur incognitæ quantitatis (x) erunt $\sqrt[2]{a}$, $\sqrt[3]{f}$, $\sqrt[\frac{5}{2}]{p}$, P; ſi vero $+ \sqrt[2]{a} + \sqrt[3]{f} + \sqrt[\frac{5}{2}]{p}$ $+ P$

$+P=0$; tum nihilo erit æqualis valor incognitæ quantitatis (x), ſin aliter vero, non. Si vero evaneſcat unus valor incognitæ quantitatis (x); tum inquirendum eſt, utrum duæ, tres, &c. radices reſultantis æquationis nihilo ſint æquales, necne.

Ex. 2. Sit data æquatio $\sqrt[5]{8x^2+2x+\frac{7}{3x}+\sqrt{64x^4-20}}+\sqrt[6]{10x+\sqrt[2]{64x^4-20}}=0$; invenire, quot dimenſiones habet data æquatio e ſurdis quantitatibus liberata.

In hâc æquatione maximæ dimenſiones incognitæ quantitatis (x) in priori irrationali quantitate erunt $\frac{2}{5}$ vel $\frac{4}{2.5}=\frac{2}{5}$; maximæ vero dimenſiones in poſteriori quantitate erunt $\frac{4}{2.6}=\frac{1}{3}$: ergo dimenſiones termini, in quo dimenſiones incognitæ quantitatis (x) inveniuntur maximæ, erunt $\frac{2}{5}$: ducantur hæ dimenſiones $\frac{2}{5}$ in contentum e ſingulis dimenſionibus, quarum radices in datâ æquatione continentur, & reſultabunt $\frac{2}{5}\times 5\times 6\times 2=24=A$.

Minimæ autem dimenſiones incognitæ quantitatis x in priori irrationali quantitate erunt $-\frac{1}{5}$, in poſteriori vero erunt nihilo æquales; ergo dimenſiones termini, in quo minimæ inveniuntur dimenſiones incognitæ quantitatis erunt $-\frac{1}{5}$, & exinde $-\frac{1}{5}\times 5\times 6\times 2=-12=-B$: & $A+B=24+12=36$. Sed, cum quantitas $\sqrt[5]{8x^2+2x+\&c.+\sqrt{64x^4-20}}$ evadat $\sqrt[5]{8x^2+2x+\&c.-\sqrt[2]{64x^4-20}}$; tum habet eaſdem dimenſiones ac quantitas $\sqrt[5]{2x+\&c.}$; hæc vero quantitas minores habet dimenſiones quam quantitas $\sqrt[6]{10x+\sqrt{64x^4-20}}$;

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præcedentis enim quantitatis dimenſiones erunt $\frac{1}{5}$; ſubſequentis vero dimenſiones erunt $\frac{4}{6\,.\,2} = \frac{1}{3}$: hinc dimenſiones ex radice $+ \sqrt[5]{8+8(16)x^3 - \&c.}$ exortæ erunt $\frac{2}{5} \times 5 \times 6 \times 1 = 12 = a$; ex radice $\sqrt[6]{8x^2 - \&c.}$ erunt $\frac{1}{3} \times 5 \times 6 \times 1 = 10 = b$; & conſequenter dimenſiones æquationis reſultantis erunt $a + b + B = 12 + 10 + 12 = 34$.

Et ſic in aliis caſibus, cum quidam evaneſcant valores terminorum, ratiocinari liceat conſimili methodo e diverſis radicibus terminorum prædictorum ſejunctim conſideratis.

2. Alia etiam deduci poteſt conſimilis methodus transformandi æquationes, ita ut irrationales quantitates exterminentur; etiamque numerum dimenſionum cujuſcunque incognitæ quantitatis *(x)* in æquatione reſultanti deducendi.

Per ſeriem infinitam terminorum juxta dimenſiones incognitæ quantitatis *(x)* progredientium, i. e. vel aſcendentium vel deſcendentium inveniantur ſinguli diverſi valores ex irrationalibus quantitatibus: ducantur hæ quantitates in ſeſe, exinde reſultabit æquatio quæſita, quæ nullas habet irrationales quantitates.

Ex. Sit æquatio $\sqrt{16x^4 + 100} + \sqrt{9x^2 + 36} + 7 + 2x = 0$: quatuor diverſi valores harum irrationalium quantitatum per infinitam ſeriem terminorum juxta dimenſiones incognitæ quantitatis x progredientium, i. e. deſcendentium, invenientur reſpective

$$4x^2 + \overline{2+3}\,x + 7 + \frac{6}{x} \text{ \&c. ad terminum } \frac{\pi}{x^6} \text{ \&c.} = 0,$$

$$4x^2 + \overline{2-3}\,x + 7 - \frac{6}{x} \text{ \&c. ad prædictum terminum, \&c.} = 0,$$

$$-4x^2 + \overline{2+3}\,x + 7 + \frac{6}{x} \text{ \&c.} \qquad = 0,$$

$-4x^2$

$-4x^2+\overline{2-3}\,x+7-\frac{6}{x}$ &c. $=0$;

ducantur hi diverſi valores in ſeſe, reſultabit æquatio quæſita

$$256x^8-416x^6-896x^5 \text{ \&c.} = 0.$$

Facile ex hac reſolutionis methodo conſtant dimenſiones, ad quas aſſurget incognita quantitas (x) in æquatione reſultanti a ſurdis liberâ.

In multiplicatione irrationalium quantitatum, quoniam dantur plures earum valores, facile eſt in errores induci, ni reſpectus habeatur ad diverſas radices; e. g. ducatur $\sqrt{-a^2}$ in ſe $\sqrt{-a^2}$, productum erit $-a^2$; ſed hoc productum etiam erit $\sqrt{-a^2}\times\sqrt{-a^2}=\sqrt{a^4}=\pm a^2$; ſed $+a^2$ in hoc caſu haud erit vera radix.

Eadem etiam applicari poſſunt ad omnes irrationales quantitates.

PROB. XXVII.

Quantitate datâ, quæ ſimplicium terminorum juxta dimenſiones (0, 1, 2, 3, &c.) *cujuſlibet literæ* (x) *progredientium ſeriem* (a + b + c + d + e + f + g + h + k + &c.) *exprimit, invenire quantitatem alternorum ſeriei terminorum* (a + c + e + g + k + &c.) *ſummæ æqualem; etiamque quantitates terminorum ſeriei propoſitæ binis vel ternis vel denique* n *intervallis a ſe invicem diſtantium ſummis reſpective æquales.*

Sint $\alpha, \beta, \gamma, \delta, \epsilon$, &c. reſpective radices $(n+1)$ poteſtatis unitatis; i. e. ſi $n+1=2$, α, β erunt reſpective 1 & -1; ſi $n+1=3$, α, β, γ erunt reſpective $1, \frac{-1+\sqrt{-3}}{2}, \frac{-1-\sqrt{-3}}{2}$.

In datâ quantitate pro literâ (x) ſubſtituantur reſpective $\alpha x, \beta x, \gamma x, \delta x, \epsilon x$, &c. quantitatum reſultantium ſumma per numerum $(n+1)$ diviſa, erit quæſita quantitas.

Ex. 1. Sit quantitas $\overline{a+x}^n$, quæ exprimit ſeriem ſequentem ſimplicium terminorum juxta dimenſiones literæ (x) progredientium

$a^n +$

$a^m + ma^{m-1}x + m \cdot \frac{m-1}{2} a^{m-2}x^2 + m \times \frac{m-1}{2} \times \frac{m-2}{3} a^{m-3}x^3 + \&c.$ invenire quantitatem alternis ſeriei terminis æqualem.

In hoc exemplo $n + 1 = 2$; & quoniam radices quadraticæ unitatis ſunt 1 & -1, pro x in datâ quantitate ſcribantur reſpective $+x$ & $-x$; reſultant duæ quantitates $\overline{a+x}^m$ & $\overline{a-x}^m$; quarum ſumma per numerum $n + 1 = 2$ diviſa, erit quæſita quantitas, i. e. $\frac{\overline{a+x}^m + \overline{a-x}^m}{2} = a^m + m \times \frac{m-1}{2} a^{m-2}x^2 + \&c.$

Ex. 2. Eâdem quantitate datâ; invenire quantitatem primi, quarti, ſeptimi, decimi, &c. terminorum ſummæ æqualem. In hoc exemplo $n + 1 = 3$, & cubicæ radices numeri (1) erunt 1, $\frac{-1+\sqrt{-3}}{2}$ & $\frac{-1-\sqrt{-3}}{2}$: in datâ igitur quantitate pro (x) ſubſtituantur x, $\frac{-1+\sqrt{-3}}{2}x$, & $\frac{-1-\sqrt{-3}}{2}x$ reſpective; reſultant quantitates, quarum ſumma per numerum $(n + 1 = 3)$ diviſa erit quæſita quantitas, i. e. $\frac{\overline{a+x}^m + \overline{a - \frac{1+\sqrt{-3}}{2}x}^m + \overline{a - \frac{1-\sqrt{-3}}{2}x}^m}{3} =$ $a^m + m \times \frac{m-1}{2} \times \frac{m-2}{3} \times a^{m-3}x^3 + \&c.$

Ex. 3. Datâ quantitate $\overline{a + bx + \overline{c + dx + ex^2}^l}^m$ exprimente ſeriem terminorum juxta dimenſiones literæ (x) progredientium; invenire quantitatem alternis hujus ſeriei terminis æqualem. Hoc exemplo $n + 1 = 2$, ergo pro x in datâ quantitate ſcribantur reſpective x & $-x$, & reſultant quantitates $\overline{a + bx + \overline{c + dx + ex^2}^l}^m$, &

& $\overline{a - bx + \overline{c - dx + ex^2}^{\,l}}^{\,m}$; quarum ſumma per numerum $(n + 1 = 2)$ diviſa, erit quæſita quantitas.

Cor. Sint $\alpha, \beta, \gamma, \delta, \epsilon, \zeta,$ &c. reſpective radices æquationis $x^n - 1 = 0$; datâ quantitate (A), quæ ſimplicium terminorum juxta dimenſiones cujuſcunque incognitæ quantitatis (x) progredientium ſummam exprimit: in datâ quantitate A pro incognitâ quantitate (x) ſcribantur reſpective $\alpha x, \beta x, \gamma x, \delta x, \epsilon x, \zeta x,$ &c. & ſint quantitates reſultantes reſpective $A, B, C, D, E, F,$ &c. tum ſumma primi termini, & terminorum, quorum diſtantiæ a primo ſint reſpective $n, 2n, 3n, 4n, 5n,$ &c. erit per problema $\frac{A+B+C+D+E+F+\&c.}{n}$.

Summa autem ſecundi termini, & terminorum, quorum diſtantiæ a ſecundo ſint reſpective $n, 2n, 3n, 4n, 5n,$ &c. erit

$$\frac{\alpha^{n-1}A+\beta^{n-1}B+\gamma^{n-1}C+\delta^{n-1}D+\epsilon^{n-1}E+\zeta^{n-1}F+\&c.}{n}.$$

Summa autem tertii termini, & terminorum, quorum diſtantiæ a tertio ſint reſpective $n, 2n, 3n, 4n, 5n,$ &c. erit

$$\frac{\alpha^{n-2}A+\beta^{n-2}B+\gamma^{n-2}C+\delta^{n-2}D+\epsilon^{n-2}E+\zeta^{n-2}F+\&c.}{n}.$$

Summa denique termini, cujus diſtantia a primo ſit r, & terminorum, quorum diſtantiæ a prædicto termino ſint reſpective $n, 2n, 3n, 4n, 5n,$ &c. erit $\frac{\alpha^{n-r}A+\beta^{n-r}B+\gamma^{n-r}C+\delta^{n-r}D+\epsilon^{n-r}E+\zeta^{n-r}F+\&c.}{n}$.

Ex. 1. Sit data quantitas $\overline{a+x}^{m} = a^m + ma^{m-1}x +$ &c. & quantitas primo, tertio, quinto, ſeptimo, &c. terminis æqualis erit per problema $\frac{\overline{a+x}^{m} + \overline{a-x}^{m}}{2}$; hoc enim in caſu $n = 2$; & exinde radices (α, β) æquationis $x^2 - 1 = 0$, erunt 1 & -1; ſubſtituantur $+x$ & $-x$ pro x, reſultant quantitates A & B reſpective $\overline{a+x}^{m}$ & $\overline{a-x}^{m}$.

O Quan-

Quantitas autem æqualis ſecundo, quarto, ſexto, octavo, &c. terminis erit $\frac{\overline{a+x}^{n}-\overline{a-x}^{n}}{2}$; erit enim hæc ſumma $\frac{\alpha^{2-1}(\alpha)A+\beta^{2-1}(\beta)B}{2}$; ſed α & β ſunt 1 & -1, A & B ſunt $\overline{a+x}^{n}$ & $\overline{a-x}^{n}$; ergo ſumma quæſita erit $\frac{\overline{a+x}^{n}-\overline{a-x}^{n}}{2}$.

Si ſeries terminorum ſecundum dimenſiones quantitatis x progredientium ſit hujuſce formulæ $x^{b}\times\overline{A+Bx+Cx^{2}+\&c.}$ & k ſit reſiduum a diviſione indicis (b) per n; tum erit ſumma termini, cujus diſtantia a primo ſit r, & terminorum, quorum diſtantiæ a prædicto ſint reſpective n, $2n$, $3n$, &c. $=\frac{\alpha^{2n-r-k}A+\beta^{2n-r-k}B+\gamma^{2n-r-k}C+\&c.}{n}$.

In omni ſerie ſecundum dimenſiones quantitatis x progrediente, erit $\frac{\alpha^{n-r}A+\beta^{n-r}B+\gamma^{n-r}C+\&c.}{n}$ æqualis ſummæ omnium terminorum in eâ contentorum, in quibus indices quantitatis x hanc habeant formulam $np+r$, ubi p eſt integer numerus.

In ſingulis quantitatibus A, B, C, &c. detegendis, eædem radices irrationalium quantitatum in datâ quantitate contentarum ſemper aſſumendæ ſunt; e. g. ſit $\sqrt[\lambda]{P}$ irrationalis quantitas prædicta; & pro eâ in quantitate A aſſumatur $\pi\sqrt[\lambda]{P}$, ubi π eſt radix æquationis $x^{\lambda}-1=0$; tum pro $\sqrt[\lambda]{P'}$ in quantitate B etiam aſſumatur $\pi\sqrt[\lambda]{P'}$; & ſic deinceps.

PROB. XXVIII.

Invenire formulas quantitatum, ex quibus in ſe invicem multiplicatis, orientur quæcunque datæ quantitates.

Per

Per problema 26. conſtat, cujus generis vel formulæ ſint irrationales quantitates, quæ in ſeſe multiplicatæ rationale contentum efficiant; & exinde facile conſtabit, cujus generis vel formulæ ſint irrationales quantitates, quæ in ſeſe ductæ irrationalem cujuſcunque generis quantitatem efficiant.

His igitur cognitis, aſſumantur quantitates talis formulæ, qualem exigit data quantitas; ducantur hæ quantitates in ſe invicem, & reſultat quæſitæ formulæ quantitas: æquatis igitur datæ & reſultantis quantitatis correſpondentibus terminis, invenientur æquationes, e quibus erui poſſunt quantitates quæſitæ.

Generaliter ita aſſumantur quantitates, ut tot vel plures ſint incognitæ quantitates, quam æquationes independentes reſultantes ex æquatis terminis.

Ex. 1. Data quantitas $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - tx^{n-5} +$ &c. oriri poteſt e multiplicatione terminorum $x-\alpha$, $x-\beta$, $x-\gamma$, $x-\delta$, $x-\epsilon$, &c. quorum numerus eſt n; hi enim termini in ſe invicem multiplicati producent quantitatem dati generis $x^n - \overline{\alpha+\beta+\gamma+\delta+\&c.}\, x^{n-1} + \overline{\alpha\beta+\alpha\gamma+\beta\gamma+\alpha\delta+\beta\delta+\gamma\delta+\&c.}\, x^{n-2}$ &c. æquatis igitur hujus & datæ quantitatis terminis, reſultabunt (n) æquationes independentes totidem incognitas quantitates habentes; ergo data quantitas $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} +$ &c. oriri poteſt e multiplicatione horum factorum $x-\alpha$, $x-\beta$, $x-\gamma$, $x-\delta$, &c.

Ex. 2. Prædicta quantitas $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} +$ &c. oriri poteſt e multiplicatione factorum $\sqrt[2]{x+A} + x + a$, $-\sqrt[2]{x+A} + x + a$, $\sqrt[2]{x+B} + x + b$, $-\sqrt[2]{x+B} + x + b$, &c. ad n terminos: reſultans enim æquatio erit quæſitæ formulæ, & n erunt æquationes (n) incognitas quantitates habentes.

Cor. Radices æquationum $\pm\sqrt{x+A} + x + a = 0$, $\pm\sqrt{x+B} + x + b = 0$, &c. erunt etiam radices æquationis $x^n - px^{n-1} + qx^{n-2}$ &c. $= 0$.

$a = \frac{1}{\overline{\alpha-\beta}\times\overline{\alpha-\gamma}\times\overline{\alpha-\delta}\times\overline{\alpha-\epsilon}\times \&c.}$, $b = \frac{1}{\overline{\beta-\alpha}\times\overline{\beta-\gamma}\times\overline{\beta-\delta}\times\overline{\beta-\epsilon}\times \&c.}$,

$c = \frac{1}{\overline{\gamma-\alpha}\times\overline{\gamma-\beta}\times\overline{\gamma-\delta}\times\overline{\gamma-\epsilon}\times \&c.}$, $d = \frac{1}{\overline{\delta-\alpha}\times\overline{\delta-\beta}\times\overline{\delta-\gamma}\times\overline{\delta-\epsilon}\times \&c.}$,

$e = \frac{1}{\overline{\epsilon-\alpha}\times\overline{\epsilon-\beta}\times\overline{\epsilon-\gamma}\times\overline{\epsilon-\delta}\times \&c.}$, &c.

Cor. Hinc inveniri poſſunt infinitæ diverſæ coefficientium cujuſcunque æquationis conſtitutiones, quæcunque ſit ejus conſtructio.

PROB. XXIX.

Datâ reſolutione, æquationem invenire.

Prob. 26. vel quinti ope ita reducatur data æquatio, ut ex eâ exterminentur irrationales quantitates, & conſit problema.

Æquatio reſultans ſæpe reduci poteſt in alias minores habentes dimenſiones; hoc fit diviſores æquationis reſultantis inveniendo, ſi nullos recipiat diviſores, tum haud reduci poteſt reſultans æquatio.

Ex. 1. Sit x incognita quantitas, & data reſolutio $x = \sqrt[3]{-\frac{1}{2}b+\sqrt{\frac{1}{4}b^2+\frac{1}{27}a^3}} + \sqrt[3]{-\frac{1}{2}b-\sqrt{\frac{1}{4}b^2+\frac{1}{27}a^3}}$; ita reducatur data æquatio, ut ex eâ exterminentur irrationales quantitates; reſultat æquatio $x^9 + 3bx^6 + \overline{3b^2+a^3}\,x^3 + b^3 = 0$; cujus cubicæ æquationes $x^3 + ax + b = 0$, $x^3 - \frac{1+\sqrt{-3}}{2}ax + b = 0$, $x^3 - \frac{1-\sqrt{-3}}{2}ax + b = 0$ erunt diviſores.

Hinc conſtat, quod reſolutio $\sqrt[3]{-\frac{1}{2}b+\sqrt{\frac{1}{4}b^2+\frac{1}{27}a^3}} +$

 $\sqrt{-\frac{1}{2}b}$

$x^n - n\,\overline{a^2-b}^{\frac{1}{n}} A x^{n-2} + n.\frac{n-3}{2}\,\overline{a^2-b}^{\frac{2}{n}} A^2 x^{n-4} - n.\frac{n-4}{2}.\frac{n-5}{3}$ $\overline{a^2-b}^{\frac{3}{n}}\times A^3 x^{n-6} + \&c. - \overline{A^n+1}\,a + \overline{A^n-1}\sqrt{b} = 0$ habet alteram æquationem ei correſpondentem $x^n - n\,\overline{a^2-b}^{\frac{1}{n}} A x^{n-2} + n.\frac{n-3}{2}$ $\overline{a^2-b}^{\frac{2}{n}} A^2 x^{n-4} - \&c. - \overline{A^n+1}\,a - \overline{A^n-1}\sqrt{b} = 0$; & erit reſolutio æquationis $2n\times n$ dimenſionum.

Ex. 4. Sit reſolutio $x = \sqrt[n]{a+\sqrt{b}} + \frac{A}{\sqrt[n]{a+\sqrt{b}}}$; exterminentur irrationales quantitates, & reſultat æquatio

$$\overline{x^n - n.Ax^{n-2} + n\times\frac{n-3}{2}A^2x^{n-4} - n\times\frac{n-4}{2}\times\frac{n-5}{3}A^3x^{n-6} + \&c.}^{\,2} - \frac{\overline{A^n+a^2-b}^{\,2}}{a^2-b}\times a - b\times\frac{\overline{A^n-a^2+b}^{\,2}}{\overline{a^2-b}^{\,2}} = 0.$$

Cor. Sit $x^n - 1 = 0$, cujus quadratici diviſores ſint $z^2 - \pi z + 1 = 0$, $z^2 - \rho z + 1 = 0$, $z^2 - \sigma z + 1 = 0$, $z^2 - \tau z + 1 = 0$, &c. quarum æquationum ſint duæ radices reſpective α, β; & γ, δ; & ϵ, ζ; & η, θ; &c. i. e. $\beta = \frac{1}{\alpha}$, $\delta = \frac{1}{\gamma}$, $\epsilon = \frac{1}{\zeta}$, $\theta = \frac{1}{\eta}$, &c. & quoniam radices e reſolutione $x = \sqrt[n]{a+\sqrt{b}} + \frac{A}{\sqrt[n]{a+\sqrt{b}}}$ erunt $\sqrt[n]{a+\sqrt{b}} + \frac{A}{\sqrt[n]{a+\sqrt{b}}}$, $\alpha\sqrt[n]{a+\sqrt{b}} + \frac{A}{\alpha\sqrt[n]{a+\sqrt{b}}}$, $\beta\sqrt[n]{a+\sqrt{b}} + \frac{A}{\beta\sqrt[n]{a+\sqrt{b}}}$, $\gamma\sqrt[n]{a+\sqrt{b}} + \frac{A}{\gamma\sqrt[n]{a+\sqrt{b}}}$; &c. correſpondentes radices e reſolutione $\sqrt[n]{a+\sqrt{b}} + A\sqrt[n]{a-\sqrt{b}}$ erunt $\sqrt[n]{a+\sqrt{b}}$

$+A$

$+A\sqrt[n]{a-\sqrt{b}}$, $\alpha\sqrt[n]{a+\sqrt{b}}+\beta A\sqrt[n]{a-\sqrt{b}}$, $\gamma\sqrt[n]{a+\sqrt{b}}+\delta A\sqrt[n]{a-\sqrt{b}}$, $\varepsilon\sqrt[n]{a+\sqrt{b}}+\zeta A\sqrt[n]{a-\sqrt{b}}$, &c.

Ex. 5. Sit reſolutio $x=\left(a+\left(b+\left(c+\left(d+e^{\frac{1}{m}}\right)^{\frac{1}{n}}\right)^{\frac{1}{p}}\right)^{\frac{1}{q}}\right)^{\frac{1}{r}}+f$; ita tranſformetur hæc æquatio, ut exterminentur irrationales quantitates, reſultat æquatio $\left(\left(\left(\left(\left(x-f\right)^{r}-a\right)^{q}-b\right)^{p}-c\right)^{n}-d\right)^{m}-e=0$.

Et ſic de pluribus hujuſcemodi exemplis.

Datâ in genere *(n)* dimenſionum æquatione $x^{n}-px^{n-1}+qx^{n-2}-rx^{n-3}+$ &c. $=0$; ejus reſolutio erit reſolutio (ut conſtat e præcedentibus) æquationis, cujus dimenſiones ſunt multo majores quam dimenſiones datæ æquationis.

2. Subſequens methodus reſolutionem æquationum præbebit.

Aſſumatur reſolutio, plures, quam datæ æquationis dimenſiones *(n)*, incognitas quantitates habens; transformetur hæc æquatio, ita ut exterminentur irrationales quantitates; reſultet æquatio, cujus data æquatio ſit diviſor; unde erui poſſunt *(n)* æquationes plures quam *(n)* incognitas quantitates habentes, quibus reductis reſultat æquatio duas vel plures incognitas quantitates habens.

Incognitarum reſultantis æquationis quantitatum inveniantur rationales radices, quibus pro ſuis valoribus ſubſtitutis, reſultant incognitæ quantitates quæſitæ.

E præcedentibus exemplis, &c. conjicere liceat juſtam reſolutionem, quam exigat quæcunque data æquatio; ſed in æquationibus quinque & ſuperiorum dimenſionum calculus requiritur fere infinite laborioſus.

LEMMA.

LEMMA.

1. Sint $\alpha, \beta, \gamma, \delta, \varepsilon$, &c. radices æquationis $z^n - 1 = 0$; tum erit $\alpha^m + \beta^m + \gamma^m + \delta^m +$ &c. $= 0$, ni $n = m$ vel n ſit diviſor quantitatis m, in quo caſu $\alpha^m + \beta^m + \gamma^m + \delta^m +$ &c. $= n$.

2. Summa omnium rectangulorum ſubſequentis generis $\alpha^m\beta^r + \beta^m\alpha^r + \alpha^m\gamma^r + \gamma^m\alpha^r + \beta^m\gamma^r + \gamma^m\beta^r +$ &c. $= 0$: ni $n = m + r$ vel n ſit diviſor quantitatis $m + r$, in quo caſu $\alpha^m\beta^r + \beta^m\alpha^r +$ &c. $= -n$; ni $n = m$ vel diviſor ſit quantitatis m, tum erit $\alpha^m\beta^r + \beta^m\alpha^r + \alpha^m\gamma^r +$ &c. $= n^2 - n$; ni etiam $m = r$ tum erit $\alpha^m\beta^r + \beta^m\alpha^r +$ &c. $= \frac{n^2 - n}{2}$.

3. 1. Summa omnium quantitatum hujuſce generis $\alpha^m\beta^r\gamma^s\delta^t$ &c. $+ \alpha^r\beta^m\gamma^s\delta^t$ &c. $+ \alpha^m\beta^r\gamma^t\delta^s$ &c. $+$ &c. $= 0$, ni $n = m + r + s + t +$ &c. vel n ſit diviſor ſummæ $m + r + s + t +$ &c.

3. 2. Sit π numerus indicum m, r, s, t, &c.; & n vel $= m + r + s + t +$ &c. vel ejus diviſor; pro $1 . 2 . 3 . 4 \ldots \overline{\pi - 2} \times \overline{\pi - 1} \times n$ ſcribatur A.

3. 3. Sit n vel æqualis vel diviſor ſummæ cujuſcunque numeri (ρ) minoris quam π prædictarum quantitatum m, r, s, t, &c.; & conſequenter vel æqualis vel diviſor ſummæ reliquarum ($\pi - \rho$) quantitatum: inveniatur aggregatum e ſingulis quantitatibus hujuſce generis $1 . 2 . 3 \ldots \overline{\rho - 2} . \overline{\rho - 1} \times 1 . 2 . 3 \ldots \overline{\pi - \rho - 2} \times \overline{\pi - \rho - 1} \times n^2$ & aggregatum inventum dicatur B.

3. 4. Sit n vel æqualis vel diviſor ſummæ cujuſcunque numeri (σ) prædictarum quantitatum m, r, s, t, &c.; etiamque vel æqualis vel diviſor ſummæ ex quocunque numero (ρ') reliquarum quantitatum; & conſequenter vel æqualis vel diviſor erit ſummæ ex omnibus ($\pi - \sigma - \rho'$) reliquis quantitatibus; tum inveniatur aggregatum e ſingulis quantitatibus hujuſce generis $1 . 2 . 3 \ldots \overline{\sigma - 2} \times \overline{\sigma - 1} \times 1 . 2 . 3 \ldots \overline{\rho' - 2} \times \overline{\rho' - 1} \times 1 . 2 . 3 \ldots \overline{\pi - \sigma - \rho' - 2} \times \overline{\pi - \sigma - \rho' - 1} \times n^3$, quod dicatur C.

3. 5. Eodem

3. 5. Eodem modo ſit n vel æqualis vel diviſor ſummæ cujuſcunque numeri (τ) prædictarum quantitatum m, r, s, t, &c.; & ſimiliter ſit n vel æqualis vel diviſor ſummæ numeri cujuſcunque (σ') reliquarum quantitatum; etiamque ſit n æqualis vel diviſor ſummæ (ϱ'') reliquarum quantitatum; tum erit n æqualis vel diviſor ſummæ $(\pi - \tau - \sigma' - \varrho'')$ reliquarum quantitatum: inveniatur agregatum e ſingulis quantitatibus hujuſce generis $1 . 2 . 3 .. \overline{\tau - 2} \times \overline{\tau - 1} \times 1 . 2 . 3 .. \overline{\sigma' - 2} \times \overline{\sigma' - 1} \times 1 . 2 . 3 .. \overline{\varrho'' - 2} \times \overline{\varrho'' - 1} \times 1 . 2 . 3 .. \overline{\pi - \tau - \sigma' - \varrho'' - 2} \times \overline{\pi - \tau - \sigma' - \varrho'' - 1} \times n^4$, quod dicatur D: & ſic deinceps.

Erit ſumma $\alpha^m \beta^r \gamma^s \delta^t$ &c. $+ \alpha^r \beta^m \gamma^s \delta^t$ &c. $+ \alpha^m \beta^r \gamma^t \delta^s$ &c. $+ \alpha^r \beta^m \gamma^t \delta^s$ &c. $+$ &c. $= \pm (A - B + C - D +$ &c.$)$; ſignum affixum erit $+$, ſi π ſit impar numerus; ſin aliter $-$.

In hoc caſu ſint (a) indices $= m$, (b) indices $= r$, (c) indices $= s$, (d) indices $= t$, &c.; tum erit ſumma prædicta $=$

$$\frac{\pm (A - B + C - D + \text{\&c.})}{1 . 2 .. a \times 1 . 2 .. b \times 1 . 2 .. c \times 1 . 2 .. d \times \text{\&c.}}.$$

THEOR. XX.

1. Sit $x = a\sqrt[3]{p} + b\sqrt[3]{p^2}$, exterminentur irrationales quantitates; reſultabit $x^3 - 3abpx - a^3p - b^3p^2 = 0$. Fiat hæc æquatio eadem ac æquatio $x^3 + qx - r = 0$, & evadet $3abp = -q$, $a^3p + b^3p^2 = r$; unde $p = \frac{-q}{3ab}$, & exinde $r = \frac{-a^2q}{3b} + \frac{bq^2}{9a^2}$; vel $b = \frac{-q}{3ap}$ & exinde $r = a^3p - \frac{q^3}{27a^3p}$; vel $a = \frac{-q}{3bp}$ & exinde $r = \frac{-q^3}{27b^3p^2} + b^3p^2$: pro unâ literâ vel a vel b vel p aſſumatur quæcunque quantitas, ſubſtituatur quantitas aſſumpta pro ejus valore in unâ e tribus præcedentibus æquationibus; reſultabunt æquationes vel quadraticæ, vel quarum formula eſt quadratica.

2. Sit reſolutio $x = a\sqrt[4]{p} + b\sqrt[4]{p^2} + c\sqrt[4]{p^3}$, ita reducatur hæc

 æquatio,

æquatio, ut exterminentur irrationales quantitates, reſultabit æquatio $x^4 - (2b^2 + 4ac)px^2 - 4(a^2bp + bc^2p^2)x - a^4p + b^4p^2 - c^4p^3 + 2a^2c^2p^2 - 4ab^2cp^2 = 0$. Fiat hæc æquatio eadem ac biquadratica $x^4 + qx^2 - rx + s = 0$, & aſſumatur $p = 1$; tum reſultabunt $2b^2 + 4ac = -q$, $4a^2b + 4bc^2 = r$, $a^4 + c^4 - 2a^2c^2 + 4ab^2c - b^4 = \overline{a^2 - c^2}^2 + 4ac \times b^2 - b^4 = \overline{a^2 + c^2}^2 - 4a^2c^2 + 4ac \times b^2 - b^4 = -s$; unde e ſecundâ æquatione $a^2 + c^2 = \frac{r}{4b}$ & e primâ $4ac = -q - 2b^2$; & exinde $\overline{a^2 + c^2}^2 - 4a^2c^2 + 4ac \times b^2 - b^4 = \frac{r^2}{16b^2} - \frac{1}{4}\overline{q + 2b^2}^2 - b^2q - 2b^4 - b^4 = -s$, & conſequenter $4b^6 + 2qb^4 + (\frac{1}{4}q^2 - s)b^2 - \frac{r^2}{16} = 0$ æquatio formulæ cubicæ; ex quâ inveniatur quantitas b, & exinde erui poſſunt quantitates a & c, & conſequenter reſolutio biquadraticæ quæſitæ.

Ex iiſdem principiis deduci poſſunt diverſæ reſolutiones biquadraticæ prædictæ $x^4 + qx^2 - rx + s = 0$; inveniri enim poſſunt æquationes, quarum radices ſunt vel a vel c vel p, &c.

3. 1. Sit $x = a\sqrt[2n]{p} + b\sqrt[2n]{p^2}$, ita reducatur hæc æquatio, ut exterminentur irrationales, reſultabit æquatio $x^{2n} - 2b^n.px^n - \frac{n}{1.2}.2nb^{n-1}a^2px^{n-1} - \frac{n.\overline{n^2-1}}{1.2.3.4} \times 2nb^{n-2}a^4px^{n-2} - \frac{n \times \overline{n^2-1} \times \overline{n^2-4}}{1.2.3.4.5.6} \times 2nb^{n-3}a^6px^{n-3} - \frac{n.\overline{n^2-1} \times \overline{n^2-4} \times \overline{n^2-9}}{1.2.3.4.5.6.7.8} \times 2nb^{n-4}a^8px^{n-4} \ldots - \frac{n.\overline{n^2-1}.\overline{n^2-4}.\overline{n^2-9}.\overline{n^2-16}\ldots\overline{n^2-\overline{n-2}^2}}{1.2.3.4\ldots\overline{2n-3}.\overline{2n-2}} \times 2na^{2n-2}bpx = a^{2n}p - b^{2n}p^2$.

3. 2. Sit æquatio $x = a\sqrt[2n+1]{p} + b\sqrt[2n+1]{p^2}$, ita reducatur hæc æquatio, ut

ut exterminentur irrationales, & reſultabit $x^{2n+1} - \overline{2n+1}\, b^n a p x^n - \frac{n.\overline{n+1}}{1.2.3} \times \overline{2n+1}\, b^{n-1} a^3 p x^{n-1} - \frac{n\times \overline{n^2-1}\times\overline{n+2}}{1.2.3.4.5}\times \overline{2n+1}\, b^{n-2} a^5 p x^{n-2} - \frac{n\times\overline{n^2-1}\times\overline{n^2-4}\times\overline{n+3}}{1.2.3.4.5.6.7}\times\overline{2n+1}\, b^{n-3}a^7 p x^{n-3} - \frac{n.\overline{n^2-1}.\overline{n^2-4}.\overline{n^2-9}\times\overline{n+4}}{1.2.3.4\ldots8.9}\times\overline{2n+1}\, b^{n-4}a^9 p x^{n-4}\ldots\ldots - \frac{n\times\overline{n^2-1}\times\overline{n^2-4}\times\overline{n^2-9}\times\overline{n^2-16}\times\ldots\overline{n^2-\overline{n-2}^2}\times\overline{2n-1}}{1.2.3.4.5.6.7.8\ldots.\overline{2n-1}}\times\overline{2n+1}\, b a^{2n-1} p x = a^{2n+1}p + b^{2n+1}p^2.$

3. 3. Sit $x = a\sqrt[m]{p} + b\sqrt[m]{p^2}$; tum æquatio ab irrationalibus quantitatibus libera, erit $x^m - m a^{m-2} b p x - m.\frac{m-3}{2} a^{m-4} b^2 p x^2 - m.\frac{m-4}{2}.\frac{m-5}{3}\times a^{m-6} b^3 p x^3 - m.\frac{m-5}{2}.\frac{m-6}{3}.\frac{m-7}{4} a^{m-8} b^4 p x^4 - m.\frac{m-6}{2}.\frac{m-7}{3}.\frac{m-8}{4}.\frac{m-9}{5} a^{m-10} b^5 p x^5 - \&c. = a^m p \pm b^m p^2$; erit $+ b^m p^2$, ſi m ſit impar numerus; ſin aliter, $-b^m p^2$.

4. 1. Sit n impar numerus, & $x = a\sqrt[n]{p} + b\sqrt[n]{p^3}$; tum erit $x^n - p\,(n a^{n-3} b x^2 + n.\frac{n-5}{2} a^{n-6} b^2 x^4 + n.\frac{n-7}{2}.\frac{n-8}{3} a^{n-9} b^3 x^6 + n.\frac{n-9}{2}.\frac{n-10}{3}.\frac{n-11}{4}\times a^{n-12} b^4 x^8 + n.\frac{n-11}{2}.\frac{n-12}{3}.\frac{n-13}{4}.\frac{n-14}{5} a^{n-15} b^5 x^{10} + n.\frac{n-13}{2}.\frac{n-14}{3}.\frac{n-15}{4}.\frac{n-16}{5}.\frac{n-17}{6} a^{n-18} b^6 x^{12} + \&c.) \pm p^2\,(n a^{\frac{n-3}{2}} b^{\frac{n+1}{2}} x - \frac{1}{2^2}\times n.\frac{n-5}{2}.\frac{n-7}{3} a^{\frac{n-9}{2}} b^{\frac{n+3}{2}} x^3 + \frac{1}{2^4}\times n.\frac{n-7}{2}.\frac{n-9}{3}.\frac{n-11}{4}.\frac{n-13}{5} a^{\frac{n-15}{2}} b^{\frac{n+5}{2}} x^5 - \frac{1}{2^6}\times n\times\frac{n-9}{2}\times\frac{n-11}{3}\times\frac{n-13}{4}\times$

$n-15$

$\frac{n-15}{5} \times \frac{n-17}{6} \times \frac{n-19}{7} a^{\frac{n-21}{2}} b^{\frac{n+7}{2}} x^7 + \frac{1}{2^8} \times n \times \frac{n-11}{2} \times \frac{n-13}{3} \times \frac{n-15}{4} \times \frac{n-17}{5} \times \frac{n-19}{6} \times \frac{n-21}{7} \times \frac{n-23}{8} \times \frac{n-25}{9} a^{\frac{n-27}{2}} b^{\frac{n+9}{2}} x^9 - \&c.) = a^n p + b^n p^3.$

Quantitas $\pm p^2$ denotat $+p^2$, ſi $\frac{n-3}{4}$ ſit integer numerus; ſin aliter $-p^2$.

4. 2. Sit n par numerus & $x = a\sqrt[n]{p} + b\sqrt[n]{p^3}$; tum erit æquatio ab irrationalibus quantitatibus liberata $x^n - p\,(na^{n-3}bx^2 + n \, . \, \frac{n-5}{2} a^{n-6} b^2 x^4 + n \, . \, \frac{n-7}{2} \times \frac{n-8}{3} a^{n-9} b^3 x^6 + n \, . \, \frac{n-9}{2} \times \frac{n-10}{3} \times \frac{n-11}{4} \times a^{n-12} b^4 x^8 + n \, . \, \frac{n-11}{2} \, . \, \frac{n-12}{3} \, . \, \frac{n-13}{4} \, . \, \frac{n-14}{5} \times a^{n-15} b^5 x^{10} + \&c.)$ $\pm p^2 \, (\tfrac{1}{2} n \times \frac{n-4}{2} a^{\frac{n-6}{2}} b^{\frac{n+2}{2}} x^2 - \frac{1}{2^3} \times n \times \frac{n-6}{2} \times \frac{n-8}{3} \times \frac{n-10}{4} a^{\frac{n-12}{2}} b^{\frac{n+4}{2}} x^4 + \frac{1}{2^5} \times n \, . \, \frac{n-8}{2} \, . \, \frac{n-10}{3} \, . \, \frac{n-12}{4} \, . \, \frac{n-14}{5} \, . \, \frac{n-16}{6} a^{\frac{n-18}{2}} b^{\frac{n+6}{2}} x^6 - \frac{1}{2^7} \times n \, . \, \frac{n-10}{2} \, . \, \frac{n-12}{3} \, . \, \frac{n-14}{4} \, . \, \frac{n-16}{5} \, . \, \frac{n-18}{6} \, . \, \frac{n-20}{7} \, . \, \frac{n-22}{8} a^{\frac{n-24}{2}} b^{\frac{n+8}{2}} x^8 + \&c.) = a^n p + b^n p^3 \pm 2a^{\frac{n}{2}} b^{\frac{n}{2}} p^2.$

Quantitates $\pm p^2$ & $\pm 2a^{\frac{n}{2}} b^{\frac{n}{2}} p^2$ denotant $+p^2$ & $+ 2a^{\frac{n}{2}} b^{\frac{n}{2}} p^2$, ſi $\frac{n-2}{4}$ ſit integer numerus; ſin aliter $-p^2$ & $- 2a^{\frac{n}{2}} b^{\frac{n}{2}} p^2$ reſpective. In his ſeriebus haud continetur negativa poteſtas quantitatis a.

5. 1. Sit $x = a\sqrt[n]{p} + b\sqrt[n]{p^{n-1}}$, & n impar numerus; tum erit $x^n - nabpx^{n-2} + n \, . \, \frac{n-3}{2} a^2 b^2 p^2 x^{n-4} - n \, . \, \frac{n-4}{2} \, . \, \frac{n-5}{3} a^3 b^3 p^3 x^{n-6} + n \, . \, \frac{n-5}{2} \, . \, \frac{n-6}{3} \, . \, \frac{n-7}{4} a^4 b^4 p^4 x^{n-8} - \&c. = a^n p + b^n p^{n-1}.$

ut exterminentur irrationales, & reſultabit $x^{2n+1} - \overline{2n+1}\, b^n a p x^n - \frac{n.\overline{n+1}}{1.2.3} \times \overline{2n+1}\, b^{n-1} a^3 p x^{n-1} - \frac{n \times \overline{n^2-1} \times \overline{n+2}}{1.2.3.4.5} \times \overline{2n+1}\, b^{n-2} a^5 p x^{n-2} - \frac{n\times\overline{n^2-1}\times\overline{n^2-4}\times\overline{n+3}}{1.2.3.4.5.6.7}\times\overline{2n+1}\,b^{n-3}a^7px^{n-3} - \frac{n.\overline{n^2-1}.\overline{n^2-4}.\overline{n^2-9}\times\overline{n+4}}{1.2.3.4\ldots8.9}\times\overline{2n+1}\,b^{n-4}a^9px^{n-4}\ldots\ldots - \frac{n\times\overline{n^2-1}\times\overline{n^2-4}\times\overline{n^2-9}\times\overline{n^2-16}\times\ldots\overline{n^2-\overline{n-2}^2}\times\overline{2n-1}}{1.2.3.4.5.6.7.8\ldots.\overline{2n-1}}\times\overline{2n+1}\,b a^{2n-1}px = a^{2n+1}p + b^{2n+1}p^2.$

3. 3. Sit $x = a\sqrt[m]{p} + b\sqrt[m]{p^2}$; tum æquatio ab irrationalibus quantitatibus libera, erit $x^m - ma^{m-2}bpx - m.\frac{m-3}{2}a^{m-4}b^2px^2 - m.\frac{m-4}{2}.\frac{m-5}{3}\times a^{m-6}b^3px^3 - m.\frac{m-5}{2}.\frac{m-6}{3}.\frac{m-7}{4}a^{m-8}b^4px^4 - m.\frac{m-6}{2}.\frac{m-7}{3}.\frac{m-8}{4}.\frac{m-9}{5}a^{m-10}b^5px^5 - \&c. = a^mp \pm b^mp^2$; erit $+ b^mp^2$, ſi m ſit impar numerus; ſin aliter, $-b^mp^2$.

4. 1. Sit n impar numerus, & $x = a\sqrt[n]{p} + b\sqrt[n]{p^3}$; tum erit $x^n - p\,(na^{n-3}bx^2 + n.\frac{n-5}{2}a^{n-6}b^2x^4 + n.\frac{n-7}{2}.\frac{n-8}{3}a^{n-9}b^3x^6 + n.\frac{n-9}{2}.\frac{n-10}{3}.\frac{n-11}{4}\times a^{n-12}b^4x^8 + n.\frac{n-11}{2}.\frac{n-12}{3}.\frac{n-13}{4}.\frac{n-14}{5}a^{n-15}b^5x^{10} + n.\frac{n-13}{2}.\frac{n-14}{3}.\frac{n-15}{4}.\frac{n-16}{5}.\frac{n-17}{6}a^{n-18}b^6x^{12} + \&c.) \pm p^2\,(na^{\frac{n-3}{2}}b^{\frac{n+1}{2}}x - \frac{1}{2^2}\times n.\frac{n-5}{2}.\frac{n-7}{3}a^{\frac{n-9}{2}}b^{\frac{n+3}{2}}x^3 + \frac{1}{2^4}\times n.\frac{n-7}{2}.\frac{n-9}{3}.\frac{n-11}{4}.\frac{n-13}{5}a^{\frac{n-15}{2}}b^{\frac{n+5}{2}}x^5 - \frac{1}{2^6}\times n\times\frac{n-9}{2}\times\frac{n-11}{3}\times\frac{n-13}{4}\times$

$n-15$

$\frac{n-15}{5} \times \frac{n-17}{6} \times \frac{n-19}{7} a^{\frac{n-21}{2}} b^{\frac{n+7}{2}} x^7 + \frac{1}{2^8} \times n \times \frac{n-11}{2} \times \frac{n-13}{3} \times \frac{n-15}{4} \times \frac{n-17}{5} \times \frac{n-19}{6} \times \frac{n-21}{7} \times \frac{n-23}{8} \times \frac{n-25}{9} a^{\frac{n-27}{2}} b^{\frac{n+9}{2}} x^9 - \&c.) = a^n p + b^n p^3$.

Quantitas $\pm p^2$ denotat $+p^2$, ſi $\frac{n-3}{4}$ ſit integer numerus; ſin aliter $-p^2$.

4. 2. Sit n par numerus & $x = a\sqrt[n]{p} + b\sqrt[n]{p^3}$; tum erit æquatio ab irrationalibus quantitatibus liberata $x^n - p\,(n a^{n-3} b x^2 + n . \frac{n-5}{2} a^{n-6} b^2 x^4 + n . \frac{n-7}{2} \times \frac{n-8}{3} a^{n-9} b^3 x^6 + n . \frac{n-9}{2} \times \frac{n-10}{3} \times \frac{n-11}{4} \times a^{n-12} b^4 x^8 + n . \frac{n-11}{2} . \frac{n-12}{3} . \frac{n-13}{4} . \frac{n-14}{5} \times a^{n-15} b^5 x^{10} + \&c.) \pm p^2\,(\frac{1}{2} n \times \frac{n-4}{2} a^{\frac{n-6}{2}} b^{\frac{n+2}{2}} x^2 - \frac{1}{2^3} \times n \times \frac{n-6}{2} \times \frac{n-8}{3} \times \frac{n-10}{4} a^{\frac{n-12}{2}} b^{\frac{n+4}{2}} x^4 + \frac{1}{2^5} \times n . \frac{n-8}{2} . \frac{n-10}{3} . \frac{n-12}{4} . \frac{n-14}{5} . \frac{n-16}{6} a^{\frac{n-18}{2}} b^{\frac{n+6}{2}} x^6 - \frac{1}{2^7} \times n . \frac{n-10}{2} . \frac{n-12}{3} . \frac{n-14}{4} . \frac{n-16}{5} . \frac{n-18}{6} . \frac{n-20}{7} . \frac{n-22}{8} a^{\frac{n-24}{2}} b^{\frac{n+8}{2}} x^8 + \&c.) = a^n p + b^n p^3 \pm 2 a^{\frac{n}{2}} b^{\frac{n}{2}} p^2$.

Quantitates $\pm p^2$ & $\pm 2 a^{\frac{n}{2}} b^{\frac{n}{2}} p^2$ denotant $+p^2$ & $+ 2 a^{\frac{n}{2}} b^{\frac{n}{2}} p^2$, ſi $\frac{n-2}{4}$ ſit integer numerus; ſin aliter $-p^2$ & $-2 a^{\frac{n}{2}} b^{\frac{n}{2}} p^2$ reſpective.

In his ſeriebus haud continetur negativa poteſtas quantitatis a.

5. 1. Sit $x = a\sqrt[n]{p} + b\sqrt[n]{p^{n-1}}$, & n impar numerus; tum erit $x^n - n a b p x^{n-2} + n . \frac{n-3}{2} a^2 b^2 p^2 x^{n-4} - n . \frac{n-4}{2} . \frac{n-5}{3} a^3 b^3 p^3 x^{n-6} + n . \frac{n-5}{2} . \frac{n-6}{3} . \frac{n-7}{4} a^4 b^4 p^4 x^{n-8} - \&c. = a^n p + b^n p^{n-1}$.

5. 2. Sit

5. 2. Sit $x = a\sqrt[n]{p} + b\sqrt[n]{p^{n-1}}$, & n par numerus; tum erit $x^n - nabpx^{n-2} + n.\frac{n-3}{2}a^2b^2p^2x^{n-4} - n.\frac{n-4}{2}.\frac{n-5}{3}a^3b^3p^3x^{n-6} + n.\frac{n-5}{2}.\frac{n-6}{3}.\frac{n-7}{4}a^4b^4p^4x^{n-8} - \&c. = a^np \pm 2a^{\frac{n}{2}}b^{\frac{n}{2}}p^{\frac{n}{2}} + b^np^{n-1}$: erit $+ 2a^{\frac{n}{2}}b^{\frac{n}{2}}p^{\frac{n}{2}}$, ſi $n = 4r + 2$; ſin aliter $- 2a^{\frac{n}{2}}b^{\frac{n}{2}}p^{\frac{n}{2}}$; ubi r eſt integer numerus.

6. 1. Sit $x = a\sqrt[n]{p} + b\sqrt[n]{p^{n-2}}$; & n impar numerus, qui non dividi poteſt per numerum 3; tum erit æquatio ab irrationalibus quantitatibus liberata $x^n - na^2bpx^{n-3} + n.\frac{n-5}{2}a^4b^2p^2x^{n-6} - n.\frac{n-7}{2}.\frac{n-8}{3}a^6b^3p^3x^{n-9} + n.\frac{n-9}{2}.\frac{n-10}{3}.\frac{n-11}{4}a^8b^4p^4x^{n-12} - n.\frac{n-11}{2}.\frac{n-12}{3}.\frac{n-13}{4}.\frac{n-14}{5}a^{10}b^5p^5x^{n-15} + n.\frac{n-13}{2}.\frac{n-14}{3}.\frac{n-15}{4}.\frac{n-16}{5}.\frac{n-17}{6}a^{12}b^6p^6x^{n-18} - \&c.$ (ad m terminos, ubi m vel ſit æqualis vel proxime major quam $\frac{n}{3}$) $- ab^{\frac{n+1}{2}}p^{\frac{n-1}{2}}(nx^{\frac{n-3}{2}} + \frac{1}{2^2}\times n.\frac{n-5}{2}.\frac{n-7}{3}a^2bpx^{\frac{n-9}{2}} + \frac{1}{2^4}\times n.\frac{n-7}{2}.\frac{n-9}{3}\times\frac{n-11}{4}\times\frac{n-13}{5}a^4b^2p^2x^{\frac{n-15}{2}} + \frac{1}{2^6}\times n\times\frac{n-9}{2}.\frac{n-11}{3}.\frac{n-13}{4}.\frac{n-15}{5}.\frac{n-17}{6}.\frac{n-19}{7}a^6b^3p^3x^{\frac{n-21}{2}} + \frac{1}{2^8}\times n\times\frac{n-11}{2}.\frac{n-13}{3}.\frac{n-15}{4}.\frac{n-17}{5}.\frac{n-19}{6}.\frac{n-21}{7}.\frac{n-23}{8}.\frac{n-25}{9}a^8b^4p^4x^{\frac{n-27}{2}} + \&c.) = A = a^np + b^np^{n-2}$.

Sit n impar numerus per 3 diviſibilis, tum erit quantitas prædicta $A = a^np + b^np^{n-2} + 3a^{\frac{n}{3}}b^{\frac{2n}{3}}p^{\frac{2n}{3}-1} + 3a^{\frac{2n}{3}}b^{\frac{n}{3}}p^{\frac{n}{3}}$.

6. 2. Sit

6. 2. Sit $x = a\sqrt[n]{p} + b\sqrt[n]{p^{n-2}}$, & n par numerus per 3 haud diviſibilis; tum erit $x^n - na^2bpx^{n-3} + n . \frac{n-5}{2} a^4b^2p^2x^{n-6} - n . \frac{n-7}{2} . \frac{n-8}{3} a^6b^3p^3x^{n-9} + n . \frac{n-9}{2} \times \frac{n-10}{3} \times \frac{n-11}{4} a^8b^4p^4x^{n-12} - n . \frac{n-11}{2} . \frac{n-12}{3} . \frac{n-13}{4} . \frac{n-14}{5} a^{10}b^5p^5x^{n-15}$ + &c. (ad prædictos m terminos) $- b^{\frac{n}{2}}p^{\frac{n}{2}-1} (2x^{\frac{n}{2}} + \frac{1}{2} n . \frac{n-4}{2} a^2bpx^{\frac{n}{2}-3} + \frac{1}{2^3} \times n . \frac{n-6}{2} . \frac{n-8}{3} . \frac{n-10}{4} a^4b^2p^2x^{\frac{n}{2}-6} + \frac{1}{2^5} \times n . \frac{n-8}{2} . \frac{n-10}{3} . \frac{n-12}{4} . \frac{n-14}{5} . \frac{n-16}{6} a^6b^3p^3x^{\frac{n}{2}-9}$ + &c.) $= A = a^n - b^np^{n-2}$.

Sit n par numerus per 3 diviſibilis; tum erit $A = a^np - b^np^{n-2} - 3a^{\frac{2n}{3}}b^{\frac{n}{3}}p^{\frac{n}{3}} + 3a^{\frac{n}{3}}b^{\frac{2n}{3}}p^{\frac{2n}{3}-1}$.

In omnibus his caſibus poteſtas quantitatis a nunquam evadit negativa.

7. Sint $\alpha, \beta, \gamma, \delta, \epsilon$, &c. radices æquationis $z^n - 1 = 0$, & data æquatio $x = a\sqrt[n]{p} + b\sqrt[n]{p^2} + c\sqrt[n]{p^3} + d\sqrt[n]{p^4} \ldots + b\sqrt[n]{p^\lambda} \ldots + k\sqrt[n]{p^\mu} \ldots + l\sqrt[n]{p^\nu} \ldots + q\sqrt[n]{p^\xi} \ldots + r\sqrt[n]{p^\varphi} \ldots + s\sqrt[n]{p^{n-4}} + t\sqrt[n]{p^{n-3}} + v\sqrt[n]{p^{n-2}} + u\sqrt[n]{p^{n-1}}$:

Diverſi valores quantitatis x erunt $a\sqrt[n]{p} \times \alpha + b\sqrt[n]{p^2} \times \alpha^2 + c\sqrt[n]{p^3} \times \alpha^3 + d\sqrt[n]{p^4} \times \alpha^4 \ldots + s\sqrt[n]{p^{n-4}} \times \alpha^{n-4} + t\sqrt[n]{p^{n-3}} \times \alpha^{n-3} + v\sqrt[n]{p^{n-2}} \times \alpha^{n-2} + u\sqrt[n]{p^{n-1}} \times \alpha^{n-1}$;

$a\sqrt[n]{p} \times \beta + b\sqrt[n]{p^2} \times \beta^2 + c\sqrt[n]{p^3} \times \beta^3 + d\sqrt[n]{p^4} \times \beta^4 \ldots + s\sqrt[n]{p^{n-4}} \times \beta^{n-4} + t\sqrt[n]{p^{n-3}} \times \beta^{n-3} + v\sqrt[n]{p^{n-2}} \times \beta^{n-2} + u\sqrt[n]{p^{n-1}} \times \beta^{n-1}$;

$a\sqrt{}$,

$a\sqrt[n]{p} \times \gamma + b\sqrt[n]{p^2} \times \gamma^2 + c\sqrt[n]{p^3} \times \gamma^3 \ldots\ldots + t\sqrt[n]{p^{n-3}} \times \gamma^{n-3} + v\sqrt[n]{p^{n-2}} \times \gamma^{n-2} + u\sqrt[n]{p^{n-1}} \times \gamma^{n-1}$;

$a\sqrt[n]{p} \times \delta + b\sqrt[n]{p^2} \times \delta^2 + c\sqrt[n]{p^3} \times \delta^3 \ldots\ldots + t\sqrt[n]{p^{n-3}} \times \delta^{n-3} + v\sqrt[n]{p^{n-2}} \times \delta^{n-2} + u\sqrt[n]{p^{n-1}} \times \delta^{n-1}$;

&c. &c. &c. &c.

& conſequenter ſumma radicum, quæ eſt coefficiens ſecundi æquationis termini, erit $a\sqrt[n]{p} \times \overline{\alpha + \beta + \gamma + \delta + \&c.}\ (o) + b\sqrt[n]{p^2} \times \overline{\alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \&c.}\ (o) + c\sqrt[n]{p^3} \times \overline{\alpha^3 + \beta^3 + \gamma^3 + \delta^3 + \&c.}\ (o) + \ldots\ldots + v\sqrt[n]{p^{n-2}} \times \overline{\alpha^{n-2} + \beta^{n-2} + \gamma^{n-2} + \delta^{n-2} + \&c.}\ (o) + u\sqrt[n]{p^{n-1}} \times \overline{\alpha^{n-1} + \beta^{n-1} + \gamma^{n-1} + \delta^{n-1} + \&c.}\ (o)$.

Aggregatum e ſingulis duobus valoribus vel radicibus æquationis quæſitæ, quod eſt coefficiens tertii quæſitæ æquationis termini, erit $a^2\sqrt[n]{p^2} \times \overline{\alpha\beta + \alpha\gamma + \beta\gamma + \alpha\delta + \&c.}\ (o) + ab\sqrt[n]{p^3} \times \overline{\alpha\beta^2 + \beta\alpha^2 + \alpha\gamma^2 + \gamma\alpha^2 + \beta\gamma^2 + \gamma\beta^2 + \alpha\delta^2 + \&c.}\ (o)$ & generaliter omnes termini nihilo erunt æquales, ni $a \times u \times p \times \overline{\alpha\beta^{n-1} + \beta\alpha^{n-1} + \alpha\gamma^{n-1} + \beta\gamma^{n-1} + \gamma\alpha^{n-1} + \gamma\beta^{n-1} + \&c.}\ (-n) + bvp \times \overline{\alpha^2\beta^{n-2} + \beta^2\alpha^{n-2} + \alpha^2\gamma^{n-2} + \gamma^2\alpha^{n-2} + \beta^2\gamma^{n-2} + \&c.}\ (-n) + ctp \times \overline{\alpha^3\beta^{n-3} + \beta^3\alpha^{n-3} + \alpha^3\gamma^{n-3} + \gamma^3\alpha^{n-3} + \beta^3\gamma^{n-3} + \gamma^3\beta^{n-3} + \&c.}\ (-n) + dsp \times \overline{\alpha^4\beta^{n-4} + \beta^4\alpha^{n-4} + \alpha^4\gamma^{n-4} + \gamma^4\alpha^{n-4} + \&c.}\ (-n) = -np\,(au + bv + ct + ds + \&c.)$: ſi $n = 2\lambda$, tum erit coefficiens contenti $b^2 p = -\frac{n}{2}$, & conſequenter aggregatum prædictum $= -np\,(au + bv + ct + ds + \ldots\ldots + \frac{1}{2}b^2)$.

Aggregatum contentorum e ſingulis tribus prædictis valoribus vel radicibus, quod eſt coefficiens quarti termini æquationis quæſitæ, erit

$a^3\sqrt[n]{p^3}\times\overline{\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta+\&c.}\,(0)+a^2b\sqrt[n]{p^4}\times\overline{\alpha\beta\gamma^2+}$
$\overline{\alpha\gamma\beta^2+\beta\gamma\alpha^2+\alpha\beta\delta^2+\&c.}\,(0)+\&c.+a^2vp\times\overline{\alpha\beta\gamma^{n-2}+\alpha\gamma\beta^{n-2}}$
$\overline{+\beta\gamma\alpha^{n-2}+\alpha\delta\beta^{n-2}+\&c.}\,(n)+abtp\times\overline{\alpha\beta^2\gamma^{n-3}+\beta\alpha^2\gamma^{n-3}+\gamma\beta^2\alpha^{n-3}}$
$\overline{+\alpha\beta^2\delta^{n-3}+\&c.}\,(1.2.n)+\&c.$: generaliter omnes quantitates in aggregato contentæ (ni quantitas contenta denotet formulam $\sqrt[n]{p^n}=p$ vel etiam hanc $\sqrt[n]{p^{2n}}=p^2$) erunt $=0$: i.e. ſi generalis terminus per hanc formulam $blk\sqrt[n]{p^{\lambda+\mu+\nu}}\times\overline{\alpha^\lambda\beta^\mu\gamma^\nu+\alpha^\lambda\beta^\nu\gamma^\mu+\alpha^\mu\beta^\lambda\gamma^\nu+\alpha^\mu\beta^\nu\gamma^\lambda+}$
$\overline{\alpha^\nu\beta^\mu\gamma^\lambda+\&c.}$ denotetur; ubi ſit $\lambda+\mu+\nu$ nec $=n$, nec $=2n$, tum erit terminus prædictus $=0$: ſit $\lambda+\mu+\nu$ vel $=n$ vel $=2n$, tum erit terminus prædictus vel $=1.2.n\times bklp$ vel $=1.2.n\times bklp^2$; ni duo indices λ, μ, ν ſint inter ſe æquales, tum dividatur terminus prædictus per 1.2; ſi vero tres ſint inter ſe æquales, i.e. $\lambda=\mu=\nu$, tum is dividatur per $1.2.3$.

Inveniantur omnes quantitates hujuſce generis, ubi $\lambda+\mu+\nu$ vel $=n$ vel $=2n$; & ſimul addantur, & dicatur earum aggregatum A.

Aggregatum contentorum e ſingulis quatuor valoribus vel radicibus prædictis, quod eſt coefficiens quinti termini quæſitæ æquationis, erit $a^4\sqrt[n]{p^4}\times\overline{\alpha\beta\gamma\delta+\alpha\beta\gamma\epsilon+\&c.}\,(0)+a^3b\sqrt[n]{p^5}\times\overline{\alpha\beta\gamma\delta^2+\alpha\beta\gamma^2\delta}$
$\overline{+\alpha\beta^2\gamma\delta+\&c.}\,(0)+\&c.$: ſit $bklq\sqrt[n]{p^{\lambda+\mu+\nu+\xi}}\times\overline{\alpha^\lambda\beta^\mu\gamma^\nu\delta^\xi+}$
$\overline{\alpha^\lambda\beta^\mu\gamma^\xi\delta^\nu+\alpha^\mu\beta^\lambda\gamma^\nu\delta^\xi+\alpha^\lambda\beta^\mu\gamma^\nu\epsilon^\xi+\&c.}$ generalis terminus; hic terminus erit $=0$, ni $\lambda+\mu+\nu+\xi=n$, vel $2n$, vel $3n$; in quibus caſibus terminus vel $=-1.2.3\,nbklqp$, vel $=-1.2.3\,nbklqp^2$, vel $=-1.2.3\,nbklqp^3$; ni $\lambda+\mu=\nu+\xi=n$, in quo caſu terminus prædictus erit $=-(1.2.3\,n-n^2)\,bklqp^2$; in hoc caſu ſi $\lambda=\nu$ & conſequenter $\mu=\xi$, tum erit $-(1.2.3\,n-1.2n^2)\,bklqp^2$; ſi vero $\lambda=\mu=\nu=\xi=\frac{n}{2}$, tum erit terminus $=-(1.2.3\,n-3n^2)\,bklqp^2$.

In-

In omnibus his caſibus, ſi duo ex indicibus λ, μ, ν, ξ ſint æquales; tum dividatur terminus prædictus per 1 . 2; ſi tres ſint æquales per 1 . 2 . 3; ſi quattuor, per 1 . 2 . 3 . 4; ſi duo bis ſint æquales, tum dividatur per 1 . 2 . 1 . 2.

Inveniatur aggregatum e ſingulis terminis hujuſce generis, quod dicatur B.

Et ſic inveniantur aggregata contentorum e ſingulis quinque, ſex, ſeptem, &c. diverſis valoribus vel radicibus quæſitæ æquationis in ſeſe ductis, quæ dicantur reſpective C, D, E, &c.; tum erit æquatio quæſita $x^n - np\,(au + bv + ct + ds + \&c.)\,x^{n-2} - Ax^{n-3} + Bx^{n-4} - Cx^{n-5} + Dx^{n-6} - \&c. = 0$.

Si reſolutio data ſit $z = n \times H + a\sqrt[n]{p} + b\sqrt[n]{p^2} + c\sqrt[n]{p^3} + \&c.$, tum vel ex principiis hic traditis; vel ſcribendo pro x in æquatione prius inventâ $z - n \times H$ facile deduci poteſt æquatio, cujus radix eſt z.

Cor. Hinc inveniri poſſunt infinitæ æquationes (n) dimenſionum, ($n - 1$) incognitas quantitates habentes; quarum reſolutiones cognoſcuntur.

Aſſumatur enim $x = H + a\sqrt[n]{p} + b\sqrt[n]{p^2} + c\sqrt[n]{p^3} \ldots k\sqrt[n]{p^{n-2}} + l\sqrt[n]{p^{n-1}}$; hæc autem reſolutio $(n + 1)$ habet incognitas quantitates; ſubſtituantur pro duabus literis duæ quæcunque quantitates; & exterminentur omnes irrationales quantitates; reſultat æquatio (n) dimenſionum ($n - 1$) incognitas quantitates habens.

Sit data æquatio $x^n + qx^{n-2} - rx^{n-3} + sx^{n-4} - tx^{n-5} + \&c. = 0$. Fiant correſpondentes datæ & reſultantis æquationis coefficientes inter ſe æquales, & (n) reſultent æquationes $(n - 1)$ ſolummodo incognitas quantitates habentes; e methodo communes diviſores inveniendi erui poſſunt incognitarum quantitatum valores, ſi modo ullos admittant; etiamque, ſi modo ita reducantur (n) reſultantes æquationes, ut exterminentur omnes incognitæ quantitates, conſtat, quo caſu reſolutionem hujuſmodi admittet data æquatio.

Cor.

Ex. Sit æquatio $x^3 + qx + r = o$, cujus generalis reſolutio eſt $x = \sqrt[3]{-\frac{1}{2}r + \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}} + \sqrt[3]{-\frac{1}{2}r - \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}}$: deinde fingatur $(r) = o$, evadet $x = \sqrt[3]{+\sqrt{\frac{q^3}{27}}} + \sqrt[3]{-\sqrt{\frac{q^3}{27}}} = \sqrt[3]{\frac{q^{\frac{3}{2}}}{3\sqrt{3}}} + \sqrt[3]{-\frac{q^{\frac{3}{2}}}{3\sqrt{3}}}$ vel $= \frac{q^{\frac{1}{2}}}{\sqrt{3}} - \frac{q^{\frac{1}{2}}}{\sqrt{3}} = o$ vel $\frac{-1 \pm \sqrt{-3}}{2} \times \frac{q^{\frac{1}{2}}}{\sqrt{3}}, + \frac{1 \pm \sqrt{-3}}{2} \frac{q^{\frac{1}{2}}}{\sqrt{3}} = \pm \sqrt{-3} \times \frac{q^{\frac{1}{2}}}{\sqrt{3}} = \pm \sqrt{-1} \times \sqrt{q} = \pm \sqrt{-q}$; quæ eſt radix æquationis $x^2 + q = o$.

Cor. Hæc reſolutio continet plures valores æquationis $x = o$, & plures valores æquationis $(x) = \pm \sqrt{-q}$.

2. 1. Sit æquatio $ax^n + p'x^{n-1} + q'x^{n-2} + r'x^{n-3} \ldots + R'x^2 + Q'x + P' = o$; ejus reſolutio ſemper continet generalem reſolutionem omnium æquationum inferiorum $(n-1, n-2,$ &c.$)$ dimenſionum; ni $a = o$, in quo caſu pro $\frac{p'}{a}, \frac{q'}{a}, \frac{r'}{a}$, &c. ſcribantur reſpective p, q, r, &c.; reſultat reſolutio æquationis $x^n + px^{n-1} + qx^{n-2} +$ &c. $= o$, in cujus denominatore haud continetur litera (a); & quæ continet in ſe generalem reſolutionem omnium æquationum $(n-1, n-2, n-3,$ &c.) dimenſionum.

2. 2. Aliter: in datâ reſolutione generalis æquationis pro literis a, p', q', &c., R', Q' & P' ſcribantur reſpective P', Q', R', &c.; q', p' & $a = o$; denominator erit functio, cujus indices ſunt affirmativi, quantitatis P': radices reſultantis reſolutionis erunt radices reciprocæ æquationis $p'x^{n-1} + q'x^{n-2} \ldots + R'x^2 + Q'x + P' = o$.

Ex. Sit æquatio $ax^2 + p'x + q' = o$, cujus reſolutio innoteſcit, viz. $x = \frac{-p' = \sqrt{p'^2 - 4aq'}}{2a}$: ſit $a = o$; tum ex hâc reſolutione haud datur reſolutio æquationis $p'x + q' = o$: in datâ reſolutione pro literis

teris a, p' & q' ſcribantur q', p' & o; reſultat $x = \frac{-p' \pm \sqrt{p'^2}}{2a} = \frac{-p'+p'}{2q'} = o$ vel $\frac{-p'-p'}{2q'} = \frac{-p'}{q'}$: reciproca igitur radix æquationis $p'x + q' = o$ erit $\frac{-p'}{q'}$; unde ejus radix erit $\frac{-q'}{p'}$.

Conſimilis methodus etiam applicari poteſt ad inveniendas radices æquationum inferiorum dimenſionum ex datis reſolutionibus correſpondentium æquationum ſuperiorum dimenſionum, cum evaneſcant coefficientes a, p', &c., P', Q', &c.

SCHOLIUM.

Reſolutio, quæ generaliter reſolvet quamcunque æquationem (n) dimenſionum, neceſſario erit reſolutio æquationis $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times \ldots \times n$ dimenſionum; continet enim radices quadraticas, cubicas, etiamque ex quartâ, quintâ, 6, 7, ... $n-2$, $n-1$ & n poteſtatibus: hæc autem reſolutio etiam inveniet, quando duæ radices, &c. æquationis n dimenſionum fiant inter ſe æquales, ergo in generali reſolutione æquationum ſuperiorum dimenſionum calculus requiritur pene infinite operoſus. Fruſtra per tales methodos radices ſuperiorum dimenſionum æquationis $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} -$ &c. $= o$ quæruntur. Hinc recentiores, Vieta primum, deinde multi alii methodos approximationum invenere: valorem quàm proximè radici æqualem primò cognitum pro conceſſo habuere, deinde valorem continuo ad radicem magis appropinquantem methodis fere iiſdem inveſtigavere omnes: minime autem pendet e ratione, quam habet quantitas aſſumpta pro radice ad radicem ipſam, ſeriei convergentia; potius vero pendet ex hoc, quod quantitas aſſumpta multo propior accedat ad unam radicem quæſitam, quam ad ullam aliam datæ æquationis radicem, ex. grat.

Sit æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} +$ &c. .... $Ax^2 - Bx + C = o$ cujus radices ſint α, β, γ, δ, ϵ, &c. ſit etiam a quantitas aſſumpta pro

pro radice α; & transformetur data æquatio in alteram, cujus radices ſint minores vel majores quam radices datæ æquationis per quantitatem (a), i. e. ſint $\alpha - a$, $\beta - a$, $\gamma - a$, $\delta - a$, $\varepsilon - a$, &c. reſultans æquatio erit

$$\begin{array}{llllll} v^n + nav^{n-1} & + n \times \frac{n-1}{2} a^2 v^{n-2} \ldots\ldots & & & na^{n-1}v & + a^n \\ -p & - \overline{n-1}\, ap \ldots\ldots & & & - \overline{n-1}\, pa^{n-2} & - pa^{n-1} \\ & + & q \ldots\ldots & & + \overline{n-2}\, qa^{n-3} & + qa^{n-2} \\ & & & & - \overline{n-3}\, ra^{n-4} & - ra^{n-3} \\ & & & & \&c. & \&c. \end{array}$$

Sint $a : \alpha :: 1 : 1 + b$, $a : \beta :: 1 : 1 + k$, $a : \gamma :: 1 : 1 + l$, $a : \delta :: 1 : 1 + m$, $a : \varepsilon :: 1 : 1 + n'$, &c. & ſi modo a multo propior accedat ad radicem α, quam ad β, vel γ, vel δ, vel ε, &c. tum b multo minor erit quam k, vel l, vel m, vel n', &c. radices reſultantis æquationis ſunt reſpective ba, ka, la, ma, $n'a$, &c. ergo $ba \times ka \times la \times ma \times n'a \times$ &c. $= \pm (a^n - pa^{n-1} + qa^{n-2} - ra^{n-3} + sa^{n-4}$ &c.$) = Q'$ contentum ſub ſingulis reſultantis æquationis radicibus; & $ka \times la \times ma \times n'a \times$ &c. $+ ba \times la \times ma \times n'a \times$ &c. $+ ba \times ka \times ma \times n'a \times$ &c. $+$ &c. $= \pm (na^{n-1} - \overline{n-1}\, pa^{n-2} + \overline{n-2}\, qa^{n-3} - \overline{n-3}\, ra^{n-4} +$ &c.$) = P'$ aggregatum e ſingulis contentis ſub $\overline{n-1}$ radicibus reſultantis æquationis: conſequenter approximatio per vulgares regulas inventa $- \frac{a^n - pa^{n-1} + qa^{n-2} - ra^{n-3} + \&c.}{na^{n-1} - \overline{n-1}\, pa^{n-2} + \overline{n-2}\, qa^{n-3} - \&c.}$: ad quantitatem quæſitam $\alpha - a = ba :: klmn'$ &c. $: klmn'$ &c. $+ blmn'$ &c. $+ bkmn'$ &c. $+$ &c. $:: \frac{Q'}{ba} : P'$.

Si vero quantitas aſſumpta multo propior accedat ad duas radices α & β datæ æquationis quam ad reliquas, i. e. ſi duæ radices reſultantis æquationis, quæ ſit $v^n \ldots\ldots Tv^4 - Sv^3 + Rv^2 - Qv + P = o$, ſint minores quam reliquæ; tum radices quadraticæ æquationis $v^2 - \frac{Q}{R}v +$

$\frac{P}{R} = o$ erunt quam proxime illæ radices quæſitæ; ſi vero tres, tum radices æquationis $v^3 - \frac{R}{S}v^2 + \frac{Q}{S}v - \frac{P}{S} = o$ erunt quam proxime radices quæſitæ; & ſic deinceps: & facile conſimili methodo inveniri poſſunt rationes, quas habent radices inventæ per has regulas ad veras radices quæſitas.

Sed plura de infinitis ſeriebus, quam ratio noſtri inſtituti exigat, jam adjiciuntur.

CAP.

CAP. IV.

De duabus vel pluribus æquationibus duas vel plures incognitas quantitates habentibus.

THEOR. XXIV.

Sint n ſimplices æquationes $ax + by + cz + dv + \&c. = A$, $a'x + b'y + c'z + d'v + \&c. = A'$, $a''x + b''y + c''z + d''v + \&c. = A''$, $a'''x + b'''y + c'''z + d'''v + \&c. = A'''$, &c.; n incognitas quantitates $(x, y, z, \&c.)$ habentes; eas in unam reducere, ita ut exterminentur $(n - 1)$ incognitæ quantitates $(y, z, \&c.)$; erit

$$x = \frac{b' \times c'' \times d''' \times \&c. \times A - b \times c'' \times d''' \times \&c. \times A' - b' \times c \times d''' \times \&c. \times A'' - b' \times c'' \times d \times \&c. \times A''' - \&c. + bc' \times d''' \times \&c. \times A'' + \&c.}{b' \times c'' \times d''' \times \&c. \times a - b \times c'' \times d''' \times \&c. \times a' - b' \times c \times d''' \times \&c. \times a'' - b' \times c'' \times d \times \&c. \times a''' - \&c. + bc' \times d''' \times \&c. \times a'' + \&c.}.$$

Numerator hujuſce fractionis evadet idem ac denominator; ſi modo in eo pro A, A', A'', A''', &c. ſcribantur reſpective a, a', a'', a''', &c.

In ſingulis contentis $(b'c''d''' \times \&c. \times a$, $bc''d''' \&c. \times a'$, &c.) denominatoris continentur (n) factores, viz. a, b', c'', d''', &c.; a', b, c'', d''', &c.; &c.

Omnis æquatio præbet unum e prædictis (n) factoribus.

Hi (n) factores erunt reſpective coefficientes (n) diverſarum incognitarum $(x, y, z, \&c.)$ in diverſis (n) æquationibus contentarum.

Omnia contenta hujuſmodi in denominatore continentur.

Signa quibuſcunque duobus contentis, e. g. $b'c''d''$ &c., $bc''d'''$ &c. affixa erunt eadem, vel non; prout impar vel par fuerit numerus diverſorum factorum in iis contentorum.

PROB. XXXI.

Duas æquationes in unam transformare, ita ut incognita quantitas exterminetur.

1. Collocentur ſingulæ æquationis termini juxta incognitæ quantitatis

:itatis exterminandæ dimenſiones, & ita ſubducatur alterutra æquatio aut ejus multiplex de alterâ, & reliquum aut ejus multiplex de ablatâ æquatione, & ſic deinceps, ut tandem exterminetur incognita quantitas; i. e. exterminetur incognita quantitas eodem modo, quo inveniuntur communes diviſores.

Ex. Sint duæ æquationes $y^2 + (3 + 2x) \times y + 4x^2 + 5x + 6 = 0$, & $y^2 + \overline{5 + 2x}y + 10x^2 + 7x + 12 = 0$, & operatio erit hujuſmodi

$$y^2 + \overline{3+2x}y + 4x^2 + 5x + 6) y^2 + \overline{5+2x}y + 10x^2 + 7x + 12) 1$$
$$y^2 + \overline{3+2x}y + 4x^2 + 5x + 6$$
$$2y + 6x^2 + 2x + 6 = 0,$$

$$2y + 6x^2 + 2x + 6) y^2 + \overline{3+2x}y + 4x^2 + 5x + 6) \tfrac{1}{2}y + \frac{x-3x^2}{2}$$
$$y^2 + \overline{3x^2+x+3}y$$
$$(x - 3x^2) \times y + 4x^2 + 5x + 6$$
$$(x - 3x^2) \times y + \frac{x-3x^2}{2} \times (6x^2 + 2x + 6)$$
$$4x^2 + 5x + 6 - (x - 3x^2) \times (3x^2 + x + 3)$$

& ita transformantur duæ æquationes in unam $(4x^2 + 5x + 6 - (x - 3x^2) \times (3x^2 + x + 3) = 0$, ut exterminetur incognita quantitas (y).

Eadem methodus etiam reducit tres vel plures (n) æquationes tres vel plures (n) incognitas quantitates $(x, y, z, \&c.)$ habentes, in (m) æquationes, ita ut exterminentur quæcunque $(n - m)$ prædictæ incognitæ quantitates: dividantur continuo omnes reliquæ æquationes & diviſor per reliquam æquationem, in quâ maximæ dimenſiones quantitatis exterminandæ minores ſint quam maximæ dimenſiones ejuſdem quantitatis in reliquis; & ſic deinceps de æquationibus, quæ ſunt reſidua reſultantia, ita ut tandem exterminetur ea incognita quantitas.

2. Sint duæ æquationes $ay^2 + by + c = 0$, & $py^2 + qy + r = 0$; eas in unam reducere, ita ut incognita quantitas (y) exterminetur.

Ducatur prior æquatio in p, poſterior vero in a coefficientes terminorum,

norum, in quibus maximæ inveniuntur dimenſiones quantitatis (y) exterminandæ, causâ fractionum evitandarum; operatio erit

$$pay^2 + pby + pc = 0$$

$$pay^2 + qay + ra = 0$$ ſubducatur poſterior æquatio de priori, & reſiduum erit $\overline{pb - qa}\,y + \overline{pc - ra} = 0$: ducantur æquationes $py^2 + qy + r = 0$, & $\overline{pb - qa}\,y + \overline{pc - ra} = 0$ causâ evitandarum fractionum in $pb - qa$ & py reſpective, reſultant

$$\overline{p^2b - pqa}\,y^2 + \overline{qpb - q^2a}\,y + pbr - qar = 0,$$ &

$$\overline{p^2b - pqa}\,y^2 + \overline{p^2c - pra} \times y = 0;$$ ſubducatur poſterior de priori, & erit reſiduum $\overline{qpb - q^2a + pra - p^2c}\,y + pbr - qar = 0$; eodem proceſſu repetito exterminabitur quantitas (y), & reſultat quantitas quæſita

$$\overline{pc - ra} \times \overline{qpb - q^2a + pra - p^2c} - \overline{pb - qa} \times \overline{pbr - qar} = 0.$$

2. 2. Eodem modo ſint duæ æquationes $ay^2 + by + c = 0$, & $py^3 + qy^2 + ry + s = 0$; & exterminetur incognita quantitas y, reſultabit æquatio $\overline{aB - bA} \times \overline{cB - asb} + \overline{sa^2 - cA}^2 = 0$, ubi $A = aq - pb$, $B = ar - pc$.

Sint etiam $ax^3 + bx^2 + cx + d = 0$, & $px^3 + qx^2 + rx + s = 0$; & ſupponantur $A = bp - aq$, $C = dp - as$, $B = cp - ar$, $P = cA^2 - aAB - bAC + aC^2$, $Q = cAB - aBB - dAC$, $R = dA^2 - bAB + aBC$; tum $PQ + RR = 0$.

Et ſic de pluribus æquationibus plures incognitas quantitates habentibus.

Cor. Æquationes ſic derivatæ plerumque plures habent dimenſiones quam neceſſario exigat problema.

1. Sint enim duæ datæ æquationes $Ay^n + By^{n-1} + Cy^{n-2} + Dy^{n-3} +$ &c. $= 0$, $ay^m + by^{m-1} + cy^{m-2} + dy^{m-3} +$ &c. $= 0$, ubi A, B, C, D, &c. a, b, c, d, &c. ſint quæcunque functiones incognitæ quantitatis (x); ducatur prior æquatio in $x^r - px^{r-1} + qx^{r-2} -$ &c. & duæ æquationes $\overline{x^r - px^{r-1} + qx^{r-2} - \text{\&c.}} \times \overline{Ay^n + By^{n-1} + Cy^{n-2} + \text{\&c.}} = 0$, & $ay^m + by^{m-1} + cy^{m-2} + dy^{m-3} +$ &c. $= 0$ habent (r) radices incognitæ

nitæ quantitatis (x), $m \times r$ radices vero incognitæ quantitatis (y), quæ haud inveniuntur in duabus datis æquationibus; omnis enim radix æquationis $x^r - px^{r-1} + qx^{r-2} - \&c. = o$ pro (x) ſubſtituta in æquatione $\overline{x^r - px^{r-1} + qx^{r-2} - \&c.} \times \overline{Ay^n + By^{n-1} + Cy^{n-2} + Dy^{n-3} + \&c.} = o$ æquationem nihilo æqualem reddet: ſi vero in poſteriori æquatione $ay^m + by^{m-1} + cy^{m-2} + dy^{m-3} + \&c. = o$ pro incognitâ quantitate (x) ſubſtituatur prædicta radix, reſultat æquatio, quæ (m) habet dimenſiones incognitæ quantitatis (y); ergo m ſunt valores incognitæ quantitatis (y), qui correſpondent unicuique e prædictis radicibus æquationis $x^r - px^{r-1} + qx^{r-2} - \&c. = o$, & conſequenter $m \times r$ invenientur valores incognitæ quantitatis (y) in his duabus æquationibus, qui in datis haud inveniuntur.

Si vero $x^r - px^{r-1} + qx^{r-2} - \&c.$ & a communem habeant diviſorem, qui ſit $x^s - Px^{s-1} + Qx^{s-2} - \&c.$ tum haud plures novæ radices incognitæ quantitatis (y) inveniuntur, quam $\overline{r-s} \times m + s \times \overline{m-1}$: ſi vero tres quantitates $x^r - px^{r-1} + qx^{r-2} - \&c.$ & a & b habeant communem diviſorem, qui ſit $x^t - \pi x^{t-1} + \varrho x^{t-2} - \&c.$ tum haud plures inveniuntur novæ radices incognitæ quantitatis (y) quam $\overline{r-s} \times m + \overline{s-t} \times \overline{m-1} + t \times \overline{m-2}$; & ſic deinceps: novæ autem radices incognitæ quantitatis (x) in omnibus his caſibus eædem erunt, erunt enim radices æquationis $x^r - px^{r-1} + qx^{r-2} - \&c. = o$, ni quædam radices æquationis $x^r - px^{r-1} + qx^{r-2} - \&c. = o$ (ut facile e prædictâ methodo conſtabit) nullos habeant correſpondentes valores incognitæ quantitatis (y).

2. Iiſdem datis æquationibus $Ay^n + By^{n-1} + \&c. = o$ & $ay^m + by^{m-1} + \&c. = o$, ducantur hæ æquationes reſpective in quantitates V & Z, ubi V & Z ſunt functiones algebraicæ quantitatum x & y; æquationes reſultantes $\overline{Ay^n + By^{n-1} + \&c.} \times V = o$ & $\overline{ay^m + by^{m-1} + \&c.} \times Z = o$ continent omnes radices, quas habent datæ æquationes $Ay^n + By^{n-1} + \&c. = o$ & $ay^m + by^{m-1} + \&c. = o$; cum omnibus radicibus æquationum $V = o$ & $ay^m + by^{m-1} + \&c. = o$; $Z = o$ & $Ay^n + By^{n-1} + \&c. = o$; & $V = o$ & $Z = o$ reſpective adjunctis.

i 3. Eadem

3. Eadem etiam applicari poſſunt ad tres vel plures æquationes $A=o$, $B=o$, $C=o$, &c. tres vel plures incognitas quantitates $(x, y, z$, &c.) habentes; ducantur hæ æquationes reſpective in V, W, Z, &c.; ubi V, W, Z, &c. ſunt etiam functiones algebraicæ prædictarum incognitarum quantitatum $(x, y, z$, &c.) tum reſultantes æquationes $V\times A=o$, $W\times B=o$, $Z\times C=o$, &c. continent omnes radices, quas habent datæ æquationes $A=o$, $B=o$, $C=o$, &c. cum radicibus æquationum $V=o$, $B=o$, $C=o$, &c.; $W=o$, $A=o$, $C=o$, &c.; $Z=o$, $A=o$, $B=o$, &c.; $V=o$, $W=o$, $C=o$, &c.; &c.; & $V=o$, $W=o$, $Z=o$, &c. reſpective adjunctis.

Cor. 1. Hinc facile deduci poſſunt quam plurimæ æquationes, quæ deprimi poſſunt in alias minorum dimenſionum. Aſſumantur quantitates A, B, C, &c., V, W, Z, &c., quæ ſint functiones quantitatum $(x, y, z$, &c.); tum æquationes $V\times A=o$, $W\times B=o$, $Z\times C=o$, &c. deprimi poſſunt in ſubſequentes æquationes $A=o$, $B=o$, $C=o$, &c.; $A=o$, $B=o$, $Z=o$, &c.; $A=o$, $W=o$, $C=o$, &c.; $A=o$, $W=o$, $Z=o$, &c.; $V=o$, $B=o$, $C=o$, &c.; $V=o$, $B=o$, $Z=o$, &c.; $V=o$, $W=o$, $C=o$, &c.; $V=o$, $W=o$, $Z=o$, &c.; &c.; hæ ſubſequentes æquationes eaſdem præbebunt radices ac præcedentes.

Singula æquatio primum reducenda eſt in communem denominatorem; deinde abjiciatur denominator, ſolummodo reſtant quantitates, quæ ſunt functiones incognitarum quantitatum $(x, y, z$, &c.) in numeratore.

Ex. Sint duæ datæ æquationes $y^n+\overline{a+bx}\,y^{n-1}+\overline{c+dx+ex^2}\,y^{n-2}+$ &c. $=o$, & $\overline{A+Bx}\,y^{n-1}+\overline{C+Dx+Ex^2}\,y^{n-2}+$ &c. $=o$; ducatur prior æquatio in $A+Bx$, poſterior vero in y; reſultant æquationes $\overline{A+Bx}\,y^n+\overline{A+Bx}\times\overline{a+bx}\,y^{n-1}+\overline{A+Bx}\times\overline{c+dx+ex^2}\,y^{n-2}+$ &c. $=o$, & $\overline{A+Bx}\,y^n+\overline{C+Dx+Ex^2}\,y^{n-1}+$ &c. $=o$, quarum differentia erit $\overline{\overline{A+Bx}\times\overline{a+bx}-C-Dx-Ex^2}\,y^{n-1}+$ &c. $=o$: duæ æquationes $\overline{\overline{A+Bx}\times\overline{a+bx}-C-Dx-Ex^2}\,y^{n-1}+$ &c. $=o$,

les ex hypotheſi emergant, aliquid abſurdi in ſe continent datæ æquationes; ſin aliter vero non. e. g. Sint æquationes $y^2 + 7x^2 + 20 = 0$, & $y^2 + 7x^2 + 40 = 0$; ſubtrahatur prior æquatio de poſteriori, & reſultat $40 - 20 = 0$, i. e. $40 = 20$, q. e. abſurdum.

Hoc conſtat ex eo, quod ſi haud aliquid abſurdi contineatur in diviſore quotiente & reſiduo, tum haud aliquid abſurdi continetur in diviſore & quantitate dividendâ; & vice verſâ ſi aliquid abſurdi contineatur in diviſore vel reſiduo, tum aliquid abſurdi continebitur in diviſore & quantitate dividendâ.

Omnia hæc etiam applicari poſſunt ad (n) æquationes (m) incognitas quantitates habentes.

Cor. Conſtat, ut hæc methodus reducendi duas vel plures æquationes duas vel plures dimenſiones habentes in unam, ita ut exterminentur incognitæ quantitates, eadem eſt ac methodus reducendi duas vel plures ſimplices æquationes unam ſolummodo dimenſionem incognitarum quantitatum habentes in unam, ita ut exterminentur incognitæ quantitates. Sint enim duæ æquationes $Ay^m + By^{m-1} + Cy^{m-2} + \&c. = P$, & $ay^n + by^{n-1} + cy^{n-2} + dy^{n-3} + \&c. = Q$; eas in unam reducere, ita ut exterminetur incognita quantitas (y).

Aſſumantur quantitates y^m, y^{m-1}, y^{m-2}, &c. y^n, y^{n-1}, y^{n-2}, &c. tanquam diverſæ incognitæ quantitates; i. e. exterminentur incognitæ quantitates eodem modo, quo ſimplices æquationes diverſas incognitas quantitates habentes reducantur; tum eadem erit methodus reducendi has æquationes ac præcedens.

Si numerus datarum æquationum major ſit per unitatem quam numerus quantitatum y^m, y^{m-1}, y^{m-2}, &c., y^n, y^{n-1}, &c.; tum per theor. 24. facile reduci poſſunt prædictæ æquationes, ita ut exterminetur incognita quantitas y.

In his methodis reducendi duas æquationes n & m dimenſionum in unam, ita ut incognita quantitas (x) exterminetur; plerumque æquatio reducta aſcendet ad plures dimenſiones, quam neceſſario exigat problema; ſed per methodos rationales diviſores inveniendi, ſemper deprimi poteſt æquatio ad dimenſiones requiſitas.

Si in iiſdem & ſubſequentibus methodis reducendi tres vel plures æquationes in unam, ita ut una vel duæ vel plures incognitæ quantitates exterminentur, ad plures dimenſiones aſcendant reductæ æquationes, quam neceſſario exigat problema: tum ex methodo in ſubſequente capite datâ i. e. rationales diviſores inveniendi methodo deprimi poſſunt reſultantes æquationes.

3. Transformare duas æquationes ex aliâ methodo in unam, ita ut incognita quantitas exterminetur.

Sint duæ æquationes

$$y^n - py^{n-1} + qy^{n-2} - ry^{n-3} + sy^{n-4} - \&c. = 0,$$
$$\& \; V - Ty + Sy^2 - Ry^3 + Qy^4 - \&c. = 0.$$

Sint etiam $\alpha, \beta, \gamma, \delta,$ &c. prioris æquationis radices (y), quarum numerus erit (n); quibus in poſteriori pro ſuo valore ſubſtitutis, reſultant quantitates

$$V - T\alpha + S\alpha^2 - R\alpha^3 + Q\alpha^4 - \&c.$$
$$V - T\beta + S\beta^2 - R\beta^3 + Q\beta^4 - \&c.$$
$$V - T\gamma + S\gamma^2 - R\gamma^3 + Q\gamma^4 - \&c.$$
$$V - T\delta + S\delta^2 - R\delta^3 + Q\delta^4 - \&c.$$
&c.

Multiplicentur hæ quantitates reſultantes continuo in ſeſe, contentum erit

$$V^n - V^{n-1}T \times \begin{matrix}\alpha\\ \beta\\ \gamma\\ \delta\\ \&c.\end{matrix} + V^{n-1}S \times \begin{matrix}\alpha^2\\ \beta^2\\ \gamma^2\\ \delta^2\\ \&c.\end{matrix} + V^{n-1}R \times \begin{matrix}\alpha^3\\ \beta^3\\ \gamma^3\\ \delta^3\\ \&c.\end{matrix} + \&c.$$

$$+ V^{n-2}T^2 \times \begin{matrix}\alpha\beta\\ \alpha\gamma\\ \beta\gamma\\ \alpha\delta\\ \beta\delta\\ \gamma\delta\\ \&c.\end{matrix} + V^{n-2}TS \times \begin{matrix}\alpha^2\beta\\ \beta^2\alpha\\ \alpha^2\gamma\\ \gamma^2\alpha\\ \beta^2\gamma\\ \gamma^2\beta\\ \&c.\end{matrix} + \&c.$$

Ope

Ope problematum primi & tertii inveniri poteſt hoc contentum, quod erit

$$V^n - V^{n-1}Tp + V^{n-2}S\times\overline{p^2-2q} + V^{n-3}R\times\overline{p^3-3pq+3r} + \&c.$$
$$+ V^{n-2}T^2\times q + V^{n-3}TS\times\overline{pq-3r} + \&c.$$

nihilo fiat æquale, tum ita transformantur duæ æquationes in unam, ut exterminetur incognita quantitas (y).

Ex. 1. Sint duæ æquationes $y^2+\overline{3+2x}\times y+4x^2+5x+6=0$,
$\&\ y^2+\overline{5+2x}\times y+10x^2+7x+12=0$.

Sint α & β prioris æquationis radices (y), quibus pro (y) in poſteriori æquatione ſubſtitutis, reſultant quantitates

$$\alpha^2+\overline{5+2x}\times\alpha+10x^2+7x+12$$

$\&\ \beta^2+\overline{5+2x}\times\beta+10x^2+7x+12$, quibus quantitatibus in ſeſe ductis, fit rectangulum

$$\alpha^2\beta^2+\overline{5+2x}\times\alpha\beta\times\overline{\alpha+\beta}+\overline{10x^2+7x+12}\times\overline{\alpha^2+\beta^2}+\overline{5+2x}^2\times\alpha\beta+$$
$$\overline{10x^2+7x+12}\times\overline{5+2x}\times\overline{\alpha+\beta}+\overline{10x^2+7x+12}^2$$

ope problematis primi prodit hoc rectangulum

$$\overline{4x^2+5x+6}^2-\overline{5+2x}\times\overline{4x^2+5x+6}\times\overline{3+2x}+\overline{10x^2+7x+12}\times$$
$$\overline{\overline{3+2x}^2-2\times\overline{4x^2+5x+6}}+\overline{5+2x}^2\times\overline{4x^2+5x+6}-\overline{10x^2+7x+12}\times$$
$$\overline{5+2x}\times\overline{3+2x}+\overline{10x^2+7x+12}^2.$$

Fiat hoc rectangulum nihilo æquale, erit æquatio quæſita.

Cor. 1. Facile conſtat, quod tot erunt valores incognitarum quantitatum, quot invenit hæc reductionis methodus; plures autem minime admittent datæ æquationes.

Cor. 2. Hac methodo ita transformentur duæ æquationes, quarum dimenſiones ſunt n & m reſpective, duas incognitas quantitates (x & y) habentes in unam, ut exterminetur una incognita quantitas (y); plures quam $(n\times m)$ dimenſiones haud habere poteſt reſultans æquatio.

Cor.

Cor. 3. Sint duæ æquationes
$My^n+\overline{a+bx}\times y^{n-1}+\overline{c+dx+ex^2}\times y^{n-2}+\overline{f+gx+hx^2+kx^3}\times y^{n-3}+\&c.=0$,
$Ny^m+\overline{A+Bx}\times y^{m-1}+\overline{C+Dx+Ex^2}\times y^{m-2}+\overline{F+Gx+Hx^2+Kx^3}\times y^{m-3}+\&c.=0$, quarum dimenſiones ſunt n & m, duas incognitas quantitates $(x$ & $y)$ habentes; & rejiciantur harum æquationum omnes termini, in quibus incognitarum quantitatum $(x$ & $y)$ dimenſio haud invenitur maxima; quantitates reſultantes erunt

$$My^n+bxy^{n-1}+ex^2y^{n-2}+kx^3y^{n-3}+\&c.$$
$$Ny^m+Bxy^{m-1}+Ex^2y^{m-2}+Kx^3y^{m-3}+\&c.$$

ſi communem habeant diviſorem hæ quantitates reſultantes, $(n\times m)$ dimenſiones non habere poteſt reſultans æquatio, cujus radix vel ſit incognita quantitas $(x$ vel $y)$; ſi vero nullum habeant communem diviſorem, $(n\times m)$ dimenſiones habebit æquatio reſultans.

Si communem diviſorem (r) dimenſionum habeant prædictæ quantitates reſultantes, tum haud plures quam $n\times m-r$ dimenſiones habere poteſt prædicta æquatio.

Sint tres vel plures (h') æquationes, totidem incognitas $(x, y, z, v$, &c.) quantitates habentes, quarum dimenſiones ſint reſpective m, n, o, l, &c. & rejiciantur harum æquationum omnes termini, in quibus incognitarum quantitatum $(x, y, z, v$, &c.) dimenſio haud invenitur maxima, & reſultent quantitates A, B, C, D, &c.: ſi quantitatum reſultantium nullus inveniatur communis diviſor, qui diverſis reſultantibus communis eſt; tum æquatio, cujus radix ſit (x vel y vel z vel v,&c.) $n\times m\times o\times l\times$ &c. dimenſiones habet; ſin aliter vero non: i. e. aſſumantur $(h'-1)$ æquationes $B=0$, $C=0$, $D=0$, &c. etiamque $(h'-1)$ aliæ æquationes $A=0$, $B=0$, $C=0$, &c.; quarum una eſt reliqua $(A=0)$, quæ in præcedentibus haud continetur; tum ex prioribus & poſterioribus $(h'-1)$ æquationibus aſſumptis inveniantur rationes, quas habent inter ſe correſpondentes valores incognitarum quantitatum $(x, y, z$, &c.); & ſi dentur nullæ rationes inter quantitates $(x, y, z$, &c.) in prioribus $(h'-1)$ æquationibus eædem ac in poſterioribus æquationibus; tum aſcendent æquationes reſultantes

tantes, quarum radices vel ſint x vel y vel z, &c. ad $n \times m \times o \times l \times$ &c. dimenſiones; ſin aliter vero non. In æquationibus $A = o$, $B = o$, $C = o$, &c. pro literis y, z, v, &c. ſcribantur reſpective αx, βx, γx, &c.; & reſultabunt (h') æquationes, in quibus haud continetur quantitas (x), quæ $(h' - 1)$ ſolummodo habent incognitas quantitates α, β, γ, &c.; tum ex methodo communes diviſores detegendi inveniantur correſpondentes valores incognitarum quantitatum α, β, γ, &c.; ſi modo ulli ſint: ſi nulli ſint correſpondentes valores prædicti; tum aſcendet æquatio, cujus radix eſt x vel y vel z, &c. ad $n \times m \times o \times l \times$ &c. dimenſiones; ſin aliter vero non.

Si vero dentur (K') diverſæ correſpondentes rationes prædictæ inter incognitas quantitates $(x, y, z$, &c.) eædem in prioribus ac in poſterioribus $(h' - 1)$ æquationibus, i. e. K' correſpondentes valores e ſingulis quantitatibus α, β, γ, &c. tum æquationes reſultantes neceſſario haud aſcendent ad plures quam $n \times m \times o \times l \times$ &c. $- K'$ dimenſiones.

Cor. 4. Si æquatio reſultans, cujus radix eſt incognita quantitas $(x$ vel $y)$, dimenſiones $(n \times m)$ habeat: ſumma radicum ſolummodo pendet e terminis duarum datarum æquationum, in quibus incognitarum quantitatum (x & y) dimenſiones inveniuntur n & $n-1$, m & $m-1$ reſpective, i. e. pendet e terminis in priori æquatione

$$My^n + bxy^{n-1} + ex^2y^{n-2} + kx^3y^{n-3} + \&c.$$
$$\& \ ay^{n-1} + dxy^{n-2} + hx^2y^{n-3} + \&c.$$

& terminis in poſteriori æquatione

$$Ny^m + Bxy^{m-1} + Ex^2y^{m-2} + Kx^3y^{m-3} + \&c.$$
$$Ay^{m-1} + Dxy^{m-2} + Hx^2y^{m-3} + \&c.$$

Summa rectangulorum ſub ſingulis binis pendet e terminis n, $n-1$ & $n-2$ dimenſionum in unâ æquatione; & terminis m, $m-1$ & $m-2$ dimenſionum in alterâ; i. e. terminis in priori æquatione

$$My^n + bxy^{n-1} + ex^2y^{n-2} + kx^3y^{n-3} + \&c.$$
$$ay^{n-1} + dxy^{n-2} + hx^2y^{n-3} + \&c.$$
$$\& \ cy^{n-2} + gxy^{n-3} + \&c.$$

terminis

terminis autem in poſteriori æquatione

$$Ny^m + Bxy^{m-1} + Ex^2y^{m-2} + Kx^3y^{m-3} + \&c.$$
$$Ay^{m-1} + Dxy^{m-2} + Hx^2y^{m-3} + \&c.$$
$$\& \; Cy^{m-2} + Gxy^{m-3} + \&c.$$

& ſic deinceps de ſummâ contentorum ſub ſingulis ternis, &c.

Et ſimiliter, ſi modo dentur tres vel plures æquationes, totidem incognitas quantitates (x, y, z, v, &c.) habentes; & æquatio, cujus radix eſt (x vel y vel z vel v, &c.) dimenſiones $n \times m \times o \times l \times$ &c. habeat; tum ſumma ejus radicum pendet e terminis n & $n-1$, m & $m-1$, o & $o-1$, l & $l-1$, &c. dimenſionum in reſpectivis æquationibus: ſumma quadratorum e ſingulis ejus radicibus pendet e terminis n, $n-1$ & $n-2$; m, $m-1$ & $m-2$; o, $o-1$ & $o-2$; &c. dimenſionum in reſpectivis æquationibus; & ſic deinceps.

Si æquatio, cujus radix eſt x vel y vel z, &c. haud plures habeat quam $n \times m \times o \times l \times$ &c. $-r$ dimenſiones; tum ſumma ejus radicum ſolummodo pendet e terminis n, $n-1$, ... $n-r-1$; m, $m-1$, ... $m-r-1$; o, $o-1$, ... $o-r-1$; l, $l-1$, ... $l-r-1$; &c. dimenſionum in reſpectivis æquationibus.

Cor. 5. Sint duæ æquationes n & m dimenſionum

$$(P) \; \overline{ax^r + bx^{r-1} + cx^{r-2} + \&c.} \times y^{n-r} + \overline{dx^{r+1} + ex^r + \&c.} \times y^{n-r-1} + \overline{fx^{r+2} + gx^{r+1} + hx^r + \&c.}\, y^{n-r-2} + \&c. = 0.$$

$$\& \; (Q) \; \overline{Ax^s + Bx^{s-1} + \&c.} \times y^{m-s} + \overline{Dx^{s+1} + Ex^s + \&c.} \times y^{m-s-1} + \&c. = 0.$$

Ita reducantur duæ æquationes in unam, ut exterminetur incognita quantitas (y); reſultans æquatio haud plures habere poteſt dimenſiones quam $(nm - sr)$: facile e prædictis conſtat, utrum totidem vel minores habeat dimenſiones, necne. *Demonſtratio.*

1. Sit æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0$, cujus radices ſint α, β, γ, δ, &c.; tum ex caſ. 3. prob. 3. conſtat in ſingulis terminis aggregati e ſingulis hujuſcemodi contentis $\alpha^a \beta^b \gamma^c \delta^d$ &c., ubi a, b, c, d, &c. integri ſint numeri; & a major ſit quam b, b quam c, c quam d, &c. haud plures quam (a) dari dimenſiones coefficientium (p, q, r, s, t, &c.); poſito quod ſingulæ coefficientes (p, q, r, s, t, &c.)

unam ſolummodo habent dimenſionem, & conſequenter dimenſiones contenti $p^\mu q^\nu r^\pi s^\varrho$ &c. ſunt $\mu + \nu + \pi + \varrho +$ &c.: plures enim quam factores (a) in ſingulis terminis prædicti aggregati e ſingulis contentis $\alpha^a \beta^b \gamma^c \delta^d$ &c. contineri nequeunt.

2. Sit æquatio $x^n - \frac{p}{b} x^{n-1} + \frac{q}{b} x^{n-2} - \frac{r}{b} x^{n-3} +$ &c. $= 0$, quoniam haud dantur plures quam (a) factores coefficientium $\left(\frac{p}{b}, \frac{q}{b}, \frac{r}{b}, \frac{s}{b}, \text{\&c.}\right)$ in ſingulis terminis aggregati e ſingulis contentis $\alpha^a \beta^b \gamma^c \delta^d$ &c.; ergo in denominatore aggregati prædicti haud dantur plures quam (a) dimenſiones quantitatis (b).

3. In prædicto aggregato inveniendo numerus dimenſionum e ſingulis contentis $p^\mu q^\nu r^\pi s^\varrho$ &c. in eo exiſtentibus, i. e. $\mu + 2\nu + 3\pi +$ &c. ſemper idem erit.

4. Sint dimenſiones quantitatum $p, q, r,$ &c. majores quam denominatoris (b) per unam, duas, tres, &c.; tum, ſi ſumma $\mu + 2\nu + 3\pi +$ &c. eadem maneat, contentum $\frac{p^\mu}{b^\mu} \times \frac{q^\nu}{b^\nu} \times \frac{r^\pi}{b^\pi} \times$ &c. ad communem denominatorem (b^a) reductum, eaſdem habet dimenſiones.

Hìc per dimenſiones quantitatum $p, q, r, s,$ &c. & b intelligo maximas dimenſiones incognitæ quantitatis in iis contentas.

4. Si $p, q, r, s,$ &c. & b prædictas habeant dimenſiones, i. e. dimenſiones quantitatum $p, q, r,$ &c. majores ſint quam dimenſiones quantitatis (b) per unam, duas, tres, &c. e. g. Sint duæ æquationes $(ax^r + bx^{r-1} +$ &c.$)\, y^{n-r} + (dx^{r+1} +$ &c.$)\, y^{n-r-1} + \ldots Fx^n + Gx^{n-1} +$ &c. $= 0$, cujus radices ſint $\alpha, \beta, \gamma, \delta,$ &c. & $(Ax^s + Bx^{s-1} +$ &c.$)\, y^{m-s} + (Dx^{s+1} +$ &c.$)\, y^{m-s-1} +$ &c. $= 0$; tum dimenſiones e ſingulis terminis in contento $(Ax^s \alpha^{m-s} + Dx^{s+1} \alpha^{m-s-1} +$ &c.$) \cdot (Ax^s \beta^{m-s} + Dx^{s+1} \beta^{m-s-1} +$ &c.$) \cdot (Ax^s \gamma^{m-s} + Dx^{s+1} \gamma^{m-s-1} +$ &c.$) \cdot$ &c. ad communem denominatorem $b^{m-s} = (ax^r + bx^{r-1} +$ &c.$)^{m-s}$ reductis, ſemper eædem evadent. Erit contentum $(Ax^s + Bx^{s-1} +$ &c.$)\, \alpha^{m-s} \times (Ax^s + Bx^{s-1} +$ &c.$)\, \beta^{m-s} \times (Ax^s + Bx^{s-1} +$ &c.$)\, \gamma^{m-s} \times$ &c. $= (Ax^s +$

Bx

$Bx^{s-1}+$ &c.$)^{n-r}\times\alpha^{m-s}\beta^{m-s}\gamma^{m-s}$ &c. $=\pm(Ax^{s}+Bx^{s-1}+$ &c.$)^{n-r}\times$ $\left(\frac{Fx^{n}+Gx^{n-1}+\&c.}{ax^{r}+bx^{r-1}+\&c.}\right)^{m-s}$; in cujus numeratore dimenſiones quantitatis (x); & conſequenter dimenſiones æquationis, cujus radix eſt (x), haud majores erunt quam $s\times\overline{n-r}+n\times\overline{m-s}=n\times m-rs$. q. e. d.

Cor. Sint $n-r$ & $m-s$ maximæ dimenſiones quantitatis (y) in reſpectivis æquationibus $P=0$ & $\mathcal{Q}=0$ contentæ; tum æquatio, cujus radix eſt incognita quantitas $(x$ vel $y)$, haud aſcendet ad majores dimenſiones quam $nm-rs$.

Ex conſimilibus principiis probari poteſt; ſi modo $n-r$ & $m-s$, $n-t$ & $n-u$ ſint maximæ dimenſiones quantitatum y & x reſpective in æquationibus $P=0$ & $\mathcal{Q}=0$; æquationem, cujus radix eſt x vel y, haud plures habere dimenſiones quam $n\times m-r\times s-t\times u$.

Cor. 6. 1. Sint tres æquationes $A=0$, $B=0$ & $C=0$ tres incognitas quantitates x, y & z habentes; quarum dimenſiones ſint reſpective n, m & o; primum ita reducantur duæ æquationes $A=0$ & $B=0$ in unam, ut exterminetur quantitas x; reſultat æquatio $P=0$ haud plurium quam $n\times m$ dimenſionum duas incognitas quantitates y & z involvens: & ſic reducantur duæ æquationes $B=0$ & $C=0$ ut exterminetur x, & reſultat æquatio haud plurium quam $n\times o$ dimenſionum: deinde ita reducantur duæ æquationes haud plurium quam $n\times m$ & $n\times o$ dimenſionum in unam, ut exterminetur y vel z; & reſultat æquatio haud plurium quam $n^{2}\times mo$: æquationes per hanc methodum reſultantes hand plures habent quam vel $nmo\times n$ vel $nmo\times m$ vel $nmo\times o$ dimenſiones.

2. In genere ſint (α) æquationes totidem (α) incognitas quantitates $(x, y; z; v; w,$ &c. $W)$ habentes; quarum dimenſiones ſint reſpective n, m, o, r, s, t, &c., quarum n minor ſit quam m, m quam o, o quam r, r quam s; &c. reduci poſſunt hæ (α) æquationes in $(\alpha-1)$ æquationes $\left(\alpha\cdot\frac{\alpha-1}{2}\right)$ diverſis modis, ita ut exterminetur incognita quantitas (x); quarum dimenſiones haud plures erunt reſpective quam $n\times m$, $n\times o$, $m\times o$, $n\times r$, $m\times r$, $o\times r$, &c.

 Præ-

Prædictæ $(\alpha - 1)$ æquationes reduci poſſunt in $(\alpha - 2)$ æquationes, ita ut exterminetur incognita quantitas (y) diverſis $\left(\alpha \cdot \overline{\alpha - 1} \cdot \frac{\alpha - 2}{2} + \alpha \cdot \frac{\alpha - 1}{2} \cdot \frac{\alpha - 2}{3} \cdot \frac{\alpha - 3}{4} = \alpha \cdot \frac{\alpha - 1}{2} \cdot \frac{\alpha - 2}{3} \cdot \frac{\alpha + 9}{4}\right)$ modis, quarum dimenſiones haud plures erunt quam $n^2 m o$, $n^2 m r$, $n^2 o r$, &c., $m^2 n o$, $m^2 n r$, $m^2 o r$, &c., &c., $n m o r$, $n m o s$, $n r o s$, &c. reſpective.

Hæ vero $(\alpha - 2)$ æquationes reduci poſſunt in $(\alpha - 3)$ æquationes, ita ut exterminetur incognita quantitas (z), quarum dimenſiones haud plures erunt quam $n^4 m^2 o r$, $m^4 n^2 o r$, $o^4 m^2 n r$, &c.; $n^3 m^3 o r$, $o^3 r^3 n m$, &c., $n^4 m o r s$, $m^4 n m o r$, &c., $n^3 m o r s t$, &c., &c.: denique hæ (α) æquationes reduci poſſunt in unam, ita ut exterminentur omnes $(x, y, z$, &c.) præter unam (W) incognitæ quantitates; æquatio reſultans haud minores habet dimenſiones quam $n^{2^{\alpha-2}} \times m^{2^{\alpha-3}} \times o^{2^{\alpha-4}} \times r^{2^{\alpha-5}} \times s^{2^{\alpha-6}} \times$ &c. ultimus factor hanc habet formulam $k^{2^{-1}}$, pro quo ſcribatur k. Coefficientes terminorum datarum æquationum, quorum dimenſiones incognitarum quantitatum in iis contentarum ſint maximæ, ita conſtitui poſſunt, ut deſtruant quoſdam primos terminos æquationis reſultantis.

Cor. Quamvis reſultans æquatio ex hâc methodo reductionis aſcendat ad majores dimenſiones incognitæ quantitatis, quam exigant datæ æquationes; e methodo rationales diviſores inveniendi ſemper deprimi poteſt in æquationem, quæ nullas alias radices habeat, præter eas, quas exigunt datæ æquationes.

4. Sint duæ datæ æquationes duas incognitas quantitates x & y habentes, inveniantur e notis regulis (ſi tales dentur regulæ) ex unâ datâ æquatione omnes diverſi valores unius incognitæ quantitatis (x) terminis vero alterius.

Subſtituantur hi diverſi valores pro incognitâ quantitate (x) in alterâ datâ æquatione, & ducantur quantitates exinde reſultantes continuo in ſeſe; ita reducuntur duæ æquationes in unam ut exterminetur incognita quantitas (x).

u Ex. 1.

Ex. 1. Sint duæ æquationes $x^2 - 2ax + a^2 - b = 0$, & $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \&c. = 0$; duæ radices prioris æquationis erunt $x = a \pm \sqrt{b}$; ſubſtituantur hæ radices pro ſuô valore (x) in æquatione $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0$, & ducantur duæ quantitates reſultantes in ſeſe, & ſcribantur

$$A = \overline{a+b^{\frac{1}{2}}}^{n} + \overline{a-b^{\frac{1}{2}}}^{n} - p\times\overline{\overline{a+b^{\frac{1}{2}}}^{n-1} + \overline{a-b^{\frac{1}{2}}}^{n-1}} + q\times\overline{\overline{a+b^{\frac{1}{2}}}^{n-2} + \overline{a-b^{\frac{1}{2}}}^{n-2}} + \&c.$$

$$B = \overline{a+b^{\frac{1}{2}}}^{n} - \overline{a-b^{\frac{1}{2}}}^{n} - p\times\overline{\overline{a+b^{\frac{1}{2}}}^{n-1} - \overline{a-b^{\frac{1}{2}}}^{n-1}} + q\times\overline{\overline{a+b^{\frac{1}{2}}}^{n-2} - \overline{a-b^{\frac{1}{2}}}^{n-2}} + \&c.$$

æquatio reſultans erit $A^2 - B^2 = 0$.

Ex. 2. Sint duæ æquationes $x^3 - 3ax^2 + \overline{3a^2+b}\,x - a^3 - ba + c = 0$; & $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - tx^{n-5} + \&c. = 0$; tres autem radices prioris æquationis erunt

$$a + \sqrt[3]{-\tfrac{1}{2}c + \sqrt{\tfrac{1}{4}c^2 + \frac{b^3}{27}}}\,(\alpha) + \sqrt[3]{-\tfrac{1}{2}c + \sqrt[2]{\tfrac{1}{4}c^2 + \frac{b^3}{27}}}\,(\beta),\ a - \frac{1+\sqrt{-3}}{2}$$

$$\alpha - \frac{1-\sqrt{-3}}{2}\beta,\ a - \frac{1-\sqrt{-3}}{2}\alpha - \frac{1+\sqrt{-3}}{2}\beta;$$ ſubſtituantur

hæ tres radices pro ſuo valore (x) in æquatione $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = 0$, & ducantur tres quantitates reſultantes continuo in ſeſe; & ſupponantur

$$A = \overline{a+\alpha+\beta}^{n} + \overline{a - \frac{1+\sqrt{-3}}{2}\alpha - \frac{1-\sqrt{-3}}{2}\beta}^{n} + \overline{a - \frac{1-\sqrt{-3}}{2}}$$

$$\overline{\alpha - \frac{1+\sqrt{-3}}{2}\beta}^{n} - p\times\overline{\overline{a+\alpha+\beta}^{n-1} + \overline{a - \frac{1+\sqrt{-3}}{2}\alpha - \frac{1-\sqrt{-3}}{2}\beta}^{n-1} +}$$

$$\overline{\overline{a - \frac{1-\sqrt{-3}}{2}\alpha - \frac{1+\sqrt{-3}}{2}\beta}^{n-1}} + q\times\overline{a+\alpha+\beta}^{n-2} + \&c.\ \&c.$$

$$B = \overline{a+\alpha+\beta}^{n} - \frac{1+\sqrt{-3}}{2}\times\overline{a - \frac{1+\sqrt{-3}}{2}\alpha - \frac{1-\sqrt{-3}}{2}\beta}^{n} - \frac{1-\sqrt{-3}}{2}$$

$$\times\overline{a - \frac{1-\sqrt{-3}}{2}\alpha - \frac{1+\sqrt{-3}}{2}\beta}^{n} - p\times\overline{a+\alpha+\beta}^{n-1} - \frac{1+\sqrt{-3}}{2}$$

$a-$

Sint duæ æquationes formulæ ſubſequentis $(az^n + bz^{n-1} + cz^{n-2} + dz^{n-3} + \&c.)\, y^r + (a'z^{n-1} + b'z^{n-2} + c'z^{n-3} + d'z^{n-4} + \&c.)\, y^{r-1} + (a''z^{n-2} + b''z^{n-3} + c''z^{n-4} + \&c.)\, y^{r-2} + (a'''z^{n-3} + b'''z^{n-4} + \&c.)\, y^{r-3} + \&c. = o$, & $(Az^m + Bz^{m-1} + Cz^{m-2} + Dz^{m-3} + \&c.)\, y^s + (A'z^{m-1} + B'z^{m-2} + C'z^{m-3} + D'z^{m-4} + \&c.)\, y^{s-1} + (A''z^{m-2} + B''z^{m-3} + C''z^{m-4} + \&c.)\, y^{s-2} + \&c. = o$; in iis pro z ſcribatur $\frac{x}{y}$, & transformabuntur hæ duæ æquationes in duas alias relationes inter x & y reſpective exprimentes, quarum dimenſiones erunt n & m reſpective.

4. 1. Sint duæ æquationes $Ay^m + By^{m-1} + Cy^{m-2} + \&c. = o$, & $ay^n + by^{n-1} + cy^{n-2} + \&c. = o$; aſſumantur duæ quantitates $Py^{m-1} + Qy^{m-2} + Ry^{m-3} + \&c.$ & $py^{n-1} + qy^{n-2} + ry^{n-3} + \&c.$

Ducatur data æquatio $Ay^m + By^{m-1} + Cy^{m-2} + \&c. = o$ in aſſumptam quantitatem $py^{n-1} + qy^{n-2} + ry^{n-3} + \&c. = o$, & ſupponatur ſingulus rectanguli reſultantis $(Apy^{m+n-1} + \overline{Aq + Bp} \times y^{m+n-2} + \overline{Ar + Bq + Cp} \times y^{m+n-3} + \&c. = o)$ terminus (ultimo excepto) æqualis ejus correſpondenti termino in æquatione e multiplicatione datæ æquationis $ay^n + by^{n-1} + cy^{n-2} + \&c.$ in aſſumptam quantitatem $Py^{m-1} + Qy^{m-2} + Ry^{m-3} + \&c.$ i. e. $aPy^{m+n-1} + \overline{aQ + bP}\, y^{m+n-2} + \overline{aR + bQ + cP}\, y^{m+n-3} + \&c.$ unde $Ap - aP = o$ & $Aq + Bp - aQ - bP = o$, &c. ita reducantur ſimplices $(n + m)$ æquationes reſultantes in unam, ut exterminentur incognitæ quantitates P, Q, R, &c., p, q, r, &c.; & reſultet æquatio $\pi = o$; tum ita reducuntur duæ datæ æquationes in unam $\pi = o$, ut exterminetur incognita quantitas (y).

Si ducantur duæ æquationes $Ay^m + By^{m-1} + \&c. = o$ & $ay^n + by^{n-1} + \&c. = o$ in duas æquationes $py^n + qy^{n-1} + \&c. = o$ & $Py^m + Qy^{m-1} + \&c. = o$, & fiant duæ æquationes reſultantes inter ſe æquales; evadant duæ æquationes $py^n + qy^{n-1} + \&c. = o$ & $Py^m + Qy^{m-1} + \&c. = o$ eædem ac duæ æquationes $ay^n + by^{n-1} + \&c. = o$ & $Ay^m + By^{m-1} + \&c. = o$: conſimilia etiam dici poteſt de his multiplicationibus, ſi modo aſſumptæ æquationes majores habeant quam n & m dimenſiones reſpective.

4. 2. Hæc

4. 2. Hæc methodus etiam applicari poteſt ad tres vel plures (b) æquationes totidem incognitas quantitates habentes.

Facile enim conſtat ex his duabus obſervationibus numerus dimenſionum, quas habere debet incognita quantitas, i. e. radix in aſſumptis æquationibus: nempe quod ſingula æquatio in ſuam correſpondentem æquationem aſſumptam ducta ſemper eaſdem conficient dimenſiones; & quod dimenſiones æquationum aſſumptarum ſimulque adjectarum æquales vel proxime majores erunt quam differentia inter prædictam ſummam & numerum $(b-1)$ datarum æquationum unitate diminutum.

In hoc enim caſu tot vel plures erunt incognitæ quantitates in aſſumptis æquationibus, quot reſultant ſimplices æquationes.

5. Eſt etiam alia methodus transformandi duas æquationes in unam, ita ut incognita quantitas exterminetur.

Per methodum ſerierum infinitarum inveniantur ſingulæ radices incognitæ quantitatis exterminandæ in unâ datâ æquatione, quibus pro ſuo valore in alterâ ſubſtitutis, & contento ex quantitatibus reſultantibus in ſeſe multiplicatis nihilo æquale ſuppoſito; reſultat quæſita æquatio.

Ex. Sint duæ æquationes $y^2 - \overline{4x+2}\,xy + 3x^2 + 4x + 4 = 0$ & $y^2 - 3xy + 4x^2 + 6 = 0$ eas in unam transformare, ita ut exterminetur incognita quantitas (y).

Incognitæ quantitatis (y) duæ radices in priori æquatione per infinitas ſeries inveniuntur, viz.

$$y = x + 1 + \frac{3}{2x} + \frac{9}{8x^3} + \&c.$$

$$y = 3x + 1 - \frac{3}{2x} - \frac{9}{8x^3} - \&c.$$

quibus radicibus pro (y) in poſteriori æquatione ſubſtitutis, reſultant quantitates

$2x^2$

$$2x^2 - x + \frac{11}{2} + \frac{3}{x} + \frac{9}{8x^2} + \&c.$$

$$\& \; 4x^2 + 3x + \frac{5}{2} - \frac{3}{x} - \frac{9}{8x^2} - \&c.$$

Ducantur hæ duæ quantitates reſultantes in ſeſe; rectangulum ſub iis contentum, erit

$$8x^4 + 2x^3 + 24x^2 + 20x + 28 = 0,$$

quod nihilo fiat æquale, & erit æquatio quæſita

$$8x^4 + 2x^3 + 24x^2 + 20x + 28 = 0;$$

cujus radices omnes erunt impoſſibiles.

Omnia corollaria e priori methodo deducta ex hâc etiam inveſtigari poſſunt.

5. 2. Sint tres vel plures (b) æquationes totidem (b) incognitas quantitates $(x, y, z, \&c.\, w)$ habentes: ita transformare has æquationes in unam per methodum ſerierum infinitarum, ut omnes præter unam (w) incognitæ quantitates exterminentur.

Ex quibuſque $(b-1)$ æquationibus inveniantur omnes correſpondentes valores quantitatum $x, y, z,$ &c. in ſerie progrediente ſecundum dimenſiones quantitatis w; i. e. ſint

$$\begin{aligned} x &= a w^{\alpha} + b w^{\alpha-\beta} + c w^{\alpha-2\beta} + \&c. \\ y &= a' w^{\alpha'} + b' w^{\alpha'-\beta'} + c' w^{\alpha'-2\beta'} + \&c. \\ z &= a'' w^{\alpha''} + b'' w^{\alpha''-\beta''} + c'' w^{\alpha''-2\beta''} + \&c. \\ \&c. &= \&c. \end{aligned}$$

$$\begin{aligned} \& \; x &= A w^{\pi} + B w^{\pi-\varrho} + C w^{\pi-2\varrho} + \&c. \\ y &= A' w^{\pi'} + B' w^{\pi'-\varrho'} + C' w^{\pi'-2\varrho'} + \&c. \\ z &= A'' w^{\pi''} + B'' w^{\pi''-\varrho''} + C'' w^{\pi''-2\varrho''} + \&c. \\ \&c. &= \&c. \qquad \&c. \end{aligned}$$

Subſtituantur correſpondentes valores quantitatum $(x, y, z, \&c.)$ pro ſuis valoribus in reliquâ æquatione; & ſint ſeries exinde reſultantes $P, Q, R, S,$ &c.; ducantur hæ ſeries in ſeſe, quantitas reſultans $P \times Q \times R \times S \; \&c. = 0$ erit æquatio, cujus radix eſt w.

Dimenſiones, ad quas aſſurgit æquatio, cujus radix eſt w, deduci poſſunt e dimenſionibus ad quas aſſurgunt numeratores & denominatores quantitatum reſultantium P, Q, R, S, &c. numeratorum vero & denominatorum prædictarum quantitatum (P, Q, R, S, &c.) dimenſiones ex terminis $(b-1)$ prædictarum æquationum, quorum dimenſiones ſunt maximæ, iis proximæ, &c. erui poſſunt.

Si modo æquatio, cujus radix eſt w, haud ſit finita; tum e methodo in ſubſequente capite traditâ inveniri poteſt finita fractio prædictæ ſeriei vel æquationi æqualis, cujus fractionis ad minimos terminos reductæ numerator erit æquatio quæſita.

E quantitatibus in terminos incognitarum quantitatum maximarum dimenſionum ductis erui poteſt denominator fractionis quæſitæ: dividatur æquatio, cujus radix eſt w, per denominatorem inventum; reſultat infinita æquatio quæſita.

5. 2. Sint duæ vel plures (b) æquationes $A=o$, $B=o$, $C=o$, &c. quarum dimenſiones ſint $(n, m, l, k$, &c.) reſpective, duas vel plures (b) incognitas quantitates $(x, y, z$, &c.) habentes; in quibus $A=o$, $B=o$, $C=o$, &c. ſint a, b, c, d, &c. maximæ dimenſiones, ad quas aſcendunt termini, in quibus continentur $(b-1)$ incognitæ quantitates $(x, z, v$, &c.); tum dimenſiones ad quas aſcendere poteſt æquatio, cujus radix eſt y, haud majores erunt quam maximum ſubſequens contentum $n \times b \times c \times d \times$ &c., $m \times a \times c \times d \times$ &c., $l \times a \times b \times d \times$ &c., $k \times a \times b \times c \times$ &c., &c.

6. Eſt etiam alia methodus hoc problema reſolvendi.

Sint duæ æquationes

$$A'y^n + \overline{a+bx}\, y^{n-1} + \overline{c+dx+ex^2}\, y^{n-2} + \&c. = o,$$

$$\& \; Ny^m + \overline{A+Bx}\, y^{m-1} + \overline{C+Dx+Ex^2}\, y^{m-2} + \&c. = o,$$

eas in unam transformare, ita ut exterminetur incognita quantitas (x).

Ducantur duæ datæ æquationes $A'y^n + \overline{a+bx}\, y^{n-1} + \&c. = o$, $Ny^m + \overline{A+Bx}\, y^{m-1} + \&c. = o$ in duas aſſumptas $m'y^{mn-n} + \overline{p+qx}\, y^{mn-n-1} + \overline{r+sx+tx^2}\, y^{mn-n-2} + \&c. = o$, & $My^{mn-m} + \overline{P+Qx} \times y^{mn-m-1} +$

R + S

$\overline{R+Sx+Tx^2} \times y^{mn-m-2} +$ &c. $= o$, reſpective; fiant æquationes reſultantes inter ſe æquales, & ita capiantur coefficientes aſſumptarum æquationum ut evaneſcant omnes termini reſultantis æquationis; iis exceptis, in quibus ſolummodo inveniuntur incognita quantitas (y), & ejus poteſtates.

Hoc perficitur ſimplicium æquationum ope.

Semper tot vel plures invenientur incognitæ quantitates, quot reſultant ſimplices æquationes.

6. 2. $N(b)^n$ denotet numerum terminorum in algebraicâ æquatione n dimenſionum, quæ continet (b) incognitas quantitates $(x, y, z, v, w,$ &c.)

1. Erit per theor. noſt. $N(b)^n = \overline{n+1} \cdot \frac{n+2}{2} \cdot \frac{n+3}{3} \cdots \frac{n+b}{b}$.

2. Numerus terminorum æquationis (n) dimenſionum, qui diviſibiles ſunt per x^p, idem erit ac numerus terminorum æquationis dimenſionum $(n-p)$, i.e. per præcedentem notationem $= N(b)^{n-p}$: hinc numerus terminorum in prædictâ æquatione dimenſionum (n) diviſibilum vel per x^p, vel y^q, vel z^r, vel v^s, vel w^t, &c. erit $=$

$$\begin{array}{llllll}
+N(b)^{n-p} & -N(b)^{n-p-q} & +N(b)^{n-p-q-r} & -N(b)^{n-p-q-r-s} & +N(b)^{n-p-q-r-s-t} & -\&c. \\
+N(b)^{n-q} & -N(b)^{n-p-r} & +N(b)^{n-p-q-s} & -N(b)^{n-p-q-r-t} & +\ \&c. & -\ \&c. \\
+N(b)^{n-r} & -N(b)^{n-p-s} & +N(b)^{n-p-r-s} & \&c. & \&c. & \\
+N(b)^{n-s} & -N(b)^{n-q-s} & +N(b)^{n-q-r-s} & \&c. & \&c. & \\
\&c. & \&c. & \&c. & & &
\end{array}$$

3. Sint duæ æquationes $(n$ & $m)$ dimenſionum duas incognitas quantitates $(x$ & $y)$ habentes; ducantur hæ æquationes reſpective in duas æquationes $(nm-n$ & $nm-m)$ dimenſionum (viz. $A = o$ & $B = o$) prædictas incognitas quantitates $(x$ & $y)$ habentes; tum ex utrâque multiplicatione reſultat æquatio dimenſionum $(n \times m)$: fiant omnes termini correſpondentes duarum reſultantium æquationum inter ſe æquales, iis exceptis, in quibus ſolummodo inveniuntur incognita quantitas (y) & ejus poteſtates: numerus horum terminorum

 erit

erit $= nm \cdot \frac{nm+3}{2} - nm$; ſed hic numerus æqualis eſſe debet numero diverſarum independentium coefficientium in duabus æquationibus $A=0$ & $B=0$ contentarum; quarum numerus per caſum præcedentem erit

$$\begin{matrix} N(2)^{nm-n} - N(2)^{nm-n-m} & \overline{nm-n} \times \frac{nm-n+3}{2} - \overline{nm-n-m} \,. \\ & = \\ N(2)^{nm-m} & \overline{nm-m} \times \frac{nm-m+3}{2} \end{matrix}$$

$\frac{nm-n-m+3}{2} = nm \times \frac{nm+3}{2} - nm$; unde æquatio, cujus radices ſunt x vel y, haud plures habet dimenſiones quam $n \times m$.

3. 2. Sint (h) æquationes dimenſionum $(n, m, l, k,$ &c.$)$ reſpective, totidem (h) incognitas quantitates involventes; ducantur hæ (h) æquationes in (h) æquationes generaliter aſſumptas, quarum dimenſiones ſunt reſpective $n \times m \times l \times k \times$ &c. $- n$, $n \times m \times l \times k \times$ &c. $- m$, $n \times m \times l \times k \times$ &c. $- l$, $n \times m \times l \times k \times$ &c. $- k$, &c. eaſdem incognitas quantitates involventes; addantur æquationes reſultantes, & fiant omnes termini correſpondentes reſultantis æquationis $(n \times m \times l \times k$ &c.$)$ dimenſionum nihilo reſpective æquales, iis exceptis, in quibus ſolummodo inveniuntur incognita quantitas (y) & ejus poteſtates: numerus horum terminorum & conſequenter reſultantium æquationum erit $= \overline{P+1} \times \frac{P+2}{2} \times \frac{P+3}{3} \ldots \frac{P+h}{h} - P = N(h)^{P} - P$, ubi $P = n \times m \times l \times k \times$ &c.; ſed hic numerus debet eſſe æqualis numero diverſarum independentium coefficientium ex æquationibus dimenſionum $P-n$, $P-m$, $P-l$, $P-k$, &c. in reſultante æquatione exortarum; unde $\overline{P+1} \times \frac{P+2}{2} \cdot \frac{P+3}{3} \ldots$

$\frac{P+h}{h} - P =$

$+P$

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$$\overline{\quad\quad} -n+1\times\frac{P-n+2}{2}\ldots\frac{P-n+b}{b}-\overline{P-n-m+1}\times\frac{P-n-m+2}{2}\ldots\frac{-n-m-l-k+b}{b}+\&c.$$

$$\overline{\quad\quad} -m+1\times\frac{P-m+2}{2}\ldots\frac{P-m+b}{b}-\overline{P-n-l+1}\times\frac{P-n-l+2}{2}\ldots$$

$$\overline{\quad\quad} -l+1\times\frac{P-l+2}{2}\ldots\frac{P-l+b}{b}-\overline{P-m-l+1}\times\frac{P-m-l+2}{2}\ldots$$

$$\overline{\quad\quad} -k+1\times\frac{P-k+2}{2}\ldots\frac{P-k+b}{b}-\overline{P-n-k+1}\times\frac{P-n-k+2}{2}\ldots,$$

&c. &c.

$$+N(b)^{p-n}-N(b)^{p-n-m}+N(b)^{p-n-m-l}-N(b)^{p-n-m-l-k}+\&$$

$$+N(b)^{p-m}-N(b)^{p-n-l}+N(b)^{p-n-m-k}-\&c.$$

$$+N(b)^{p-l}-N(b)^{p-m-l}+N(b)^{p-n-l-k}\quad\&c.$$

$$+N(b)^{p-k}-N(b)^{p-n-k}+N(b)^{p-m-l-k}\quad\&c.$$

$$+\&c.\quad -\&c.\quad +\&c.$$

e æquatio, cujus radix eſt x vel y vel z, &c. haud plures ha
n $n\times m\times l\times k$ &c. dimenſiones.

vero æquationes datæ (n, m, l, k, &c.) dimenſionum reſpectiv
rales æquationes majorum quam $n\times m\times l\times k$ &c. — n, $n\times m\times$
— m, $n\times m\times l\times k$ &c. — l, $n\times m\times l\times k$ &c. — k, &c.; dimenſion
ntur; tum numerus prædictarum coefficientium (H) major
n numerus independentium æquationum (K); ergo ad libit
ſſumi poſſunt coefficientes, quot ſit differentia $(H-K)$: ſi æq
cujus radix eſt y, plures habeat radices; quam eas, quas nece
exigat problema; tum per methodum ejus rationales diviſc
gendi, ſemper deprimi poteſt æquatio.

ı multis caſibus haud neceſſe eſt, ut datæ æquationes ducantur
lictas generales æquationes dimenſionum $n\times m\times l\times k$ &c. —
$n\times l\times k$ &c. — m, $n\times m\times l\times k$ &c. — l, $n\times m\times l\times k\times$ &c. — k, &
n æquationes particularis formulæ e formulis datarum æquat
ı deductæ; & exinde facilius detegi poteſt æquatio, cujus ra
: vel y vel z, &c.: forſan vero æquationes particularis formı
d aſcendant ad dimenſiones prædictas.

Ducan

Ducantur datæ æquationes in prædictos multiplicatores, & ſi in æquatione $(\pi = o)$ ex aggregato quantitatum reſultantium exortâ continentur quidam termini multiplicatorum e reliquis independentes; vel e quibuſdam terminis multiplicatorum inveniri poſſunt æquationes, quæ deſtruant omnes terminos reſultantis æquationis $\pi = o$ præter eos, in quibus ſolummodo continetur incognita quantitas; tum rejiciantur omnes reliqui multiplicatorum termini.

Ex. 1. Sint (b) æquationes $(n, m, l, k, \&c.)$ dimenſionum reſpective totidem incognitas quantitates $(x, y, z, \&c.)$ involventes; ſint p, q, r, s, &c.; p', q', r', s', &c.; p'', q'', r'', s'', &c.; p''', q''', r''', s''', &c.; &c.; maximæ dimenſiones, ad quas aſcendunt incognitæ quantitates x, y, z, v, &c. in reſpectivis æquationibus $(n, m, l, k, \&c.)$ dimenſionum; tum æquatio, cujus radix eſt x vel y vel z, &c. haud aſcendet ad plures quam $n \times m \times l \times k \times \&c. - (n-p) \times (m-p') \times (l-p'') \times (k-p''') \times \&c. - (n-q) \times (m-q') \times (l-q'') \times (k-q''') \times \&c. - (n-r) \times (m-r') \times (l-r'') \times (k-r''') \times \&c. - (n-s) \times (m-s') \times (l-s'') \times (k-s''') \times \&c. - \&c.$ dimenſiones.

Ex. 2. Iiſdem poſitis ac in præcedente exemplo, dimenſiones quantitatum $(x \,\&\, y)$ ſimul ſumptarum haud ſuperent dimenſiones a, a', a'', a''', &c., in prædictis datis æquationibus reſpective; tum, ſi omnes præter unam (x) exterminentur incognitæ quantitates, reſultans æquatio haud plures habet quam $n \times m \times l \times k \times \&c. - (n-p) \times (m-p') \times (l-p'') \times (k-p''') \times \&c. - (n-q) \times (m-q') \times (l-q'') \times (k-q''') \times \&c. - (n-r) \times (m-r') \times (l-r'') \times (k-r''') \times \&c. - (n-s) \times (m-s') \times (l-s'') \times (k-s''') \times \&c. - \&c. + (n-a) \times (m-a') \times (l-a'') \times (k-a''') \times \&c. - (p+q-a) \times (p'-a') \times (p''-a'') \times (p'''-a''') \times \&c. - (p'+q'-a') \times (p-a) \times (p''-a'') \times (p'''-a''') \times \&c. - (p''+q''-a'') \times (p-a) \times (p'-a') \times (p'''-a''') \times \&c. - (p'''+q'''-a''') \times (p-a) \times (p'-a') \times (p''-a'') \times \&c. - \&c.$ dimenſiones.

7. Sint m, l, k, &c. reſpectivæ dimenſiones $(b-1)$ datarum æquationum, $A = o$, $B = o$, $C = o$, &c., incognitas (b) quantitates $(x, y, z, v, \&c.)$ involventium: ſit æquatio $y^n - py^{n-1} + qy^{n-2} - \&c. = o$, (i) dimenſiones habens, ubi i minor ſit quam m vel l vel k, &c.; etiamque coeffi-

$\overline{\quad\quad} \overline{-n+1} \times \frac{P-n+2}{2} \ldots \frac{P-n+b}{b} - \overline{P-n-m+1} \times \frac{P-n-m+2}{2} \ldots$

$\frac{-n-m-l-k+b}{b} + \&c.$

$\overline{\quad\quad} \overline{-m+1} \times \frac{P-m+2}{2} \ldots \frac{P-m+b}{b} - \overline{P-n-l+1} \times \frac{P-n-l+2}{2} \ldots$

$\overline{\quad\quad} \overline{-l+1} \times \frac{P-l+2}{2} \ldots \frac{P-l+b}{b} - \overline{P-m-l+1} \times \frac{P-m-l+2}{2} \ldots$

$\overline{\quad\quad} \overline{-k+1} \times \frac{P-k+2}{2} \ldots \frac{P-k+b}{b} - \overline{P-n-k+1} \times \frac{P-n-k+2}{2} \ldots,$

&c. &c.

$N(b)^{p-n} - N(b)^{p-n-m} + N(b)^{p-n-m-l} - N(b)^{p-n-m-l-k} + \&$

$N(b)^{p-m} - N(b)^{p-n-l} + N(b)^{p-n-m-k} - \&c.$

$N(b)^{p-l} - N(b)^{p-m-l} + N(b)^{p-n-l-k} \quad \&c.$

$N(b)^{p-k} - N(b)^{p-n-k} + N(b)^{p-m-l-k} \quad \&c.$

&c. — &c. + &c.

æquatio, cujus radix eſt x vel y vel z, &c. haud plures ha
$n \times m \times l \times k$ &c. dimenſiones.

vero æquationes datæ (n, m, l, k, &c.) dimenſionum reſpectiv
ales æquationes majorum quam $n \times m \times l \times k$ &c. — n, $n \times m \times$
— m, $n \times m \times l \times k$ &c. — l, $n \times m \times l \times k$ &c. — k, &c.; dimenſion
itur; tum numerus prædictarum coefficientium (H) major
numerus independentium æquationum (K); ergo ad libit
ſumi poſſunt coefficientes, quot ſit differentia $(H-K)$: ſi æq
ujus radix eſt y, plures habeat radices; quam eas, quas nece
xigat problema; tum per methodum ejus rationales diviſo
endi, ſemper deprimi poteſt æquatio.

multis caſibus haud neceſſe eſt, ut datæ æquationes ducantu
ictas generales æquationes dimenſionum $n \times m \times l \times k$ &c. —
$\times l \times k$ &c. — m, $n \times m \times l \times k$ &c. — l, $n \times m \times l \times k \times$ &c. — k, &
æquationes particularis formulæ e formulis datarum æquat
deductæ; & exinde facilius detegi poteſt æquatio, cujus ra
vel y vel z, &c.: forſan vero æquationes particularis form
aſcendant ad dimenſiones prædictas.

Ducan

Ducantur datæ [illegible]
æquatione ($\pi =$ [illegible]
continentur quidam [illegible]
tes; vel e quibusdam [illegible]
æquationes, quæ [illegible]
$\pi = o$ præter eas [illegible]
tas; tum rejiciantur [illegible] n
Ex. 1. Sint [illegible] a-
totidem incognitas [illegible] B,
s, &c.; p', [illegible] α'',
mæ dimensiones [illegible] $-\beta Q$
&c. in respectu [illegible] &c.:
æquatio [illegible] α', α'',
quam [illegible] π [illegible] $\gamma R +$
× &c. — [illegible] &c. $= C$,
(m — [illegible]
(k — [illegible]

Ex. 2 [illegible] a ſe invicem
[illegible] involventes;
[illegible]; ſint quanti-
[illegible] gnita quantitas
[illegible] $+ b'B + c'C +$
[illegible] æquationes a ſe
[illegible] es incognitarum
[illegible] æquationibus ii-
[illegible]

[illegible] *quantitates* (x, y, z,
[illegible] b B + c C + d D +
[illegible] *contineant, diviſorem*

Sit

coefficientes p, q, r, s, &c. ſint functiones rationales incognitarum $(b-1)$ quantitatum $(x, z, v$, &c.): hujuſce æquationis incognitæ quantitatis (y) ſint radices $\alpha, \beta, \gamma, \delta$, &c.: ducantur $(b-1)$ datæ æquationes $A=o$, $B=o$, $C=o$, &c. reſpective in generales æquationes $m \times l \times k \times$ &c. $-m=P-m$, $P-l$, $P-k$, &c., dimenſionum, & reſultent quantitates $\pi, \varrho, \sigma, \tau$, &c.: ſcribantur $\alpha, \beta, \gamma, \delta$, &c. reſpective pro y in quantitate $\pi+\varrho+\sigma+\tau+$ &c., & reſultent quantitates X, Y, Z, V, &c.; ducantur hæ quantitates reſultantes in ſeſe; & exinde e prob. 5. capit: primi inveniatur contentum $X \times Y \times Z \times V \times$ &c. $=\Phi=o$, in quo (Φ) ſolummodo continentur incognitæ quantitates $(x, z, v$, &c.); deinde fiant coefficientes e ſingulis terminis reſultantis æquationis $(\Phi=o)$ nihilo æquales, iis exceptis, in quibus ſolummodo continetnr incognita quantitas x & ejus poteſtates; plerumque e ſimplicibus æquationibus exinde reſultantibus reduci poſſunt datæ æquationes $(m, l, k$, &c. & $i)$ dimenſionum in unam, ita ut exterminentur omnes præter unam (x) incognitæ quantitates.

Sæpe præſtat, ut datæ æquationes vel ducantur in æquationes haud generales prædictarum dimenſionum $P-m$, $P-l$, $P-k$, &c.; vel in æquationes dimenſionum prædictis inferiorum, quarum formulæ e formulis datarum æquationum innoteſcent.

Hæc methodus reducendi plures (b) æquationes in unam, ita ut $(b-1)$ incognitæ quantitates exterminentur, plerumque haud tam operoſa erit, quam ea in præcedente caſu tradita.

THEOR. XXV.

Sint (b) æquationes $A=o$, $B=o$, $C=o$, $D=o$, &c. incognitas (b) quantitates $(x, y, z$, &c.) involventes; ducantur prædictæ quantitates A, B, C, D, &c., reſpective in quantitates a, a', a'', &c.; b, b', b'', &c.; c, c', c'', &c.; d, d', d'', &c.; in quibus $(a, a'$, &c.; b, b', &c.; c, c', &c.; &c.) haud continetur vel x vel y vel z, &c.; reſultent b æquationes a ſe invicem independentes $aA+bB+cC+dD+$ &c. $=o$, $a'A+b'B+c'C+d'D+$ &c. $=o$, $a''A+b''B+c''C+d''D+$ &c. $=o$,

$= o$, &c.; tum hæ poſteriores quantitates; vel earum termini, qui maximas habent dimenſiones; nullum habebunt communem diviſorem, qui ſit functio quantitatum $(x, y, z$, &c.$)$, quem haud habent priores (b) æquationes.

Si enim quantitates A, B, C, D, &c. communem habeant diviſorem, tum quantitates $aA + bB + cC +$ &c. $= P$, $a'A + b'B + c'C +$ &c. $= Q$, $a''A + b''B + c''C +$ &c. $= R$, &c., eundem habent communem diviſorem; & vice verſâ ſi quantitates P, Q, R, S, &c. communem habeant diviſorem, tum eundem diviſorem habebunt quantitates A, B, C, &c.; ducantur enim quantitates P, Q, R, &c. reſpective in $\alpha, \alpha', \alpha''$, &c.; β, β', β'', &c.; $\gamma, \gamma', \gamma''$, &c., &c.; reſultent quantitates $\alpha P + \beta Q + \gamma R +$ &c., $\alpha' P + \beta' Q + \gamma' R +$ &c., $\alpha'' P + \beta'' Q + \gamma'' R +$ &c., &c.: ex ſimplicibus æquationibus ita deduci poſſunt coefficientes $\alpha, \alpha', \alpha''$, &c.; β, β', β'', &c.; $\gamma, \gamma', \gamma''$, &c.; &c. ut evadant $\alpha P + \beta Q + \gamma R +$ &c. $= A$, $\alpha' P + \beta' Q + \gamma' R +$ &c. $= B$, $\alpha'' P + \beta'' Q + \gamma'' R +$ &c. $= C$, &c. q. e. d.

THEOR. XXVI.

Sint (b) æquationes $A = o$, $B = o$, $C = o$, $D = o$, &c. a ſe invicem independentes (b) incognitas quantitates $(x, y, z$, &c.$)$ involventes; tum, ſi modo a, a', a'', &c.; b, b', b'', &c.; c, c', c'', &c.; &c.; ſint quantitates invariabiles, in quibus haud continetur vel incognita quantitas x vel y vel z, &c.; & $aA + bB + cC +$ &c. $= o$, $a'A + b'B + c'C +$ &c. $= o$, $a''A + b''B + c''C +$ &c. $= o$, &c. ſint (b) æquationes a ſe invicem independentes; erunt radices, i. e. valores incognitarum quantitatum $(x, y, z$, &c.$)$ in his (b) poſterioribus æquationibus iidem ac in prioribus æquationibus.

PROB. XXXII.

Datis (h) *quantitatibus* A, B, C, D, &c. *incognitas quantitates* (x, y, z, &c.) *involventibus*; *invenire utrum ſumma* a A + b B + c C + d D + &c., *ubi litera* a, b, c, d, &c. *nec* x *nec* y *nec* z, &c. *contineant, diviſorem recipiat, necne.*

Sit

Sit quantitas $aA + bB + cC + dD +$ &c. $= px^n + qx^{n-1} + rx^{n-2} \ldots \gamma x^2 + \beta x + \alpha$: inveniantur diviſores quantitatum p & α, qui ſint p' & α'; aſſumantur duæ quantitates $p'x^r + q'x^{r-1} + s'x^{r-2} +$ &c. $\ldots + \beta'x + \alpha'$ & $\frac{p}{p'}x^{n-r} + Qx^{n-r-1} +$ &c. $+ Lx + \frac{\alpha}{\alpha'}$ pro diviſoribus quæſitis; ducantur hi duo diviſores quæſiti in ſeſe, & quantitatis reſultantis $px^n + \frac{p}{p'} \times q' + p'Qx^{n-1} + \ldots + \alpha'L + \beta' \times \frac{\alpha}{\alpha'}x + \alpha$ omnes termini in eaſdem dimenſiones incognitarum quantitatum (x, y, z, &c.) ducti fiant æquales eorum correſpondentibus terminis quantitatis $px^n + qx^{n-1} + \ldots + \beta x + \alpha$; ex æquationibus reſultantibus conſtant formulæ quantitatum p', q', β', α', Q, L, &c. & exinde per methodum communem diviſores inveniendi detegi poteſt, annon poſſibile ſit: ſi poſſibile ſit, tum dividi poteſt data quantitas in duos ejus diviſores, ſin aliter diviſores haud recipiet.

Et ſimiliter detegi poſſunt diviſores quarumcunque datarum quantitatum.

PROB. XXXIII.

Datis duabus vel pluribus (h) *æquationibus* A = o, B = o, C = o, &c. *duas vel plures incognitas quantitates habentibus; etiamque duabus vel pluribus* (h) *æquationibus* P = o, Q = o, R = o, &c. *duas vel plures incognitas quantitates habentibus; invenire, utrum omnes radices incognitarum quantitatum priorum æquationum* A = o, B = o, C = o, &c. *ſint eædem ac correſpondentes radices incognitarum quantitatum poſteriorum æquationum, necne.*

Ducantur æquationes $P = o$, $Q = o$, $R = o$, &c. reſpective in a, b, c, d, &c.; a', b', c', d', &c.; a'', b'', c'', d'', &c.; &c.; ubi a, b, c, &c.; a', b', &c.; a'', b'', &c. invariabiles denotant quantitates: fiant reſpective $aP + bQ + cR +$ &c. $= A$, $a'P + b'Q + c'R +$ &c. $= B$, $a''P + b''Q + c''R +$ &c. $= C$, &c.; & ex æquatis correſpondentibus inter ſe utrarum partium ſingularum æquationum terminis facile detegi poteſt,

teſt, annon æquationes reſultantes ſunt inter ſe contradictoriæ: ſi modo æquationes ſint inter ſe contradictoriæ, tum correſpondentes radices incognitarum quantitatum priorum & poſteriorum æquationum haud erunt eædem: ſi vero haud ſint contradictoriæ inter ſe æquationes prædictæ, tum facile erui poſſunt e ſimplicibus æquationibus coefficientes a, b, c, &c.; a', b', c', &c.; a'', b'', &c.; &c.

THEOR. XXVII.

Sint duæ vel plures (n) independentes æquationes $A=o$, $B=o$, $C=o$, &c. tot incognitas quantitates $(x, y, z, \&c.)$ involventes; tum duæ vel plures (n) independentes æquationes $P\times A+Q\times B+R\times C+\&c.=o$, $P'\times A+Q'\times B+R'\times C+\&c.=o$, $P''\times A+Q''\times B+R''\times C+\&c.=o$, &c., (ubi literæ P, Q, R, &c.; P', Q', R', &c.; P'', Q'', R'', &c.; &c.; reſpective denotant functiones incognitarum quantitatum x, y, z, &c.) omnes radices quantitatum $(x, y, z, \&c.)$ habebunt, quas habent æquationes $A=o$, $B=o$, $C=o$, &c. cum pluribus adjunctis; ni quantitates P, Q, R, &c.; P', Q', R', &c.; &c., & earum correſpondentes factores A, B, C, &c. communem habeant diviſorem: correſpondentes enim radices incognitarum æquationum $A=o$, $B=o$, $C=o$, &c. quantitatum $(x, y, z, \&c.)$ pro quantitatibus ipſis $(x, y, z, \&c.)$ in prædictis æquationibus ſcribantur; nihilo evadent æquales quantitates A, B, C, &c. reſpective; ergo nihilo evadent æquales quantitates $PA+QB+RC+\&c.$, $P'A+Q'B+R'C+\&c.$, $P''A+Q''B+R''C+\&c.$, &c.; & conſequenter omnes prædictæ radices æquationum $A=o$, $B=o$, $C=o$, &c. incognitarum quantitatum $(x, y, z, \&c.)$ erunt etiam radices earundem quantitatum æquationum $PA+QB+RC+\&c.=o$, $P'A+Q'B+R'C+\&c.=o$, $P''A+Q''B+R''C+\&c.=o$, &c.

Cor. Datis (n) æquationibus $P\times A+Q\times B+R\times C+\&c.=o$, $P'A+Q'B+R'C+\&c.=o$, $P''A+Q''B+R''C+\&c.=o$, &c., (n) incognitas quantitates $(x, y, z, \&c.)$ habentibus; tum radices incognitarum quantitatum $(x, y, z, \&c.)$ in æquationibus $A=o$,

 $B=o$,

$B = o$, $C = o$, &c. contentarum, erunt etiam radices earundem incognitarum quantitatum in datis æquationibus contentarum.

Cor. 2. E prob. 31. ſæpe dici poſſunt dimenſiones, ad quas aſſurgit æquatio, cujus radix eſt x, ex æquationibus $P \times A + Q \times B + R \times C +$ &c. $= o$, $P' \times A + Q' \times B + R' \times C +$ &c. $= o$, $P'' \times A + Q'' \times B + R'' \times C +$ &c. $= o$, &c.; etiamque ex æquationibus $A = o$, $B = o$, $C = o$, &c. deducta: exinde conſtabit numerus radicum quantitatis (x) in prioribus æquationibus contentarum, quas non habent poſteriores æquationes : e. g. ſint duæ æquationes $(P)\ y^n + \overline{a + bx}\, y^{n-1} + (c + dx + ex^2)\, y^{n-2} +$ &c. $= o$ & $(Q)\ y^m + (A + Bx)\, y^{m-1} + (C + Dx + Ex^2)\, y^{m-2} +$ &c. $= o$, quarum incognitæ quantitates (x & y) habeant ($n \times m$) radices reſpective; tum duæ æquationes $\alpha x \times P + \beta y \times Q = o$ & $\gamma x \times P + \delta y \times Q = o$ haud habebunt plures quam $\overline{n+1} \times \overline{m+1} - nm = n + m + 1$ radices, quas non habent prædictæ $P = o$ & $Q = o$.

THEOR. XXVIII.

Sint algebraicæ æquationes $A = o$, $B = o$, $C = o$, &c.; quæ ſint functiones incognitarum quantitatum (x, y, z, v, &c.); quantitates A, B, C, D, &c. recipiant communem diviſorem π; tum erit $\pi = o$ æquatio, quæ recipiet indeterminatos valores quantitatum (x, y, z, &c.)

2. Sint $\frac{A}{\pi} = a$, $\frac{B}{\pi} = b$, $\frac{C}{\pi} = c$, &c.; tum erunt æquationes $a = o$, $b = o$, $c = o$, &c. æquationes, quarum radices eædem ſunt ac radices æquationum $A = o$, $B = o$, $C = o$, &c.

PROB. XXXIV.

Datâ algebraicâ quantitate (σ), invenire ejus valorem.

1. Sit (σ) data functio quantitatum (x, y, z, &c.); in eâ pro (x, y, z, &c.) ſcribantur earum aſſignati valores (a, b, c, &c.); exinde e diverſis

verſis valoribus irrationalium quantitatum in datâ functione (σ) contentarum, erui poſſunt diverſi valores functionis (σ).

2. Sit (σ) fractio, cujus unus vel duo vel plures (n) valores numeratoris nihilo evadunt æquales, cum quantitates (x, y, z, &c.) evadant reſpective a, b, c, d, &c. & correſpondentes ejus denominatores haud nihilo evadant æquales; tum, ſi modo (m) ſit numerus diverſorum valorum irrationalium quantitatum in denominatore contentarum, quæ in numeratore haud inveniuntur, $m \times n$ erunt valores functionis σ nihilo æquales.

3. Si prædictæ fractionis (σ) unus vel duo vel plures (n) valores denominatoris nihilo evadant æquales, cum numeratoris valores haud evaneſcant, & ſit (m) numerus e diverſis valoribus irrationalium quantitatum numeratoris, qui in denominatore haud inveniuntur; tum ($n \times m$) valores fractionis (σ) evadent abſurdi, i. e. non erunt finiti.

4. 1. Sit rationalis functio quantitatis (x) $= \frac{(x-\alpha)^m}{(x-\alpha)^n} \times \frac{P}{Q} = \sigma$, cujus numerator & denominator evaneſcit, cum $x = \alpha$; & $x - \alpha$ nec ſit diviſor quantitatis P nec Q; tum, ſi m major ſit quam n, erit valor functionis (σ) $= o$, cum $x = \alpha$: ſi modo m minor ſit quam n, tum erit prædictus valor, cum $x = \alpha$, infinitus, i. e. abſurdus: ſi vero $n = m$, erit valor functionis σ (cum $x = \alpha$) $= \frac{P}{Q}$.

4. 2. Sit $\sigma = \frac{A}{B}$ irrationalis algebraica fractio, cujus numerator & denominator nihilo ſunt reſpective æquales, cum $x = \alpha$; per prob. 26. inveniantur factores (A', A'', A''', &c., B', B'', B''', &c.), qui ducti in A & B reſpective præbent rationale contentum: ſi in prædictis (A', A'', &c.; B', B'', &c.) haud detur factor nihilo æqualis, & ducatur & numerator & denominator datæ fractionis $\left(\frac{A}{B}\right)$ in $A' \times A'' \times A''' \times$ &c. $\times B' \times B'' \times B''' \times$ &c.; tum fractio reſultans erit quan-

 titas

titas hujuſce formulæ $\frac{\overline{x-\alpha}^{m}}{\overline{x-\alpha}^{n}} \times \frac{P}{Q} \times \frac{B' \times B'' \times B''' \times \&c.}{A' \times A'' \times A''' \times \&c.}$; ubi $\overline{x-\alpha}^{m} \times P = A \times A' \times A'' \times A''' \times \&c.$, & $\overline{x-\alpha}^{n} \times Q = B \times B' \times B'' \times B''' \times \&c.$; & P & Q haud diviſibiles ſunt per $x-\alpha$; tum, ſi m major ſit quam n, erit $\frac{A}{B} = o$: ſi m minor ſit quam n, erit $\frac{A}{B} =$ infinita quantitas, vel quod idem eſt, abſurda: ſi $m = n$, tum erit $\frac{A}{B} = \frac{P \times B' \times B'' \times B''' \times \&c.}{Q \times A' \times A'' \times A''' \times \&c.}$.

Si detur factor in prædictis (A', A'', A''', &c.; B', B'', B''', &c.) nihilo æqualis; tum per hanc methodum haud reſolvi poteſt problema.

Cor. Ex numero radicum in quantitatibus A & B contentarum, quæ nihilo ſunt æquales, cum $x = \alpha$, facile conſtabunt indices m & n, & conſequenter valor fractionis $\left(\frac{A}{B}\right) = \frac{\dot{A}}{\dot{B}}$ vel $\frac{\ddot{A}}{\ddot{B}}$ vel $\frac{\dddot{A}}{\dddot{B}}$ &c. prout una, duæ, tres, &c. radices ſint α.

PROB. XXXV.

Datâ æquatione unam vel plures incognitas quantitates habente, invenire conſtitutionem ejus coefficientium.

1. Datâ æquatione ($x^{n} - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \&c. = o$,) unam incognitam quantitatem (x) habente, invenire ejus coefficientium (p, q, r, s, &c.) conſtitutionem.

Sint α, β, γ, δ, &c. radices datæ æquationis, quarum numerus ſit n: quibus pro ſuo valore (x) ſubſtitutis, reſultant æquationes $\alpha^{n} - p\alpha^{n-1} + q\alpha^{n-2} - r\alpha^{n-3} + s\alpha^{n-4} - \&c. = o$, & $\beta^{n} - p\beta^{n-1} + q\beta^{n-2} - r\beta^{n-3} + s\beta^{n-4} - \&c. = o$, & $\gamma^{n} - p\gamma^{n-1} + q\gamma^{n-2} - r\gamma^{n-3} + s\gamma^{n-4} - \&c. = o$, & $\delta^{n} - p\delta^{n-1} + q\delta^{n-2} - r\delta^{n-3} + s\delta^{n-4} - \&c. = o$, &c. reducantur hæ

æquationes in unam, ita ut exterminentur omnes coefficientes præter unam (p), cujus conſtitutio quæritur; & invenietur $p = \alpha + \beta + \gamma + \delta +$ &c.; & eodem modo invenietur $q = \alpha\beta + \alpha\gamma + \beta\gamma + \alpha\delta +$ &c.; $r = \alpha\beta\gamma + \alpha\beta\delta +$ &c. & ſic deinceps.

2. Datâ æquatione $(y^n + \overline{a + bx}\, y^{n-1} + \overline{c + dx + ex^2}\, y^{n-1} + \overline{f + gx + hx^2 + kx^3}\, y^{n-3} +$ &c. $= 0)$ duas incognitas quantitates $(x$ & $y)$ habente, invenire ejus coefficientium conſtitutionem.

Sint incognitæ quantitatis (y) radices $\alpha, \beta, \gamma, \delta$, &c. & incognitæ quantitatis (x) correſpondentes radices π, ρ, σ, τ, &c. & ſingulæ incognitæ quantitatis $(x$ & $y)$ ſit numerus correſpondentium radicum reſpective $n \times \frac{n+3}{2}$; ſubſtituantur hæ correſpondentes radices pro ſuis reſpectivis valoribus $(x$ & $y)$, reſultant æquationes

$$\alpha^n + \overline{a + b\pi} \times \alpha^{n-1} + \overline{c + d\pi + e\pi^2} \times \alpha^{n-2} + \overline{f + g\pi + h\pi^2 + k\pi^3} \times \alpha^{n-3} + \text{\&c.} = 0,$$
$$\beta^n + \overline{a + b\rho} \times \beta^{n-1} + \overline{c + d\rho + e\rho^2} \times \beta^{n-2} + \overline{f + g\rho + h\rho^2 + k\rho^3} \times \beta^{n-3} + \text{\&c.} = 0,$$
$$\gamma^n + \overline{a + b\sigma} \times \gamma^{n-1} + \overline{c + d\sigma + e\sigma^2} \times \gamma^{n-2} + \overline{f + g\sigma + h\sigma^2 + k\sigma^3} \times \gamma^{n-3} + \text{\&c.} = 0,$$
$$\delta^n + \overline{a + b\tau} \times \delta^{n-1} + \overline{c + d\tau + e\tau^2} \times \delta^{n-2} + \overline{f + g\tau + h\tau^2 + k\tau^3} \times \delta^{n-3} + \text{\&c.} = 0,$$
&c.

Harum æquationum numerus per hypotheſin æqualis eſt fractioni $n \times \frac{n+3}{2}$, & numerus incognitarum coefficientium $(a, b, c, d, e,$ &c.$)$ eædem quantitati æqualis eſt; reducantur hæ $\left(n \times \frac{n+3}{2}\right)$ æquationes in unam, ita ut exterminentur omnes præter unam (a) incognitæ coefficientes, & exinde conſtat e quot & qualibus membris, i. e. ex quâ functione correſpondentium radicum incognitarum quantitatum iſta una coefficiens conſtat; & ſic de reliquis.

Cor. 1. Sit numerus correſpondentium radicum incognitarum quantitatum $(x$ & $y)$ æquationis, cujus dimenſiones ſunt (n), major quam $n \times \frac{n+3}{2}$; e quibuſque $\left(n \times \frac{n+3}{2}\right)$ correſpondentibus incogni-

cognitarum quantitatum $(x$ & $y)$ radicibus oritur æquationis conſtitutio, i.e. coefficientes quæſitæ æquationis.

Cor. 2. Sit numerus (m) correſpondentium radicum minor quam $\left(n \times \frac{n+3}{2}\right)$; & infinitis modis dari poſſunt diverſæ conſtitutiones æquationis, vel pro $\left(n \times \frac{n+3}{2} - m\right)$ coefficientibus ſcribi poſſunt quælibet quantitates.

Cor. 3. Sint duæ æquationes duas incognitas quantitates x & y habentes, quarum dimenſiones ſunt n & m reſpective, & ſit n minor quam m; tum æquationis n dimenſionum ita aſſumi poſſunt coefficientes, ut quæcunque $\left(n \times \frac{n+3}{2}\right)$ quantitates ſint radices incognitæ quantitatis (x), & quæcunque etiam $\left(n \times \frac{n+3}{2}\right)$ quantitates ſint correſpondentes radices incognitæ quantitatis (y): æquationis vero m dimenſionum coefficientes infinitis modis variari poſſunt.

Sint n & m inter ſe æquales; tum haud aſſumi poſſunt coefficientes utriuſque æquationis, ita ut quæcunque $\left(n \times \frac{n+3}{2}\right)$ quantitates ſint radices incognitæ quantitatis (x), & quæcunque etiam $\left(n \times \frac{n+3}{2}\right)$ quantitates ſint correſpondentes incognitæ quantitatis (y) radices: ſi enim hoc ſit, utraque æquatio eadem fiet: ſed aſſumi poſſunt coefficientes infinitis modis, ita ut quæcunque $\left(n \times \frac{n+3}{2} - 1\right)$ quantitates ſint reſpective correſpondentes valores incognitarum quantitatum $(x$ & $y)$.

Ex. Sint duæ quadraticæ æquationes duas incognitas quantitates habentes $y^2 + \overline{a + bx}\,y + c + dx + ex^2 = 0$ &
$$y^2 + \overline{A + Bx}\,y + C + Dx + Ex^2 = 0,$$
in

in utrâque æquatione ſunt quinque $\left(2 \times \frac{2+3}{2}\right)$ incognitæ coefficientes, ſubſtituantur pro incognitis quantitatibus *(x & y)* quæcunque quinque quantitates reſpective in utrâque æquatione, reſultabunt ex utrâque eædem quinque ſimplices æquationes; unde conſtat utraſque æquationes neceſſario eaſdem reſultare: ergo ſolummodo ſubſtitui poſſunt in utrâque æquatione quatuor correſpondentes valores incognitarum quantitatum *(x & y)*, quibus ſubſtitutis reſultant ex utrâque quadraticâ quatuor ſimplices æquationes quinque incognitas coefficientes habentes, & conſequenter in utrâque pro unâ incognitâ coefficiente aſſumi poteſt quæcunque quantitas; at cavendum eſt, ne eadem quantitas pro eâdem coefficiente in utriſque æquationibus aſſumatur.

Sint quatuor valores incognitæ quantitatis (y), 0, 2, 4, 2; & quatuor correſpondentes valores incognitæ quantitatis (x) reſpective 0, 1, 1, 2; quibus ſubſtitutis & in utrâque quadraticâ pro unâ coefficiente $(b$ & $B)$ aſſumptis reſpective numeris 4 & -2; reſultant e primâ quadraticâ quatuor ſimplices æquationes $c = o$, $4 + 2a + 8 + e + d = o$, $16 + 4a + 16 + e + d = o$, $4 + 2a + 16 + 4e + 2d = o$, e quibus invenietur prima æquatio $y^2 + \overline{4x - 10} \times y - 8x^2 + 16x = o$: & eodem modo inveniri poteſt ſecunda $y^2 - \overline{2x + 4} \times y - 2x^2 + 10x = o$.

Numerus correſpondentium radicum in omni caſu incognitarum quantitatum *(x & y)*, quæ generaliter aſſumi poſſunt, æqualis eſt numero incognitarum datæ æquationis coefficientium.

Eadem eſt methodus ratiocinandi de pluribus æquationibus plures incognitas quantitates habentibus, ſed obſervandum eſt æquationem n dimenſionum m incognitas quantitates involventem, $\overline{n+1} \times \frac{n+2}{2} \times \frac{n+3}{3} \ldots \times \frac{n+m}{m} - 1$ incognitas coefficientes habere poſſe.

Facile

Facile conſtabit infinitas dari poſſe conſimiles harum æquationum conſtitutiones.

THEOR. XXIX.

1. Sint n æquationes independentes m incognitas quantitates habentes, tum aſſumi poſſunt $(m-n)$ incognitæ quantitates e prædictis ad libitum.

2. Sint n æquationes m incognitas quantitates habentes; & ſi ita reduci poſſint prædictæ (n) æquationes, ut reſultent (r) æquationes inter ſe invicem independentes, & cognitas quantitates ſolummodo involventes; tum in prædictis (n) æquationibus aſſumi poſſunt $(m-n+r)$ incognitæ quantitates e prædictis ad libitum.

Hujuſce caſus exemp. dantur in ex. 2, & 3. prob. 22.

PROB. XXXVI.

Sint duæ æquationes duas incognitas quantitates (x & y) *habentes; ſint* $\alpha, \beta, \gamma, \delta, \epsilon,$ *&c. radices incognitæ quantitatis* (y), & $\pi, \rho, \sigma, \tau, \upsilon,$ *&c. correſpondentes radices incognitæ quantitatis* (x), & m & n *dati indices, invenire ſummam* $\alpha^m\pi^n + \beta^m\rho^n + \gamma^m\sigma^n + \delta^m\tau^n +$ &c.

Supponatur $y^m \times x^n = z$; & in datis æquationibus pro (y) ſubſtituatur ejus valor $\left(\frac{z^{\frac{1}{m}}}{x^{\frac{n}{m}}}\right)$; & ita transformentur duæ æquationes reſultantes in unam, ut exterminetur incognita quantitas (x); reſultabit æquatio, cujus radix eſt z, & cujus radicum ſumma eſt ſumma quæſita.

Cor. 2. Hæc radicum ſumma ſæpe facilius inveſtigari poteſt: haud raro pendet e terminis datæ æquationis, quorum dimenſiones ſunt maximæ; & ſic deinceps.

PROB.

PROB. XXXVII.

1. *Iiſdem literis eaſdem quantitates denotantibus; & poſitis* m, n, o, r, s, *&c.* & a, b, c, d, e, *&c. datis indicibus; invenire ſummam e ſingulis hujuſmodi contentis*

$$\alpha^m \pi^a \beta^n \rho^b \gamma^o \sigma^c \text{ \&c.} + \alpha^m \pi^a \beta^n \rho^b \delta^o \tau^c \text{ \&c.} + \beta^m \rho^a \alpha^n \pi^b \gamma^o \sigma^c \text{ \&c.} + \text{\&c.}$$

ſcribatur

$$p^m \mathfrak{Q}^a = \alpha^m \pi^a + \beta^m \rho^a + \gamma^m \sigma^a + \delta^m \tau^a + \varepsilon^m \upsilon^a + \text{\&c.}$$
$$p^n \mathfrak{Q}^b = \alpha^n \pi^b + \beta^n \rho^b + \gamma^n \sigma^b + \delta^n \tau^b + \varepsilon^n \upsilon^b + \text{\&c.}$$
$$p^o \mathfrak{Q}^c = \alpha^o \pi^c + \beta^o \rho^c + \gamma^o \sigma^c + \delta^o \tau^c + \varepsilon^o \upsilon^c + \text{\&c.}$$
$$p^r \mathfrak{Q}^d = \alpha^r \pi^d + \beta^r \rho^d + \gamma^r \sigma^d + \delta^r \tau^d + \varepsilon^r \upsilon^d + \text{\&c.}$$
&c. &c. &c.

$$p^{m+n} \mathfrak{Q}^{a+b} = \alpha^{m+n} \pi^{a+b} + \beta^{m+n} \rho^{a+b} + \gamma^{m+n} \sigma^{a+b} + \text{\&c.}$$
$$p^{m+o} \mathfrak{Q}^{a+c} = \alpha^{m+o} \pi^{a+c} + \beta^{m+o} \rho^{a+c} + \gamma^{m+o} \sigma^{a+c} + \text{\&c.}$$
$$p^{n+o} \mathfrak{Q}^{b+c} = \alpha^{n+o} \pi^{b+c} + \beta^{n+o} \rho^{b+c} + \gamma^{n+o} \sigma^{b+c} + \text{\&c.}$$
$$p^{m+r} \mathfrak{Q}^{a+d} = \alpha^{m+r} \pi^{a+d} + \beta^{m+r} \rho^{a+d} + \gamma^{m+r} \sigma^{a+d} + \text{\&c.}$$
&c. &c.

$$p^{m+n+o} \mathfrak{Q}^{a+b+c} = \alpha^{m+n+o} \pi^{a+b+c} + \beta^{m+n+o} \rho^{a+b+c} + \gamma^{m+n+o} \sigma^{a+b+c} + \text{\&c.}$$
$$p^{m+n+r} \mathfrak{Q}^{a+b+d} = \alpha^{m+n+r} \pi^{a+b+d} + \beta^{m+n+r} \rho^{a+b+d} + \gamma^{m+n+r} \sigma^{a+b+d} + \text{\&c.}$$
$$p^{n+o+r} \mathfrak{Q}^{b+c+d} = \alpha^{n+o+r} \pi^{b+c+d} + \beta^{n+o+r} \rho^{b+c+d} + \gamma^{n+o+r} \sigma^{b+c+d} + \text{\&c.}$$
&c. &c.

$$p^{m+n+o+r} \mathfrak{Q}^{a+b+c+d} = \alpha^{m+n+o+r} \pi^{a+b+c+d} + \beta^{m+n+o+r} \rho^{a+b+c+d} + \gamma^{m+n+o+r} \sigma^{a+b+c+d} + \text{\&c.}$$
$$p^{m+n+o+s} \mathfrak{Q}^{a+b+c+e} = \alpha^{m+n+o+s} \pi^{a+b+c+e} + \beta^{m+n+o+s} \rho^{a+b+c+e} + \gamma^{m+n+o+s} \sigma^{a+b+c+e} + \text{\&c.}$$
&c. &c.

Et ſic progredi licet, obſervatâ problematis tertii ſubſtitutionis lege.

Scribatur etiam

$$A = P^m \mathfrak{Q}^a \times P^n \mathfrak{Q}^b \times P^o \mathfrak{Q}^c \times P^r \mathfrak{Q}^d \times \text{\&c.}$$

$$B = \frac{P^{m+n} \mathfrak{Q}^{a+b} \times A}{P^m \mathfrak{Q}^a \times P^n \mathfrak{Q}^b} + \frac{P^{m+o} \mathfrak{Q}^{a+c} \times A}{P^m \mathfrak{Q}^a \times P^o \mathfrak{Q}^c} + \frac{P^{n+o} \mathfrak{Q}^{b+c} \times A}{P^n \mathfrak{Q}^b \times P^o \mathfrak{Q}^c} + \text{\&c.}$$

$$C = \frac{P^{m+n+o} \mathfrak{Q}^{a+b+c} \times A}{P^m \mathfrak{Q}^a \times P^n \mathfrak{Q}^b \times P^o \mathfrak{Q}^c} + \frac{P^{m+n+r} \mathfrak{Q}^{a+b+d} \times A}{P^m \mathfrak{Q}^a \times P^n \mathfrak{Q}^b \times P^r \mathfrak{Q}^d} + \frac{P^{n+o+r} \mathfrak{Q}^{b+c+d} \times A}{P^n \mathfrak{Q}^b \times P^o \mathfrak{Q}^c \times P^r \mathfrak{Q}^d} + \text{\&c.}$$

$D =$

$$D = \frac{P^{m+n+o+r} Q^{a+b+c+d} \times A}{P^m Q^a \times P^n Q^b \times P^o Q^c \times P^r Q^d} + \frac{P^{m+n+o+s} Q^{a+b+c+e} \times A}{P^m Q^a \times P^n Q^b \times P^o Q^c \times P^s Q^e} + \frac{P^{n+o+r+s} Q^{b+c+d+e} \times A}{P^n Q^b \times P^o Q^c \times P^r Q^d \times P^s Q^e} + \&c.$$

$$BB = \frac{P^{m+n} Q^{a+b} \times P^{o+r} Q^{c+d} \times A}{P^m Q^a \times P^n Q^b \times P^o Q^c \times P^r Q^d} + \frac{P^{m+o} Q^{a+c} \times P^{n+r} Q^{b+d} \times A}{P^m Q^a \times P^o Q^c \times P^n Q^b \times P^r Q^d} + \&c.$$

&c. &c. &c.

obſervatâ problematis prædicti ſubſtitutionis lege; ſumma quæſita erit

$$A - B + 2C - 1 \times 2 \times 3 D + 1 \times 2 \times 3 \times 4 E - \&c.$$
$$+ 1 \times 1 \times BB - 1 \times 1 \times 2 BC + \&c.$$

Eadem erit hæc ac problematis tertii ſeries.

1.2. Obſervatis methodis in prob. 3, & 4. traditis, exinde conſtabit methodus inveniendi aggregatum e ſingulis valoribus cujuſcunque functionis aut rationalis aut irrationalis radicum incognitarum quantitatum (x & y), etiamque infinitam radicum poteſtatum ſeriem.

Sæpe vero facilius e coefficientibus progredi liceat; facile etiam inveniri poſſunt maximæ dimenſiones, ad quas aſſurgat quæcunque incognita quantitas in reſultanti aggregato.

2. Sint plures æquationes, quæ totidem incognitas quantitates (x, y, z, v, &c.) habent; ſint α, β, γ, δ, &c. radices incognitæ quantitatis (x); π, ρ, σ, τ, &c. reſpective radices correſpondentes quantitatis (y); κ, λ, μ, ν, &c. correſpondentes radices incognitæ quantitatis (z); Γ, Δ, E, Z, &c. correſpondentes incognitæ quantitatis (v) valores; &c. ſint etiam a, f, l, p, &c. dati indices; invenire ſummam $\alpha^a \pi^f \kappa^l \Gamma^p$ &c. $+ \beta^a \rho^f \lambda^l \Delta^p$ &c. $+ \gamma^a \sigma^f \mu^l E^p$ &c. $+ \delta^a \tau^f \nu^l Z^p$ &c. $+$ &c.

Supponatur $x^a y^f z^l v^p$ &c. $= w$; & transformentur hæ æquationes reſultantes in unam, ita ut exterminentur omnes incognitæ quantitates (x, y, z, v, &c.) præter aſſumptam (w); reſultat æquatio, cujus radix eſt (w), & cujus radicum ſumma eſt ſumma quæſita.

Hoc etiam per alias methodos ſæpe perfici poteſt.

2.2. Per ſeriem, quæ eandem legem, quam obſervat præcedens hujus problematis ſeries, & per prædictas methodos in prob. 3, & 4. traditas inveſtigari poteſt aggregatum e ſingulis valoribus cujuſcunque

que functionis, aut rationalis, aut irrationalis, radicum incognitarum quantitatum (x, y, z, v, &c.)

2. 3. Et idem dici poteſt de aggregatis functionum radicum diverſarum & adventitiarum incognitarum quantitatum (x, y, z, v, &c.) conjunctarum inveniendis.

Sit α vera radix incognitæ quantitatis (x), ſubſtituatur hæc radix (α) in omnibus æquationibus pro ejus valore (x); per radices adventitias eas intelligo, quæ per hanc ſubſtitutionem in quaſdam æquationes irrepſerunt, omnibuſque æquationibus non ſunt communes.

Multæ etiam hujuſmodi propoſitiones inveniri poſſunt e tranſpoſitione terminorum ex alterâ in alteram datæ æquationis partem; ex addendo, ſubtrahendo, ducendo datas æquationes in ſe invicem, &c. & e ſcribendo in reſultantibus æquationibus pro x, y, z, &c. earum correſpondentes radices α, π, κ, &c.; β, ρ, λ, &c.; γ, σ, μ, &c.; & ſic deinceps.

PROB. XXXVIII.

Tranſformare duas æquationes duas incognitas quantitates (x & y) *habentes in unam, cujus incognita quantitas quamcunque habet algebraicam relationem ad radices incognitarum duarum æquationum quantitatum.*

1. Aſſumantur radices incognitarum quantitatum tanquam cognitæ, i. e. ſint radices incognitæ quantitatis (x) reſpective α, β, γ, δ, &c. & correſpondentes radices incognitæ quantitatis (y) reſpective π, ρ, σ, τ, &c. ex datâ relatione (ſi modo haud per æquationes exprimatur relatio) inveniantur radices quæſitæ quantitatis; & conſequenter per prob. præcedens ſumma ejus radicum, quæ eſt coefficiens ſecundi æquationis termini.

Ducantur quæque duæ, tres, &c. radices quæſitæ quantitatis in ſeſe; & duorum præcedentium problematum ope inveniri poſſunt reſultantium contentorum aggregata, quæ ſunt reſpective coefficientes tertii, quarti, &c. quæſitæ æquationis terminorum; unde ſequuntur quæſitæ æquationis coefficientes, & conſequenter æquatio ipſa.

2. Si vero per æquationes quaſcunque datas, inter radices incognitarum datarum æquationum quantitatum & quaſcunque aſſumptas incognitas quantitates exprimatur relatio: tum aſſumantur radices incognitarum quantitatum datarum æquationum tanquam cognitæ, & exinde inveniantur ſingulæ diverſæ æquationes relationem exprimentes; multiplicentur omnes valores ex ſingulis hiſce æquationibus continuo reſpective in ſe invicem; & per duo problemata præcedenta inveniantur aggregata harum æquationum reſultantium, quæ minime diverſas ejuſdem incognitæ quantitatis radices contineant; hæc aggregata erunt æquationes quæſitæ.

Cor. 1. Sint tres vel plures æquationes tres vel plures incognitas quantitates habentes; eodem modo inveniri poſſunt æquationes, quarum radices quamcunque habeant algebraicam relationem ad radices datarum æquationum.

Cor. 2. Si relatio inter incognitas datarum & quæſitarum æquationum quantitates, ſolummodo correſpondentes valores incognitarum quantitatum involvat; tum ſemper inveniri poſſunt quæſitæ æquationes e reductione plurium æquationum in unam, ita ut incognitæ quantitates exterminentur.

THEOR. XXX.

Duæ vel tres vel plures æquationes $A=o$, $B=o$, $C=o$, $D=o$, &c. duas vel tres, &c. incognitas quantitates $(x, y, z, \&c.)$ involvant; ex earum additione, ſubductione, multiplicatione, &c. in rationales quantitates conficiatur æquatio $Z=o$, ubi (Z) aggregatum eſt e compluribus poſſibilibus quadratis, i. e. $Z=P^2+Q^2+R^2+\&c.$, & P, Q, R, &c. reſpective denotant rationales functiones quantitatum $(x, y, z, \&c.)$; tum haud continentur correſpondentes poſſibiles valores e ſingulis quantitatibus $(x, y, z, \&c.)$ in datis æquationibus $A=o$, $B=o$, $C=o$, $D=o$, &c.; ni æquationes $P=o$, $Q=o$, $R=o$, &c. habeant correſpondentes valores e ſingulis incognitis quantitatibus $(x, y, z, \&c.)$, qui etiam erunt correſpondentes valores præ-

prædictarum quantitatum $(x, y, z,$ &c.$)$ in æquationibus $A=0$, $B=0$, $C=0$, &c. contenti.

Cor. Hinc omnis quantitas (x) nullam habet poſſibilem radicem (α) in prædictis æquationibus $A=0$, $B=0$, $C=0$, $D=0$, &c. ni duo vel plures valores prædictæ radicis (α) ſint inter ſe æquales.

Hinc facile inveniri poſſunt infinitæ æquationes hujuſmodi, quæ nullas habent poſſibiles radices.

THEOR. XXXI.

Sint duæ vel plures æquationes $A=0$, $B=0$, $C=0$, &c. duas vel plures incognitas quantitates $(x, y,$ &c.$)$ involventes; e ſingulis æquationibus inveniantur caſus, in quibus incognita quantitas $(x, y,$ &c.$)$ vel nullas vel duas vel quatuor, &c. vel omnes habet radices impoſſibiles; exinde deduci poſſunt limites, inter quos conſiſtant poſſibiles prædictarum æquationum radices.

Ex. 1. Sint duæ æquationes $y^2+\overline{a+bx}\,y+cx^2+dx+e=0$ & $y^2+\overline{p+qx}\,y+rx^2+sx+t=0$: 1mo. ſit b^2-4c affirmativa quantitas, tum radices quantitatis y in priori æquatione erunt poſſibiles, ſi quantitas (x) haud ponatur inter duas radices $(\pi$ & $\varrho)$ æquationis $(b^2-4c)x^2+(2ab-4d)x+a^2-4e=0$: 2do. ſit b^2-4c negativa quantitas; tum radices quantitatis y erunt poſſibiles, ſi quantitas (x) inter duas radices $(\pi$ & $\varrho)$ interponatur. Et ſimiliter, ſi q^2-4r ſit affirmativa quantitas; radices quantitatis y erunt poſſibiles, ſi quantitas (x) haud inter duas $(\alpha$ & $\beta)$ radices æquationis $(q^2-4r)x^2+(2pq-4s)x+p^2-4t=0$ ponatur: ſi q^2-4r ſit negativa quantitas; tum erit quantitas (y) poſſibilis, ſi (x) ponatur inter $(\alpha$ & $\beta)$.

Hinc, ſi utræque b^2-4c & q^2-4r ſint affirmativæ; tum (y) erit poſſibilis, ſi (x) nec ponatur inter π & ϱ, nec inter α & β: ſit b^2-4c affirmativa & q^2-4r negativa quantitas; tum (y) erit poſſibilis, ſi x ponatur inter α & β, haud vero inter π & ϱ. Et ſic de reliquis caſibus:

ſibus: deinde inveniantur limites, inter quos ponuntur omnes valores quantitatis (y), qui correſpondent omnibus valoribus quantitatis (x), inter limites prius deductos poſitis.

Ex eâdem methodo primo inveniantur limites quantitatis y, inter quos ponuntur poſſibiles valores quantitatis (x); deinde inveniantur limites, inter quos ponuntur poſſibiles valores quantitatis x, qui correſpondent omnibus valoribus quantitatis y inter limites prius deductos poſitis.

Tùm ex comparandis inter ſe limitibus inventis haud raro detegi poteſt; utrum radices incognitarum quantitatum datarum æquationum ſint poſſibiles, necne.

THEOR. XXXII.

Datis duabus vel pluribus æquationibus duas vel plures incognitas quantitates $(x, y, z,$ &c.$)$ habentibus; & his æquationibus in alias, quarum radices ſunt quadrata vel quæcunque algebraica functio incognitarum datæ æquationis quantitatum, transformatis: ex principiis in cap. 2. traditis; & numero impoſſibilium, &c. radicum incognitarum quantitatum in datis æquationibus contentarum; ſæpe erui poteſt numerus impoſſibilum, &c. radicum incognitarum quantitatum in reſultantibus æquationibus contentarum, &c. conſtat e prædictis principiis.

THEOR. XXXIII.

1. Datâ quantitate exprimente ſeriem terminorum ſecundum dimenſiones quantitatum $(x, y, z,$ &c.$)$ progredientium. Sint $\alpha, \beta, \gamma, \delta,$ &c. radices æquationis $x^n - 1 = 0$; $\lambda, \mu, \nu,$ &c. radices æquationis $x^m - 1 = 0$; $\pi, \rho, \sigma, \tau,$ &c. radices æquationis $x^r - 1 = 0$: ſcribantur in datâ æquatione pro $x, y, z,$ &c. reſpective $\alpha x, \lambda y, \pi z,$ &c.; $\alpha x, \mu y, \pi z,$ &c.; $\alpha x, \mu y, \rho z,$ &c.; $\beta x, \lambda y, \pi z,$ &c.; &c.; reſultabunt $n \times m \times r \times$ &c. diverſæ quantitates $A, B, C, D, E,$ &c.; tum erit

$\alpha^{n-1}\times$

$$\frac{\alpha^{n-d}\times\lambda^{m-e}\times\pi^{r-f}\times \&c.\times A+\alpha^{n-d}\times\mu^{m-e}\times\pi^{r-f}\times \&c.\times B+\alpha^{n-d}\times\mu^{m-e}\times\varrho^{r-f}\times \&c.\times C+\beta^{n-d}\times\lambda^{m-e}\times\pi^{r-f}\times \&c.\times D+\&c.}{n\times m\times r\times \&c.}$$

= aggregato e ſingulis prædictæ ſeriei terminis hujuſce formulæ $x^{pn+d}\times y^{p'm+e}\times z^{p''r+f}\times$ &c., ubi p, p', p'', &c. integros denotant affirmativos numeros.

2. Sit æquatio $P=o$, ubi P ſit functio incognitarum quantitatum x, y, z, &c.; pro y, z, &c. ſcribantur reſpective αy, λz, &c.; αy, μz, &c.; βy, μz, &c.; & ſic deinceps: ubi α, β, γ, &c.; λ, μ, ν, &c. ſint radices æquationum $x^n-1=o$ & $x^m-1=o$, &c.; reſultabunt $n\times m\times$ &c. æquationes: ex his æquationibus inveniantur correſpondentes valores incognitæ quantitatis (x), in terminis ſecundum dimenſiones quantitatum $(y, z,$ &c.$)$ progredientium; qui ſint reſpective A, B, C, D, &c.; tum erunt

$$\frac{\alpha^{n-d}\times\lambda^{m-e}\times \&c.\times A+\alpha^{n-d}\times\mu^{m-e}\times \&c.\times B+\beta^{n-d}\times\mu^{m-e}\times \&c.\times C+\&c.}{n\times m\times \&c.}$$

= aggregato e ſingulis prædictæ ſeriei terminis hujuſce formulæ $y^{pn+d}\times z^{p'm+d}\times$ &c.; ubi p & p', &c. integri ſunt numeri.

Et ſic de pluribus æquationibus plures incognitas quantitates, &c. habentibus.

PROB. XXXIX.

Datis duabus vel pluribus æquationibus duas vel plures incognitas quantitates habentibus; invenire illas incognitarum quantitatum radices, quæ datam inter ſe habeant relationem, vel dantur.

1. Sint radices incognitarum quantitatum $(x, y,$ &c.$)$ ſigillatim datæ.

Subſtituantur earum valores pro incognitis quantitatibus $(x, y,$ &c.$)$ per methodum communes diviſores inveniendi deduci poſſunt radices, ſi modo ullæ ſint, quæſitæ.

Ex. Sint duæ æquationes

$y^n +$

$$y^n + \overline{a + bx}\, y^{n-1} + \overline{c + dx + ex^2} \times y^{n-2} + \&c. = 0,$$

$$\& \; y^m + \overline{A + Bx} \times y^{m-1} + \overline{C + Dx + Ex^2} \times y^{m-2} + \&c. = 0,$$

invenire utrum ſint tres radices incognitæ quantitatis *(y)* in arithmeticâ progreſſione.

Sit arithmetica progreſſio α, $\alpha + \beta$, $\alpha + 2\beta$; quibus valoribus in datis æquationibus pro ſuo valore *(y)* ſubſtitutis, reſultant ſex æquationes quinque ſolummodo incognitas quantitates habentes. E. G. Sint correſpondentes valores incognitæ quantitatis (x) reſpective π, ϱ, σ; ſex æquationes reſultantes erunt

$$\alpha^n + \overline{a + b\pi} \times \alpha^{n-1} + \overline{c + d\pi + e\pi^2} \times \alpha^{n-2} + \&c. = 0;$$

$$\overline{\alpha + \beta}^{\,n} + \overline{a + b\varrho} \times \overline{\alpha + \beta}^{\,n-1} + \overline{c + d\varrho + e\varrho^2} \times \overline{\alpha + \beta}^{\,n-2} + \&c. = 0;$$

$$\alpha^m + \overline{A + B\pi} \times \alpha^{m-1} + \overline{C + D\pi + E\pi^2} \times \alpha^{m-2} + \&c. = 0;$$

$$\overline{\alpha + \beta}^{\,m} + \overline{A + B\varrho} \times \overline{\alpha + \beta}^{\,m-1} + \overline{C + D\varrho + E\varrho^2} \times \overline{\alpha + \beta}^{\,m-2} + \&c. = 0;$$

$$\overline{\alpha + 2\beta}^{\,n} + \overline{a + b\sigma} \times \overline{\alpha + 2\beta}^{\,n-1} + \overline{c + d\sigma + e\sigma^2} \times \overline{\alpha + 2\beta}^{\,n-2} + \&c. = 0;$$

$$\& \; \overline{\alpha + 2\beta}^{\,m} + \overline{A + B\sigma} \times \overline{\alpha + 2\beta}^{\,m-1} + \overline{C + D\sigma + E\sigma^2} \times \overline{\alpha + 2\beta}^{\,m-2} + \&c. = 0.$$

Quarum quinque incognitæ quantitates erunt α, β, π, ϱ, σ; inveniatur, utrum hæ æquationes ſint inter ſe contradictoriæ, necne; & conficietur exemplum.

Ex. 2. Invenire, utrum ulla ſit radix incognitæ quantitatis *(x)*, quæ æqualis eſt correſpondenti alterius incognitæ quantitatis *(y)* radici.

Sit radix quæſita (α); in datis æquationibus pro x & y ſcribatur (α), & inveniatur, utrum reſultantes æquationes quoſdam valores admittant, necne.

2. 2. Invenire, utrum ulla ſit radix incognitæ quantitatis *(x)*, quæ æqualis eſt radici incognitæ quantitatis *(y)*. Subſtituantur α & π, ϱ & α in duabus vel pluribus datis æquationibus pro x & y reſpective; reſultant æquationes, e quibus, juxta methodum communes diviſores inveniendi, detegi poteſt radix α, ſi modo ulla detur.

Ex. 3. Sint duæ æquationes $a+bx+cy+dx^2+exy+fy^2+gx^3+hx^2y+kxy^2+ly^3+\&c.=0$, & $A+Bx+Cy+Dx^2+Exy+Fy^2+Gx^3+Hx^2y+Kxy^2+Ly^3+\&c.=0$, invenire numerum correſpondentium valorum incognitarum quantitatum x & y nihilo æqualium: evaneſcant a & A termini utriuſque æquationis, in quibus nulla invenitur dimenſio; erit correſpondens valor incognitarum quantitatum $(x$ & $y)$ nihilo reſpective æqualis: evaneſcant prædicti termini A & a, termini etiamque bx & cy & Bx & Cy; & quatuor valores incognitæ quantitatis (x) nihilo erunt æquales, hi autem quatuor valores quatuor etiam habebunt valores incognitæ quantitatis (y) nihilo æquales: evaneſcant termini a, A, bx, cy, Bx, Cy, dx^2, exy, fy^2, Dx^2, Exy, Fy^2; & novem erunt correſpondentes valores incognitarum quantitatum $(x$ & $y)$ nihilo æquales: & ſic deinceps.

Sint termini $(r-1, r-2, r-3\,..\,0)$ dimenſionum unius æquationis, termini vero $(s-1, s-2, s-3\,..\,0)$ dimenſionum alterius, &c. nihilo æquales; & erunt $(s\times r)$ radices æquationis, cujus radix eſt x vel y, nihilo æquales.

Sint tres vel plures æquationes tres vel plures incognitas quantitates $(x, y, z, \&c.)$ habentes, in quibus ſint termini $(r, r-1, r-2, \&c.)$ dimenſionum primæ æquationis, termini vero $(s, s-1, s-2, \&c.)$ dimenſionum ſecundæ æquationis, termini vero $(t, t-1, t-2, \&c.)$ dimenſionum tertiæ æquationis, &c. nihilo æquales: tum erunt $r\times s\times t\times\&c.$ radices æquationis, cujus radix eſt x vel y vel z &c. nihilo æquales. Hæc deduci poſſunt ex principiis in * prop. algebraicarum curvarum traditis; nempe quod curvæ, quas deſignant æquationes prædictæ, punctum habent duplex, triplex, quadruplex, &c.; cum x & $y=0$; &c.

Sin vero termini, quorum dimenſiones ſint minimæ, habeant diviſorem (xy); tum quoad terminos minimarum dimenſionum quam plurimi correſpondentes valores incognitarum quantitatum nihilo

* Vid. meas Propr. Alg. Curv.

$nMy^{n-1}+\overline{n-1}\times\overline{a+bx}\,y^{n-2}+\overline{n-2}\times\overline{c+dx+ex^2}\,y^{n-3}+\&c.=o$, & $mNy^{m-1}+\overline{m-1}\times\overline{A+Bx}\,y^{m-2}+\overline{m-2}\times\overline{C+Dx+Ex^2}\,y^{m-3}+\&c.=o$, etiamque $by^{n-1}+\overline{d+2ex}\,y^{n-2}+\overline{g+2hx+3kx^2}\,y^{n-3}+\&c.=o$, & $By^{m-1}+\overline{D+2Ex}\,y^{m-2}+\overline{G+2Hx+3Kx^2}\,y^{m-3}+\&c.=o$, & præterea $\overline{n-1}\,by^{n-2}+\overline{n-2}\,d+2\,\overline{n-2}\,exy^{n-3}+\&c.=o$, $\overline{m-1}\,By^{m-2}+\overline{m-2}\,D+2\,\overline{m-2}\,Exy^{m-3}+\&c.=o$, cum duo valores incognitæ quantitatis (x), etiamque duo correſpondentes valores incognitæ quantitatis (y) fiant æquales.

Hoc exemp. facile deduci poteſt e principiis in prob. 6. theor. 3, & 4. traditis.

Cum vero radices haud ſint æquales, limites inter radices inveniant hæ reſultantes æquationes.

Ex. 7. Sint duæ æquationes $y^n+\overline{a+bx}\,y^{n-1}+\overline{c+dx+ex^2}\,y^{n-2}+\&c.+Ax^m+Bx^{m-1}+Cx^{m-2}+Dx^{m-3}+\&c.=o$, & $y^n+\overline{a+bx}\,y^{n-1}+\overline{c+dx+ex^2}\,y^{n-2}+\&c.+Px^r+Qx^{r-1}+Rx^{r-2}+\&c.=o$, i. e. omnes termini, in quibus continetur (y), ſint iidem in utrâque æquatione; (n) erunt valores incognitæ quantitatis (x) ſemper inter ſe æquales: valores enim incognitæ quantitatis x erunt radices æquationis $Ax^m+Bx^{m-1}+\&c.-Px^r-Qx^{r-1}-\&c.=o$, quæ ſint α, β, γ, &c.; ſcribantur hæ radices pro x in datâ vel datis æquationibus; reſultant diverſæ æquationes, in quibus dimenſiones quantitatis y inveniuntur (n), & conſequenter n valores quantitatis y correſpondent ſingulæ radici quantitatis x; ergo n valores quantitatis x erunt inter ſe æquales.

Omnia hæc ad plures æquationes plures incognitas quantitates habentes applicari poſſunt.

Ex. 8. Datis duabus æquationibus duas incognitas quantitates x & y habentibus; invenire utrum duo vel (m) ſint valores quantitatis y, qui correſpondent uni valori quantitatis x, necne.

Pro x in datis æquationibus ſcribatur a; & ita reducantur duæ datæ æqua-

æquationes $A = o$ & $B = o$ per methodum communes diviſores inveniendi, ut tandem reſultent duæ quadraticæ æquationes $\alpha y^2 + \beta y + \gamma = o$ & $\pi y^2 + \varrho y + \sigma = o$, vel $\alpha y^m + \beta y^{m-1} + \gamma y^{m-2} + \delta y^{m-3} +$ &c. $= o$ & $\pi y^m + \varrho y^{m-1} + \sigma y^{m-2} + \tau y^{m-3} +$ &c. $= o$; in quibus æquationibus $\alpha, \beta, \gamma, \delta$, &c.; $\pi, \varrho, \sigma, \tau$, &c. ſint functiones quantitatis a; tum fiant inter ſe æquales correſpondentes harum æquationum coefficientes, i. e. $\frac{\beta}{\alpha} = \frac{\varrho}{\pi}$; $\frac{\gamma}{\alpha} = \frac{\sigma}{\pi}$; $\frac{\delta}{\alpha} = \frac{\tau}{\pi}$; &c.; & ex his æquationibus reſultantibus per methodum communes diviſores inveniendi detegi poteſt valor (a) incognitæ quantitatis (x), ſi modo ſit valor (a) incognitæ quantitatis (x) in duabus prædictis æquationibus $A = o$ & $B = o$, cui reſpondent duo vel (m) valores incognitæ quantitatis y.

2. Si relatio exprimatur per æquationes: e reductione plurium æquationum in unam, ita ut incognitæ quantitates exterminentur, inveniri poſſunt illæ radices; ſi modo ullæ ſint.

3. Si vero data ſit æquatio, quæ exprimit relationem inter diverſas ejuſdem incognitæ quantitatis (x) radices; tum ita reducantur datæ æquationes ut exterminentur omnes præter incognitam quantitatem (x); & inveniatur, utrum data æquatio ſit diviſor æquationis reſultantis, necne.

4. Datis quibuſdam functionibus incognitarum quantitatum radicum datam inter ſe habentibus relationem; e datis æquationibus inveniantur hæ functiones; tum e reſultantibus & datis æquationibus inveniri poſſunt radices incognitarum quantitatum quæſitæ, ſi modo ullæ ſint.

Cor. 1. Datâ quâcunque quantitate e diverſis irrationalibus quantitatibus ($A + B + C + D + E +$ &c.) conflatâ; invenire, utrum ullos recipiat poſſibiles valores, necne.

Aſſumatur $a + p\sqrt{-1}$ pro radice irrationalis quantitatis A, $b + q\sqrt{-1}$ pro radice irrationalis quantitatis B, $c + r\sqrt{-1}$ pro radice irrationalis quantitatis C, $d + s\sqrt{-1}$ pro radice irrationalis quantitatis D, &c.

O Sup-

Supponatur $p+q+r+s+$ &c. $=o$, & per problema inveniatur; utrum æquationes reſultantes ex aſſumptis valoribus talem habeant relationem ($p+q+r+s+$ &c. $=o$), necne.

Cor. 2. Datis duabus vel pluribus æquationibus duas vel plures incognitas quantitates (x, y, z, &c.) habentibus; ſit impoſſibilis quantitas $a+\sqrt{-b^2}$ radix incognitæ quantitatis (x), & $a-\sqrt{-b^2}$ etiam erit radix ejuſdem incognitæ quantitatis (x).

Conſtat e ſubſtitutione horum valorum pro incognitâ quantitate (x).

Haud abs re forſan erit ſequentia adjungere.

1. Sint duæ vel plures æquationes, quæ duas vel plures incognitas quantitates (x, y, z, &c.) habent; eadem invenietur irrationalitas in correſpondentibus incognitarum quantitatum (x, y, z, &c.) valoribus; ni duo vel plures valores incognitæ quantitatis ſint inter ſe æquales.

2. Datis duabus æquationibus duas incognitas quantitates (x & y) habentibus; ſi radix ($a+\sqrt{-b^2}$) incognitæ quantitatis (x) ſit impoſſibilis, tum ejus correſpondens valor incognitæ quantitatis (y) etiam erit impoſſibilis; ni duo vel tres, &c. valores incognitæ quantitatis (y) ſint inter ſe æquales, qui correſpondent prædictæ radici incognitæ quantitatis (x). Et ſic de pluribus incognitis quantitatibus in pluribus æquationibus contentis.

3. Sint (n) æquationes (n) incognitas quantitates (x, y, v, w, z, &c.) habentes; invenire numerum poſſibilium valorum correſpondentium radicum (x, y, z, v, &c.) inveniatur numerus poſſibilium valorum, quos habet quæcunque incognita quantitas (x), totidem erunt poſſibiles correſpondentes valores e ſingulis incognitis quantitatibus (y, z, v, &c.); ni duo vel tres vel plures valores incognitæ quantitatis ſint inter ſe æquales; in quo caſu inveniatur e præcedente problemate, numerus poſſibilium valorum incognitarum quantitatum (y, z, v, &c.) qui reſpondeant duobus vel pluribus æqualibus vel impoſſibilibus

poſſibilibus valoribus incognitæ quantitatis (x) : & id, quod quæſitum eſt, facile acquiri poteſt.

4. 1mo. E ſubſtitutione quantitatum pro radicibus in datis æquationibus; 2do. e quantitatibus ex datis æquationibus deductis; 3^tio^. e transformatione æquationum datarum in alias, quarum radices aſſignabilem habent relationem ad radices datarum æquationum; 4to. ex inveniendo, quando duæ radices cujuſcunque incognitæ quantitatis ſemel, bis, ter, &c. fiant æquales, deduci poſſunt regulæ impoſſibiles datarum æquationum radices detegendi; 5to. transformentur datæ æquationes in alias, quarum radices aſſignabilem habeant relationem ad radices datarum æquationum; & e numero impoſſibilium radicum reſultantium æquationum inveniri poteſt numerus impoſſibilium radicum datarum æquationum.

5. Hæc etiam vel conſimilia applicari poſſunt ad numerum affirmativarum & negativarum radicum incognitarum quantitatum inveſtigandum.

PROB. XL.

Sint duæ æquationes duas incognitas quantitates (x & y) *habentes; invenire, quot reſpondentes valores incognitarum quantitatum* (x & y) *eadem habeant ſigna, quot autem diverſa.*

Inveniatur æquatio, cujus radix ſit $\frac{x}{y}$; quot ejus radices ſint affirmativæ, tot ſigna reſpondentium incognitarum quantitatum $(x$ & $y)$ erunt eadem; quot autem negativæ, tot erunt diverſa: ni ulla radix impoſſibilis incognitæ quantitatis (x) ſit $a+b\sqrt{-1}$, ejus vero correſpondens radix incognitæ quantitatis (y) ſit $ma+mb\sqrt{-1}$.

Facile etiam inveniri poſſunt æquationes, quarum radices ſint omnes negativæ, vel omnes affirmativæ.

Omnis æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} +$ &c. $= 0$, (n) dimenſionum erit differentia inter duas æquationes eaſdem (n) vel plures

plures dimenſiones habentes; quarum omnes radices ſint poſſibiles; vel m radices ſint poſſibiles & affirmativæ, (r) negativæ, cæteræ vero impoſſibiles; & quæ æquationes infinitis modis variari poſſint.

Quædam ex his etiam ad ſummam duarum vel plurium æquationum (n) dimenſionum applicari poſſunt.

Et conſimilia applicari poſſunt ad duas vel plures æquationes duas vel plures incognitas quantitates habentes.

PROB. XLI.

1. *Datis duabus æquationibus* A = 0 & P = 0 *duas incognitas quantitates* (x & y) *habentibus, diviſores recipiant hæ æquationes, i. e. ſint* $A = \alpha \times \beta \times \gamma \times$ &c. & $P = \pi \times \varrho \times \sigma \times \tau \times$ &c. *deprimere datas æquationes in alias minores habentes dimenſiones.*

Supponantur ſinguli unius datæ æquationis diviſores, ſinguli vero alterius nihilo æquales; exinde reſultant æquationes reductæ, i. e. erunt æquationes reductæ $\alpha = 0$, & $\pi = 0$; $\alpha = 0$, & $\varrho = 0$; $\alpha = 0$, & $\sigma = 0$; $\alpha = 0$, & $\tau = 0$; &c. $\beta = 0$, & $\pi = 0$; $\beta = 0$, & $\varrho = 0$; $\beta = 0$, & $\sigma = 0$; $\beta = 0$, & $\tau = 0$; &c. $\gamma = 0$, & $\pi = 0$; $\gamma = 0$, & $\varrho = 0$; $\gamma = 0$, & $\sigma = 0$; $\gamma = 0$, & $\tau = 0$; &c. $\delta = 0$, & $\pi = 0$; &c. utrum diviſores admittant datæ æquationes, necne; e ſubſequenti capite inveniri poteſt.

Cor. 1. Sint duæ æquationes $y^n + \overline{a + bx}\, y^{n-1} + \&c. = 0$ & $y^m + \overline{A + Bx}\, y^{m-1} + \&c. = 0$, quarum diviſores ſint reſpective $y + \alpha x + \beta$, $y + \gamma x + \delta$, $y + \epsilon x + \zeta$, &c. & $y + \pi x + \varrho$, $y + \sigma x + \tau$, $y + \upsilon x + \mu$, &c. ſint prioris æquationis diviſores (n), poſterioris vero (m). Supponantur ſinguli unius æquationis diviſores, & ſinguli alterius nihilo æquales, i. e. $y + \alpha x + \beta = 0$, & $y + \pi x + \varrho = 0$ (e quibus æquationibus colligitur $x = \frac{\varrho - \beta}{\alpha - \pi}$, & $y = \frac{\pi\beta - \alpha\varrho}{\alpha - \pi}$): $y + \alpha x + \beta = 0$, & $y + \sigma x + \tau = 0$, (ex hiſce æquationibus colligitur $x = \frac{\tau - \beta}{\alpha - \sigma}$,

&c.

$y^2 - \overline{11x - 1}\,y + 14x^2 - 9x + 4 = 0$; hæ æquationes nullum recipient diviſorem; ſubtrahatur poſterior æquatio de bis priori, & dividatur reſiduum per 11; reſultat æquatio $y^2 + \overline{x - 1}\,y - 2x^2 + 7x - 6 = \overline{y - x + 2} \times \overline{y + 2x - 3} = 0$.

Addantur prior æquatio & quinquies ſecunda, & dividatur ſumma per 11; & reſultat æquatio $y^2 - 5xy + 6x^2 - x - 1 = \overline{y - 2x + 1} \times \overline{y - 3x - 1} = 0$; unde correſpondentes radices incognitarum quantitatum $(x$ & $y)$ erunt reſpective $-1,\ 1,\ -\frac{3}{2},\ \frac{2}{5}$; & $-3,\ 1,\ -\frac{7}{2},\ \frac{11}{5}$.

Cor. Hinc facile inveniri poſſunt infinitæ hujuſmodi æquationes. Aſſumantur enim duæ æquationes, quæ diviſores admittunt, & quæ nullos habent communes inter ſe diviſores; ducantur hæ æquationes in quaſdam aſſumptas quantitates; e ſummis reſultantium æquationum, &c. conſtari poſſunt æquationes, quæ nullos admittant diviſores.

Et ſic de pluribus æquationibus plures incognitas quantitates habentibus.

2. Sint tres vel plures (h) æquationes $A = 0$, $B = 0$, $C = 0$, $D = 0$, &c. tres vel plures (h) incognitas quantitates $(x, y, z, v$, &c.) habentes, & diviſores recipiant hæ æquationes, i. e. ſint $A = \alpha \times \beta \times \gamma \times \delta \times$ &c., $B = \alpha' \times \beta' \times \gamma' \times \delta' \times$ &c., $C = \alpha'' \times \beta'' \times \gamma'' \times \delta'' \times$ &c., &c. deprimere datas æquationes in alias minores habentes dimenſiones.

Supponatur diviſor e ſingulis æquationibus $A = 0$, $B = 0$, $C = 0$, &c. nihilo reſpective æqualis; i. e. $\alpha = 0$, $\alpha' = 0$, $\alpha'' = 0$, &c. $\alpha = 0$, $\beta' = 0$, $\alpha'' = 0$, &c. $\alpha' = 0$, $\beta = 0$, $\beta' = 0$, &c. &c. radices incognitarum quantitatum reſultantium æquationum etiam erunt radices earundem incognitarum quantitatum datarum æquationum.

Cor. Sint tres vel plures æquationes $(n, m, l, k$, &c.) dimenſionum, primæ æquationis ſint (n) diviſores reſpective $y + \alpha x + \beta z + \gamma v$

 + &c.

$+$ &c. $+ a$, $y + \alpha' x + \beta' z + \gamma' v +$ &c. $+ a'$, $y + \alpha'' x + \beta'' z + \gamma'' v +$ &c. $+ a''$, &c.; ſecundæ æquationis ſint m diviſores $y + \kappa x + \lambda z + \mu v +$ &c. $+ b$, $y + \kappa' x + \lambda' z + \mu' v +$ &c. $+ b'$, $y + \kappa'' x + \lambda'' z + \mu'' v$ $+$ &c. $+ b''$, &c.; tertiæ æquationis ſint l diviſores $y + \pi x + \varrho z + \sigma v$ $+$ &c. $+ c$, $y + \pi' x + \varrho' z + \sigma' v +$ &c. $+ c'$, $y + \pi'' x + \varrho'' z + \sigma'' v +$ &c. $+ c''$, &c.; & ſic deinceps. Supponatur omnis diviſor e ſingulis æquationibus nihilo reſpective æqualis, i. e. $y + \alpha x + \beta z +$ &c. $+ a = o$, $y + \kappa x + \lambda z +$ &c. $+ b = o$, $y + \pi x + \varrho z +$ &c. $+ c = o$, &c.; & $y + \alpha x + \beta z +$ &c. $+ a = o$, $y + \kappa' x + \lambda' z +$ &c. $+ b' = o$, $y + \pi x + \varrho z +$ &c. $+ c = o$, &c.; & $y + \alpha x + \beta z +$ &c. $+ a = o$, $y + \kappa x +$ $+ \lambda z +$ &c. $+ b = o$, $y + \pi' x + \varrho' z +$ &c. $= o$, &c.; & ſic deinceps; tum ex omnibus hiſce æquationibus reſultantibus deduci poſſunt valores correſpondentes e ſingulis incognitis quantitatibus y, x, z, v, &c.

Hinc $n \times m \times l \times k \times$ &c. poſſunt eſſe diverſi correſpondentes valores incognitarum quantitatum (x; y, z, v, &c.); & exinde æquatio, cujus radix eſt (vel y, vel x, vel z, &c.) aſcendere poteſt ad $n \times m \times l \times k \times$ &c. dimenſiones; unde, quoniam maximæ dimenſiones, ad quas aſcendit æquatio, cujus radix eſt x vel y vel z, &c. pendent e terminis quorum dimenſiones ſunt maximæ in reſpectivis æquationibus, conſtat (ſi modo dentur quæcunque (b) æquationes, quarum dimenſiones ſunt n, m, l, k, &c. reſpective, totidem incognitas quantitates (x, y, z, v, &c.) habentes; & reducantur hæ (b) æquationes in unam, ita ut exterminentur omnes præter unam (y) incognitæ quantitates) reſultantem æquationem ($b \times m \times l \times k \times$ &c.) & haud plures dimenſiones habere poſſe.

Hæ æquationes non poſſunt habere ($n \times m \times l \times k \times$ &c.) correſpondentes valores ad libitum aſſumptos e ſingulis incognitis quantitatibus (y, x, z, &c.).

Si dentur ($n \times m \times l \times k \times$ &c. $+ 1$) diverſi correſpondentes valores e ſingulis incognitis quantitatibus (y, x, z, v, &c.); tum ex multiplicando ſingulas datas (b) æquationes in quaſcunque aſſumptas quantitates, ita ut reſultent (b) diverſæ æquationes, quarum incognitarum

quantitatum $(y, x, z,$ &c.$)$ radices eædem ſunt ac radices earundem quantitatum datarum æquationum, deduci poſſunt duæ vel plures ex reſultantibus æquationibus, quæ communem habent diviſorem.

3. Sint tres vel plures (b) æquationes $A = o$, $B = o$, $C = o$, &c., quarum dimenſiones ſint $(n, m, l, k,$ &c.$)$ reſpective, tres vel plures (b) incognitas quantitates $(y, x, v, v,$ &c.$)$ habentes, in quibus maximæ dimenſiones ad quas aſcendit quantitas (y) ſint reſpective a, a', a'', &c.: ſupponatur etiam, quod numerus $n \times m \times l \times k \times$ &c. $- (n - a) \times (m - a') \times (l - a'') \times (k - a''') \times$ &c. ſit maxima dimenſio, ad quam aſcendere poteſt æquatio, cujus radix eſt y; & dentur $(b + 1)$ æquationes $A' = o$, $B' = o$, $C' = o$, &c.; $H = o$, quarum dimenſiones ſint $(n, m, l, k,$ &c.; $N)$ reſpective, $(b + 1)$ incognitas quantitates $(y, x, z, x,$ &c.; $V)$ habentes, in quibus maximæ dimenſiones, ad quas aſſurgit quantitas y, ſint reſpective a, a', a'', &c., $a^{b'}$; tum æquatio, cujus radix eſt y, haud aſſurget ad majores quam $n \times m \times l \times k \times$ &c. $\times N - (n - a) \times (m - a') \times (l - a'') \times (k - a''') \times$ &c. $\times (N - a^{b'})$ dimenſiones.

Aſſumatur enim æquatio $H = o$ $(V + \alpha x + \beta z +$ &c.$) \times (V + \alpha' x + \beta' z +$ &c.$) \times (V + \alpha'' x + \beta'' z +$ &c.$) \times$ &c. $\times (y^{a^{b'}} + (\pi x + \rho z + \sigma v +$ &c. $+ \tau) \times y^{a^{b'} - 1} +$ &c.$)$, in quâ numerus factorum hujuſce generis $V + \alpha x + \beta z +$ &c., $V + \alpha' x + \beta' z +$ &c., &c., ſit $N - a^{b'}$: ſupponantur $V + \alpha x + \beta z +$ &c. $= o$, $V + \alpha' x + \beta' z +$ &c. $= o$, &c. & ſcribantur pro V ejus valores exinde reſultantes in (b) prioribus æquationibus $A' = o$, $B' = o$, $C' = o$, &c.; tum e ſingulis hiſce (b) æquationibus reſultantibus per hypotheſin æquatio, cujus radix eſt y, &c., haud plures habet quam $n \times m \times l \times k \times$ &c. $- (n - a) \times (m - a') \times (l - a'') \times (k - a''') \times$ &c. radices; ergo ex $(N - a^{b'})$ ſubſtitutionibus pro V in prædictis æquationibus reſultant haud plures quam $(N - a^{b'}) \times (n \times m \times l \times k \times$ &c. $- (n - a) \times (m - a') \times (l - a'') \times (k - a''') \times$ &c.$)$ diverſæ radices quantitatis y; ſed ex (b) æquationibus $A' = o$, $B' = o$, $C' = o$, &c. & æquatione $y^{a^{b'}} + (\pi x + \rho x + \sigma v +$ &c. $+ \tau) \times y^{a^{b'} - 1} +$ &c. $= o$ haud reſultant plures quam $a^{b'} \times n \times m \times l \times k \times$ &c. diverſæ radices quantitatis y; & conſequenter æquatio, cujus radix

eſt

eſt y, haud plures quam $(N - a^{b'}) \times (n \times m \times l \times k \times \&c. - (n - a) \times (m - a') \times (l - a'') \times (k - a''') \times \&c.) + a^{b'} \times n \times m \times l \times k \times \&c. = N \times n \times m \times l \times k \times \&c. - (N - a^{b'}) \times (n - a) \times (m - a') \times (l - a'') \times (k - a''')$ $\times$ &c. dimenſiones habet.

In prædictis æquationibus $A' = o$, $B' = o$, $C' = o$, $D' = o$, &c., $H = o$, ſint a, a', a'', a''', &c. maximæ dimenſiones quantitatis (y); quamobrem deprimetur æquatio, cujus radix eſt y, &c. per $(n - a) \times (m - a') \times (l - a'') \times (k - a''') \times$ &c. dimenſiones; & ſimiliter ſint b, b', b'', b''', &c.; c, c', c'', c''', &c.; d, d', d'', d''', &c.; &c. maximæ dimenſiones quantitatum x, z, v, &c. reſpective, in prædictis æquationibus $A' = o$, $B' = o$, $C' = o$, &c.; tum etiam deprimetur æquatio, cujus radix eſt y, &c. per $(n - b) \times (m - b') \times (l - b'') \times (k - b''') \times$ &c., $(n - c) \times (m - c') \times (l - c'') \times (k - c''') \times$ &c., $(n - d) \times (m - d') \times (l - d'') \times (k - d''') \times$ &c.; ergo æquatio, cujus radix eſt y, &c. haud plures habet dimenſiones quam

$$n \times m \times l \times k \times \&c. - (n - a) \times (m - a') \times (l - a'') \times (k - a''') \times \&c.$$
$$- (n - b) \times (m - b') \times (l - b'') \times (k - b''') \times \&c.$$
$$- (n - c) \times (m - c') \times (l - c'') \times (k - c''') \times \&c.$$
$$- (n - d) \times (m - d') \times (l - d'') \times (k - d''') \times \&c. \ \&c.$$

Et ſic facile in multis caſibus e terminis maximarum dimenſionum in datis æquationibus contentarum per principia prius tradita erui poſſunt maximæ dimenſiones æquationis, cujus radix eſt incognita quantitas (y) in datis æquationibus contenta.

Si vero termini datarum æquationum, quorum dimenſiones ſunt maximæ, communem habeant diviſorem; vel ex iis per additionem, ſubductionem, multiplicationem in incognitas quantitates formari poſſunt æquationes independentes, quæ communem diviſorem (r) dimenſionum habeant; tum ſaltem dimenſiones æquationis, cujus radix eſt incognita quantitas (y), erunt minores per r quam dimenſiones prius traditæ.

Ex methodo infinitarum ſerierum prius traditâ ſemper detegi poſſunt dimenſiones, per quas diminui debent dimenſiones incognitæ quantitatis prius inventæ.

Reduci poſſunt æquationes, ita ut incognitæ quantitates exterminentur, & inveniri poſſunt æquationum diviſores ita dividendo æquationem hâc methodo deductam in duas partes, ut ex utrâque parte eadem radix extrahatur: etiamque hæc perfici poſſunt e transformatione datarum æquationum in alias, quarum radices datam habeant aſſignabilem relationem ad radices datarum æquationum; hâc etiam methodo tolli poſſunt ex hiſce æquationibus duo vel plures termini æque ac ex æquatione unam ſolummodo incognitam quantitatem habente: & inveniri poſſunt æquationes, quas reducere licet; transformentur enim datæ æquationes, ita ut reſultantes plures habeant dimenſiones quam datæ, & rurſus reducantur transformatæ æquationes in datas; & ſic deinceps. Et ex eodem modo, quo prius docetur inveſtigandi conſtitutionem transformatæ æquationis unam ſolummodo incognitam quantitatem habentis methodus, inveſtigari etiam poſſunt coefficientes transformatarum æquationum plures incognitas quantitates habentium.

PROB. XLII.

Invenire duas vel tres, &c. (h) *æquationes duas vel tres, &c.* (h) *incognitas quantitates* (x, y, z, &c.) *habentes, quæ reſolvi poſſunt.*

1. Aſſumantur quæcunque algebraicæ quantitates reſpective pro reſolutione incognitarum x, y, z, &c.: inveniantur e principiis in præcedente capite traditis numeri valorum diverſorum e ſingulis irrationalibus quantitatibus in prædictis reſolutionibus contentis, qui ſint reſpective (π, ϱ, σ, τ, &c.); tum aſſumantur *(h)* æquationes formulæ $y^n + (\alpha + bx + cz +$ &c.$)\, y^{n-1} + (A + Bx + Cz +$ &c. $+ Ex^2 + Fxz + Gz^2 +$ &c.$)\, y^{n-2} +$ &c. $= 0$: in his æquationibus pro incognitis quantitatibus (y, x, z, &c.) ſcribantur reſpective earum aſſumpti valores; deinde in ſingulis æquationibus reſultantibus fiant coefficientes omnis diverſæ irrationalis quantitatis, etiamque omnes

rationales

rationales termini ſimul adjuncti, reſpective nihilo æquales; ſi modo poſſibile ſit: ex ſimplicibus æquationibus exinde reſultantibus erui poſſunt coefficientes a, b, c, d, &c.; A, B, C, D, &c., &c.

Si tot ſint ſimplices reſultantes æquationes, quæ ſunt independentes; quot coefficientes a, b, c, &c.; A, B, C, &c. in iis continentur; tum ex datis æquationibus inveniri poſſunt coefficientes a, b, c, &c.; A, B, C, &c.; &c.: ſi numerus coefficientium major ſit quam numerus independentium æquationum; tum tot coefficientes ad libitum aſſumi poſſunt, quot ſit differentia inter numerum coefficientium & numerum independentium æquationum.

Hæ reſolutiones plerumque haud erunt generales reſolutiones æquationum reſultantium; i.e. haud continebunt omnes radices earum incognitarum quantitatum.

In his caſibus ſemper cavendum eſt, ne duæ vel tres vel plures e prædictis *(b)* æquationibus ſint eædem; vel generaliter, ne ſit æquatio, quæ e reliquis pendet.

Numerus diverſarum irrationalium quantitatum in ſingulis æquationibus contentarum haud ſuperat $\pi \times \varrho \times \sigma \times \tau \times$ &c.

2. Si valores aſſumpti diverſarum literarum $(x, y, z$, &c.) haud eaſdem irrationales quantitates contineant; tum haud raro facilius erit aſſumere æquationes hujuſce formulæ $Ay^n + ay^{n-1} +$ &c. $+ Bx^n + bx^{n-1} +$ &c. $+ Cx^n + cx^{n-1} +$ &c. $= o$, in quarum terminis haud datur contentum hujuſce generis $y^\alpha x^\beta z^\gamma$ &c., ubi α, β, γ, &c. ſunt integri numeri: e.g. ſint $y = \sqrt{p} + q$, $x = \sqrt{p'} + q'$ & $z = \sqrt{p''} + q''$, &c.; aſſumantur æquationes hujuſce formulæ $y^2 + ay + bx^2 + cx + dz^2 + ez + f = o$: in his æquationibus aſſumptis pro y, x & z ſcribantur earum prius aſſumpti valores; reſultat $p + q^2 + aq + b \times \overline{p' + q'^2} + cq' + d\overline{p'' + q''^2} + eq'' + f + (2q + a)\sqrt{p} + (2bq' + c)\sqrt{p'} + (2dq'' + e)\sqrt{p''} = o$; fiant omnes termini, in quibus irrationales quantitates $(\sqrt{p}, \sqrt{p'}, \sqrt{p''})$ continentur nihilo reſpective æquales; unde reſultant æquationes $2q + a = o$ & exinde $a = -2q$; $2bq' + c = o$ & exinde $c = -2bq'$ & conſequenter b ad libitum aſ-

ſumi

ſumi poteſt: $2dq'' + e = 0$ & exinde $e = -2dq''$, ubi e ad libitum aſſumi poteſt: & $p + q^2 + aq + b \times \overline{p' + q'^2} + cq' + d\overline{p'' + q''^2} + eq'' + f = 0$; unde deduci poteſt coefficiens f, & conſequenter æquatio aſſumpta: & ſimiliter aſſumi poſſunt duæ aliæ diverſæ & independentes æquationes hujuſce formulæ; & conſequenter reſultant tres æquationes tres incognitas quantitates habentes, quarum nonnullæ radices, i. e. valores quantitatum x, y & z ſunt datæ quantitates.

Si dati valores duarum vel plurium incognitarum quantitatum (x, y, z, &c.) eaſdem irrationales quantitates involvant; tum plerumque præſtat aſſumere æquationes, in quibus continentur quantitates $x^{\alpha} y^{\beta}$ &c., ubi α, β, &c. ſunt integri numeri.

THEOR. XXXIV.

Sint $y = a\sqrt[n]{p} + b\sqrt[n]{p^2} + c\sqrt[n]{p^3} \ldots \sqrt[n]{p^{n-1}} + A\sqrt[m]{q} + B\sqrt[m]{q^2} \ldots \sqrt[m]{q^{m-1}} + H\sqrt[n]{p} \times \sqrt[m]{q} + K\sqrt[n]{p} \times \sqrt[m]{q^2} + L\sqrt[n]{p^2} \times \sqrt[m]{q} +$ &c. $+ \alpha\sqrt[k]{r} + \beta\sqrt[k]{r^2} +$ &c. $+ \pi\sqrt[n]{p^i} \times \sqrt[m]{q^o} \times \sqrt[k]{r^u} +$ &c.

$x = a'\sqrt[n]{p} + b'\sqrt[n]{p^2} \ldots \sqrt[n]{p^{n-1}} + A'\sqrt[m]{q} + B'\sqrt[m]{q^2} \ldots \sqrt[m]{q^{m-1}} + H'\sqrt[n]{p} \times \sqrt[m]{q} + K'\sqrt[n]{p} \times \sqrt[m]{q^2} +$ &c. $+ \alpha'\sqrt[k]{r} +$ &c.

$z = a''\sqrt[n]{p} +$ &c., &c.: ubi literæ a, b, c, &c.; A, B, &c.; H, K, L, &c. α, β, &c., π, &c.; &c.; a', b', &c.; A', B', &c.; H', K', &c.; α', &c.; α'', &c.; p, q, r, &c. datas quantitates; literæ x, y, z, &c. incognitas quantitates, quarum numerus eſt (h); & literæ n, m, r, s, &c. integros numeros reſpective denotant.

Inveniatur ſumma e ſingulis numeris $m-1$, $n-1$, $r-1$, &c. rectangulorum e ſingulis duobus; contentorum ſub quibuſque tribus, quatuor, quinque, &c. prædictis numeris $m-1$, $n-1$, $r-1$, &c.; dicantur hæ ſummæ reſpective P, Q, R, S, T, &c. aſſumantur (h) æquationes hujuſce formulæ $\Gamma y^l + \overline{\delta + \epsilon x +\text{ &c.}}\, y^{l-1} + \overline{\zeta + \eta x + \theta x^2 +\text{ &c.}}\, y^{l-2} +$

$y^{i-2} + \&c. = 0$, $\Gamma' y^i + \overline{\delta' + \epsilon' x + \&c.}\, y^{i-1} + \&c. = 0$, &c.: ita quidem ut $i \cdot \frac{i+1}{2} \cdot \frac{i+2}{3} \dots \frac{i+b-1}{b} - 1$ vel æqualis vel major ſit quam ſumma $P + Q + R + S + \&c.$; in his æquationibus pro y, x, z, &c. ſcribantur earum prædicti valores; deinde fiant coefficientes e ſingulis irrationalibus quantitatibus in reſultantibus æquationibus nihilo reſpective æquales; & ex æquationibus exinde reſultantibus deduci poſſunt coefficientes Γ, δ, ϵ, &c.; Γ', δ', ϵ', &c. (b) diverſarum independentium æquationum; & conſequenter (b) diverſæ independentes æquationes, quarum incognitæ quantitates $(x, y, z, \&c.)$ habent valores prius aſſignatos.

Quantitates x, y, z, &c. plerumque plures habent quam aſſignatos valores.

Hæc methodus præbebit reſolutionem duarum vel plurium æquationum duas vel plures incognitas quantitates involventium, maxime generalem adhuc cognitam.

THEOR. XXXV.

Sint duæ vel tres, &c. (b) datæ æquationes duas vel tres, &c. (b) incognitas quantitates involventes; in his æquationibus contineantur quæcunque irrationales quantitates; tum ita reducantur per prob. 26. omnes hæ æquationes reſpective, ut exterminentur irrationales quantitates; reſultant æquationes, quarum radices incognitarum quantitatum eædem erunt ac radices earundem incognitarum quantitatum datarum æquationum.

Cor. Hinc etiam deprimi poſſunt æquationes reſultantes ad datas: & per ſubſtitutionem conſimilem eæ prius datæ in prob. 30. deduci poteſt reductio duarum vel plurium æquationum duas vel plures incognitas quantitates habentium ad inferiores generalior quam quævis alia adhuc cognita.

PROB.

Hic viſum eſt adjungere nonnullas methodos æquationes vel reſolvendi vel reducendi.

1. Sint duæ vel plures æquationes datæ duas vel plures incognitas quantitates $(x, y, z,$ &c.$)$ habentes; ex quibuſdam datis æquationibus inveniantur quantitates, quæ vel in reliquis continentur, vel e quibus facile deduci poſſunt quantitates in reliquis contentæ: ſubſtituantur hæ quantitates in reliquis pro ſuis valoribus, & nonnunquam reſultat reductio quæſita.

1. 2. Sæpe ex hujuſmodi ſubſtitutione exterminari poſſunt e reliquis æquationibus quædam incognitæ quantitates.

2. Ex æquatis diverſis valoribus ejuſdem quantitatis e diverſis æquationibus petitis, ſæpe deduci poteſt reſolutio vel reductio quæſita.

3. Ex additione, ſubtractione, multiplicatione, &c. diverſarum æquationum ſæpe deduci poteſt valor quantitatis, quâ quidem inventâ, facile deduci poteſt reductio quæſita.

4. Inveniatur, annon quædam e datis æquationibus recipiant diviſores; vel ex additione, ſubductione, multiplicatione, &c. datarum æquationum in quaſcunque quantitates ductarum, erui poſſunt æquationes, quæ recipiant diviſores: ſi diviſores recipiant æquationes reſultantes, tum deprimi poſſunt.

5. Sit æquatio duas vel plures (n) incognitas quantitates $(x, y, z,$ &c.$)$ involvens, in quâ ſolummodo termini n & $n-1$, vel n, $n-1$ & $n-2$ dimenſionum, & ſic deinceps, continentur: vel ſi modo talis æquatio ex additione, ſubductione, &c. datarum æquationum deduci poteſt: in eâ pro x, z, &c. ſcribantur πy, ϱy, &c.; reſultat æquatio ex quâ facile deduci poteſt quantitas y.

6. Nonnunquam e diverſis transformationibus conſtabit æquationum reductio vel reſolutio.

7. Ex (n) datis æquationibus (n) incognitas quantitates habentibus $(x, y, z,$ &c.$)$ nonnunquam conſtabit formula æquationis reſultantis, cujus radix eſt x vel y vel z, &c. vel ex iis quædam functio.

Si talis proponatur formula æquationis reſultantis, ut ejus duæ vel

plures radices datam inter ſe habeant relationem: ſubſtituantur duæ quantitates generaliter aſſumptæ datam relationem inter ſe habentes pro ſuis valoribus in datis æquationibus; & ſi ex utriſque ſubſtitutionibus eædem reſultent æquationes; tum æquatio reſultans habet formulam propoſitam.

Ex. Sint æquationes datæ $y^2+xz=a$, $x^2+yz=b$, $z^2+yx=c$; invenire, annon omnis affirmativus valor incognitarum quantitatum (x, y, z) habet negativum ei æqualem.

In datis æquationibus pro x, y & z ſcribantur reſpective α, λ & π; etiamque earum negativæ $-\alpha$, $-\lambda$ & $-\pi$; reſultant eædem æquationes $\alpha^2+\lambda\pi=a$, $\lambda^2+\alpha\pi=b$, $\pi^2+\alpha\lambda=c$, ex utrâque ſubſtitutione; ergo omnis affirmativus valor e ſingulis incognitis quantitatibus habet negativum ei æqualem.

Æquatio, cujus radix eſt $(y$, &c.$)$ habeat formulam prædictam, quamvis e prædictis ſubſtitutionibus haud conſtet.

Ex. 2. Sint duæ æquationes $x^2-3y^2+7y=a$ & $1-3y^2x^2+7yx^2=ax^2$; invenire, annon æquatio, cujus radix eſt x, ſit recurrens.

Si æquatio ſit recurrens; tum, ſi α ſit radix, $\frac{1}{\alpha}$ etiam erit radix: ſcribantur igitur α & $\frac{1}{\alpha}$ pro x in duabus datis æquationibus; & e priori ſubſtitutione reſultant æquationes $\alpha^2-3y^2+7y=a$ & $1-3\alpha^2y^2+7\alpha^2y=a\times\alpha^2$; e poſteriori vero $\frac{1}{\alpha^2}-3y^2+7y=a$ (unde $1-3\alpha^2y^2+7\alpha^2y=a\alpha^2$) & $1-\frac{3y^2}{\alpha^2}+\frac{7y}{\alpha^2}=\frac{a}{\alpha^2}$ (unde $\alpha^2-3y^2+7y=a$); conſequenter poſteriores æquationes eædem ſunt ac priores; ergo æquatio erit recurrens, cujus radix eſt x.

8. Si dentur irrationales quantitates in datis æquationibus, quæ ſunt inter ſe independentes, & pro iis ſubſtituantur quæcunque novæ aſſumptæ quantitates; exinde ſæpe deduci poteſt magis facilis reſolutio.

Ex. Sint

Ex. Sint quatuor æquationes $(x^\alpha + y^\alpha + z^\alpha + w^\alpha)^m \times (x^\alpha + y^\alpha + z^\alpha)^n = a$, $(x^\beta + y^\beta + w^\beta)^r \times (x^\beta + y^\beta + z^\beta + w^\beta)^s = b$, $(x^\alpha + y^\alpha + z^\alpha + w^\alpha)^m + (x^\beta + y^\beta + w^\beta)^r = c$, $(x^\beta + y^\beta + z^\beta + w^\beta)^s + (x^\alpha + y^\alpha + z^\alpha)^n = d$, pro irrationalibus $(x^\alpha + y^\alpha + z^\alpha + w^\alpha)^m$, $(x^\alpha + y^\alpha + z^\alpha)^n$; $(x^\beta + y^\beta + z^\beta + w^\beta)^s$ & $(x^\beta + y^\beta + w^\beta)^r$ ſcribantur reſpective A, B, C & D; reſultant $A \times B = a$, $C \times D = b$, $A + D = c$, $B + C = d$: unde facile conſtant A, B, C & D; & exinde y, x, z & w.

9. Invenire æquationes, quæ deprimi poſſunt: aſſumantur æquationes, ita transformentur hæ æquationes ut plures habeant dimenſiones æquationes transformatæ quam datæ, & transformatæ facile deprimi poſſunt in datas.

Hæ methodi haud multum diverſæ ſunt ab iis in capite præcedenti contentis, in quibus tractatum eſt de unâ æquatione unam ſolummodo incognitam quantitatem habente.

10. Haud nunquam in perfectas poteſtates reduci poſſunt datæ æquationes, & exinde radices extrahendo reducentur æquationes.

Hæ poteſtates ſemper facile perfectæ reddantur, ſi modo fieri poſſint, ex additione vel ſubductione, &c.

11. Si vero ita transformentur datæ æquationes in alias, ut transformatæ æquationes eaſdem habeant dimenſiones, quas habeant datæ; & termini duarum vel plurium æquationum, in quibus dimenſiones incognitarum quantitatum inveniuntur maximæ, communem habeant diviſorem; vel evaneſcant termini, in quibus nullæ inveniuntur dimenſiones incognitarum quantitatum; vel magis generaliter quædam radices nihilo fiant æquales; vel ita transformentur datæ æquationes in alias, quarum dimenſiones eædem ſunt ac datæ, & quædam radices reſultantium æquationum æquales inter ſe fiant; tum deprimi poſſunt dimenſiones datarum æquationum incognitarum quantitatum.

Hactenus de methodis deprimendi datas æquationes duas vel plures incognitas quantitates habentes in alias, vel minores dimenſiones habentes; vel quarum formulæ vel ſint minores; vel quæ formulam habeant, cujus reſolutio vel reductio cognoſcitur.

Ex. 1. Sint æquationes $x^n + n \times \frac{n-1}{2} x^{n-2} y^2 + n \,.\, \frac{n-1}{2} \times \frac{n-2}{3} \,.\, \frac{n-3}{4} x^{n-4} y^4 + \&c. = a$, & $x^{n-1} y + \frac{n-1}{2} \times \frac{n-2}{3} \times x^{n-3} y^3 + \frac{n-1}{2} \,.\, \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times x^{n-5} y^5 + \&c. = B$; erit $x + y = \sqrt[n]{A + nB}$, & $x - y = \sqrt[n]{A - nB}$: erunt enim (n) valores ſummæ $(x+y)$ & differentiæ $(x-y)$ ſemper inter ſe æquales; conſequenter æquatio, cujus radix ſit $x+y$ vel $x-y$, haud plures quam n dimenſiones; æquatio autem, cujus radix ſit x vel y, habet $n \times n$ dimenſiones.

Ex. 2. Sint æquationes $\overline{x+y} \times \overline{x^{n-1} - a x^{n-2} y + \overline{n-2} \times \frac{n-1}{2} x^{n-3}}$ $\overline{y^2 - \frac{n-2}{2} \times \frac{n-3}{3} a x^{n-4} y^3} + \&c. = A$, & $\overline{x+y} \times \overline{x^{n-2} y + \frac{\overline{n-2}}{2} \times}$ $\overline{\frac{n-3}{3} x^{n-4} y^3} + \&c. = B$; erit $x + y = \sqrt[n]{A + \overline{a + \overline{n-1}} B}$.

Ex. 3. Sint $\overline{x-y} \times \overline{x^{n-1} - a x^{n-2} y + \overline{n-2} \times \frac{n-1}{2} x^{n-3} y^2 - \frac{n-3}{2}}$ $\overline{\times \frac{n-2}{3} a x^{n-4} y^3} + \&c. = A$, & $\overline{x-y} \times \overline{x^{n-2} y + \frac{\overline{n-3}}{2} \times \frac{\overline{n-2}}{3} x^{n-4} y^3} + \&c. = B$, & erit $x - y = \sqrt[n]{A + \overline{a - \overline{n-1}} B}$.

Et ſic de infinitis hujuſcemodi æquationibus.

Ex. 4. Sint datæ æquationes $z^m y^r z^s v^t \&c. = A$, $\overset{\prime}{x}{}^m \overset{\prime}{y}{}^r \overset{\prime}{z}{}^s \overset{\prime}{v}{}^t \&c. = B$, $\overset{\prime\prime}{x}{}^m \overset{\prime\prime}{y}{}^r \overset{\prime\prime}{z}{}^s \overset{\prime\prime}{v}{}^t \&c. = C$, $\overset{\prime\prime\prime}{x}{}^m \overset{\prime\prime\prime}{y}{}^r \overset{\prime\prime\prime}{z}{}^s \overset{\prime\prime\prime}{v}{}^t \&c. = D$, &c.

Facile conſtat æquationes, quarum radices vel ſint x, vel y, vel z, vel v, &c. eſſe quoad earum formulas ſimplices: & ex his additione vel ſubtractione, &c. infinitæ formari poſſe æquationes, quæ retro in has æquationes facile reduci poſſunt.

THEOR.

THEOR. XXXVI.

1. Sint *(b)* æquationes $p+q+r+s+t+$ &c. $=o$, $p'+q'+r'+s'+$ &c. $=o$, $p''+q''+r''+s''+$ &c. $=o$, $p'''+q'''+r'''+$ &c. $=o$, &c.; totidem *(b)* incognitas quantitates $(x, y, z, v, w,$ &c.$)$ habentes, in quibus ſint $p, q, r, s, t,$ &c. homogeneæ functiones incognitarum quantitatum $(x, y, z,$ &c.$)$, quarum dimenſiones ſunt reſpective n, $n-m$, $n-2m$, $n-3m$, &c.: & ſimiliter ſint p', q', r', s', &c.; p'', q'', r'', &c.; p''', q''', r''', &c.; &c.; homogeneæ functiones earundem incognitarum quantitatum $(x, y, z,$ &c.$)$, quarum dimenſiones ſunt reſpective n', $n'-m$, $n'-2m$, $n'-3m$, &c.; n'', $n''-m$, $n''-2m$, &c.; n''', $n'''-m$, $n'''-2m$, $n'''-3m$, &c.; &c.: tum æquatio, cujus radix eſt x, habet formulam $x^n+Px^{n-m}+Qx^{n-2m}+Rx^{n-3m}+$ &c. $=o$: conſimiles formulas etiam habent æquationes, quarum radices ſunt incognitæ quantitates $(y, z,$ &c.$)$. Per homogeneas functiones quantitatum $(x, y, z,$ &c.$)$ intelligo functiones in ſingulis quarum terminis eædem continentur dimenſiones prædictarum quantitatum $(x, y, z,$ &c.$)$

2. Sint duæ æquationes $x^n-px^{n-1}+qx^{n-2}-$ &c. $=o$ & $y^m-Py^{m-1}+Qy^{m-2}-$ &c. $=o$, ubi $p, q,$ &c.; $P, Q,$ &c.; ſunt datæ quantitates; ſi utcunque ex his æquationibus deducantur duæ æquationes ſimul involventes quantitates x & y, quarum radices ſint radices datarum æquationum; tum in æquationibus reſultantibus ſemper dabuntur *(n)* iidem valores quantitatis y, qui correſpondent ſingulo valori quantitatis x; & m iidem valores quantitatis x, qui correſpondent ſingulo valori quantitatis y.

THEOR. XXXVII.

1. Sint duæ $A=o$ & $B=o$, vel plures æquationes duas incognitas quantitates $(x$ & $y)$ habentes & ſimiliter involventes; reducantur hæ æquationes in unam, ita ut exterminetur altera incognita quantitas

titas (y); æquatio reſultans, cujus radix eſt x, duplas habet dimenſiones æquationis, cujus radix eſt $x+y$, vel x^2+y^2, vel x^3+y^3, vel xy, vel quæcunque rationalis functio quantitatum x & y, in quâ (x & y) ſimiliter involvuntur.

DEMONSTRATIO.

Sint radices incognitæ quantitatis (x) reſpective $\alpha, \beta, \gamma, \delta, \ldots \pi, \rho, \sigma, \tau$; correſpondentes vero radices incognitæ quantitatis (y) erunt reſpective $\tau, \sigma, \rho, \pi, \ldots \delta, \gamma, \beta, \alpha$: incognitæ quantitatis $(z=x+y)$ reſpectivi valores erunt $\alpha+\tau$, $\beta+\sigma$, $\gamma+\rho$, $\delta+\pi$, ... $\pi+\delta$, $\gamma+\rho$, $\beta+\sigma$, $\alpha+\tau$; ſit enim α valor quantitatis x, & τ ejus correſpondens valor incognitæ quantitatis y; tum erit τ valor quantitatis x & α ejus correſpondens valor quantitatis y: ſcribantur in æquationibus $A=0$ & $B=0$, &c. pro x & y reſpective α & τ, vel τ & α, & eædem reſultabunt æquationes; conſequenter, ſi α & τ ſint correſpondentes valores quantitatum (x & y), erunt (τ & α) correſpondentes valores earundem quantitatum (x & y): unde omnis valor incognitæ quantitatis $(z=x+y)$ alterum habet ſibi ipſi æqualem ($\alpha+\tau$, $\tau+\alpha$, $\beta+\sigma$, $\sigma+\beta$, &c.); & conſequenter incognita quantitas (x aut y) duplum habet dimenſionum numerum, quem habet incognita quantitas $(x+y)$; & æquatio, cujus radix eſt (x aut y), duplas habet dimenſiones æquationis, cujus radix eſt $x+y$; &c.

Ex. 1. Sint duæ æquationes $(x^2+y^2)\times(x^4+y^4)=b$, & $(x^2+y^2)\times(x+y)=a$; utriuſque æquationis terminorum, in quibus incognitarum quantitatum (x vel y) dimenſiones ſunt maximæ, communis eſt diviſor (x^2+y^2); unde ſequitur æquationem, cujus radix eſt (x vel y), dimenſiones ($6\times3=18$) non habere; ejus dimenſiones eædem erunt ac dimenſiones æquationis, cujus radix eſt x vel y, ex æquationibus $(x^2+y^2)\times(x+y)=a$ & $a(x^4+y^4)=b\overline{x+y}$ deductæ; (ſcribatur enim pro x^2+y^2 in primâ æquatione $\left(\frac{a}{x+y}\right)$ ejus valor e poſteriori

ſteriori deductus) quæ erunt (12): ſed hæ duæ æquationes incognitas quantitates $(x \,\&\, y)$ ſimiliter involvunt; & conſequenter æquatio, cujus radix eſt $(z = x + y)$ ſolummodo ſex habet dimenſiones; & e theor. præced. patet ejus formulam eſſe quadraticam, viz. $a z^6 + \overline{2b - 2a^2}\, z^3 - a^3 = 0$.

Ex. 2. Sint duæ æquationes $x^4 + x^3 y + x^2 y^2 + xy^3 + y^4 = a$ & $(x^4 + x^3 y + x^2 y + xy^3 + y^4) \times (x^4 - x^3 y + x^2 y^2 - xy^3 + y^4) = b$; harum æquationum termini, in quibus incognitarum quantitatum $(x$ aut $y)$ dimenſiones ſunt maximæ, communem habent diviſorem $(x^4 + x^3 y + x^2 y^2 + xy^3 + y^4)$; & conſequenter æquatio, cujus radix eſt $(x$ aut $y)$, non habet $(4 \times 8 = 32)$ dimenſiones; habet vero $(4 \times 4 = 16)$ dimenſiones: ſint enim α, β, γ & δ radices (x) æquationis $x^4 + x^3 y + x^2 y^2 + xy^3 + y^4 = a$; ſcribantur hæ radices α, β, γ & δ reſpective pro x in æquatione $(x^4 + x^3 y + x^2 y^2 + xy^3 + y^4)\,(x^4 - x^3 y + x^2 y^2 - xy^3 + y^4) = b$, & reſultant æquationes $a\,(\alpha^4 - \alpha^3 y + \alpha^2 y^2 - \alpha y^3 + y^4) - b = 0$, $a\,(\beta^4 - \beta^3 y + \beta^2 y - \beta y^3 + y^4) - b = 0$, &c. & exinde ſequitur dimenſiones æquationis, cujus radix eſt y, eſſe $4 . 4 = 16$: vel aliter; in poſteriori æquatione pro $x^4 + x^3 y + x^2 y^2 + xy^3 + y^4$ ſcribatur valor ejus (a) ex priori æquatione deductus; eadem, ut e prædictis conſtat, erit formula æquationis, cujus radix eſt x vel y ac octo dimenſionum æquatio, i. e. ejus negativæ & affirmativæ radices erunt inter ſe æquales: ſed hæ duæ æquationes incognitas quantitates $(x \,\&\, y)$ ſimiliter involvunt; & exinde æquatio, cujus radix eſt $(z = x^2 + y^2)$, non habere poteſt plures quam $\left(\frac{16}{4} = 4\right)$ dimenſiones; etiamque conſtat ejus formulam eſſe quadraticam, viz. $4a^2 z^4 - \overline{2a^2 + 2b} \times a z^2 = \overline{a^2 - b}^2$.

Ex. 3. Sint duæ æquationes $(x^4 + y^4) \times (x^2 + y^2) = b x^2 y^2$, & $(x^2 + y^2) \times (x + y) = a x y$; quoniam in his duabus æquationibus $(x^2 + y^2)$ erit communis quadraticus diviſor terminorum, in quibus maximæ inveniuntur (ſex & tres) dimenſiones; & duodecim radices nihilo inveniuntur æquales; ex eo, quod evaneſcunt termini unius

unius æquationis, in quibus nulla vel una invenitur dimenſio; alte rius vero evaneſcunt termini, in quibus inveniuntur nulla, una, duæ, & tres dimenſiones; & terminorum duarum dimenſionum unius æquationis, & quatuor dimenſionum alterius (xy) eſt diviſor; ergo æquatio, cujus radix ſit x vel y, $6 \times 3 - 12 - 2 = 4$ ſolummodo habet dimenſiones; & exinde, quoniam ſimiliter involvuntur termini x & y, æquatio, cujus radix ſit $x + y = z$, duas ſolummodo habebit dimenſiones; erit $z^2 + \frac{b}{a}z + \frac{b - a^2}{2} = 0$: aliter hæ duæ æquationes ita ſcribi poſſunt $a \times (x^4 + y^4) = bxy \times (x + y)$ & $(x^2 + y^2) \times (x + y) = axy$, in quibus octo radices quantitatum $(x$ & $y)$ nihilo evadent æquales.

Ex. 4. Sint duæ æquationes $x^3 + x^2y + xy^2 + y^3 = axy$, & $(x^3 + x^2y + xy^2 + y^3) \times (x^6 - x^5y + x^3y^3 - xy^5 + y^6) = bx^3y^3$; conſtat e prædictis rationibus æquationem, cujus radix ſit x vel y habere poſſe $9 \times 3 - 9 \times 2 - 3 = 6$ dimenſiones; & conſequenter æquationem, cujus radix ſit $x + y = z$, habere tres dimenſiones; erit $z^3 - az^2 - \overline{\frac{a^2}{3} + \frac{2b}{3a}}z + \frac{a^3}{3} - \frac{b}{3} = 0$.

Ex. 5. Sint duæ æquationes $x^3 + x^2y + xy^2 + y^3 = axy$, & $x^{3n} + x^{2n}y^n + x^ny^{2n} + y^{3n} = bx^ny^n$: ſit 1mo. $n = 6m + 3$; æquatio, cujus radix ſit x vel y, habet $3n \times 3 - 3n \times 2 - 3 = 3n - 3 = 18m + 6$ dimenſiones; duæ enim æquationes habent diviſorem trium dimenſionum inter ſe communem; & æquatio, cujus radix ſit $x + y$, habet $9m + 3$ dimenſiones: ſi vero $n = 2m$; tum æquatio cujus radix ſit x vel y, habet $3n \times 3 - 3 \times n \times 2 - 2 = 3n - 2 = 6m - 2$ dimenſiones; communem enim habent prædictæ æquationes quadraticum diviſorem; & æquatio, cujus radix ſit $x + y$, habet $3m - 1$ dimenſiones: ſit $n = 6m \pm 1$; & æquatio, cujus radix ſit x vel y, habet $3n - 1$ dimenſiones; & æquatio, cujus radix ſit $x + y$, habet $\frac{3n - 1}{2} = \frac{18m \pm 3 - 1}{2} = 9m - 2$ vel $9m + 1$; ubi m ſemper denotat integrum numerum. Et in genere

a

nere æquatio, cujus radix ſit $z = x + y$, erit $z^n + \overline{a - z}^{\,n} - \frac{nz^3}{2z + a} \times \overline{z^{n-2} + \overline{a - z}^{\,n-2}} + n \times \overline{\frac{n-3}{2}} \times \frac{z^6}{\overline{2z + a}|^2} \times \overline{z^{n-4} + \overline{a - z}^{\,n-4}} + \&c. = b$: lex hujuſce ſeriei conſtabit e problemate primo.

Ex. 6. Sint duæ æquationes $x^2 + zx + z^2 + zy + y^2 = az$, ubi $z^2 = xy$; & $x^4 + x^3y + x^2y^2 + xy^3 + y^4 = bxy$; facile conſtat æquationem, cujus radix ſit z, duas ſolummodo habere dimenſiones: erit $z^2 + \overline{\frac{1}{2}a + \frac{b}{2a}} \times z - \overline{\frac{a}{2} - \frac{b}{2a}}|^2 = o$. Æquatio autem $v = \frac{a}{2} - \frac{b}{2a}$, cujus radix $v = x + y$, duas habet radices æquales.

Et ſic ſint æquationes $x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5 = ax^2y^2$, & $x^{10} + x^8y^2 + x^6y^4 + x^4y^6 + x^2y^8 + y^{10} = bx^4y^4$; æquatio, cujus radix ſit x vel y, habet $5 \times 10 - 5 \times 8 = 10$ dimenſiones; æquatio autem, cujus radix ſit $x + y$, habet $\frac{10}{2} = 5$ dimenſiones.

Sint etiam duæ æquationes $x^7 + x^6y + x^5y^2 + x^4y^3 + x^3y^4 + x^2y^5 + xy^6 + y^7 = ax^3y^3$, & $x^{14} + x^{12}y^2 + x^{10}y^4 + x^8y^6 + x^6y^8 + x^4y^{10} + x^2y^{12} + y^{14} = bx^6y^6$; & æquatio, cujus radix ſit x vel y, habet 14 dimenſiones; ergo æquatio, cujus radix ſit $x + y$, habet 7 dimenſiones: & ſic ad infinitas conſimiles æquationes progredi licet; ſed de his ſatis.

2. Sint duæ æquationes $A = o$ & $B = o$, in quibus x & y ſimiliter involvuntur; etiamque ſint $\alpha, \beta, \gamma, \delta, \ldots \pi, \rho, \sigma, \tau$; radices quantitatis x; & ejus correſpondentes radices quantitatis y reſpective $\tau, \sigma, \rho, \pi, \ldots \delta, \gamma, \beta, \alpha$; tum erunt correſpondentes valores quantitatis $x - y$ reſpective $\alpha - \tau$, $\beta - \sigma$, $\gamma - \rho$, $\delta - \pi$, ... $\pi - \delta$, $\rho - \gamma$, $\sigma - \beta$, $\tau - \alpha$; unde numerus radicum quantitatis $(x - y)$ æqualis erit numero radicum vel quantitatis x vel y: at omnis affirmativa radix quantitatis $x - y$ habet negativam ei æqualem, i. e. affirmativa radix $\alpha - \tau$ habet

correſpondentem $\tau-\alpha$ negativam & ei æqualem; & ſimiliter $\beta-\sigma$ & $\sigma-\beta$; $\gamma-\varrho$ & $\varrho-\gamma$; $\delta-\pi$ & $\pi-\delta$; &c.; conſequenter æquatio, cujus radix eſt $x-y$, & cujus affirmativæ & negativæ radices ſunt inter ſe æquales, habet formulam æquationis, cujus dimenſiones ſunt dimidiæ dimenſiones æquationis, cujus radix eſt vel x vel y.

Idem etiam generaliter affirmari poteſt de quâcunque rationali functione (P) quantitatum $(x$ & $y)$, cujus quadratum (P^2) eſt functio, in quâ ſimiliter involvuntur quantitates x & y; hoc enim in caſu duo ſunt valores $+P$ & $-P$ inter ſe æquales, quorum unus eſt affirmativus, alter vero negativus; ergo æquatio, cujus radix eſt P, habet formulam æquationis, cujus radices ſunt dimidiæ dimenſiones æquationis, cujus radix eſt x vel y.

Si vero data functio ſint fractiones rationales, tum reducendæ ſunt ad communem denominatorem; & ſi in numeratore & denominatore vel eorum quadratis ſimiliter involvantur quantitates $(x$ & $y)$; tum æquatio, cujus radix eſt x vel y, duplas habet dimenſiones æquationis, cujus radix eſt data functio; e. g. ſit fractio $\frac{x}{y}+\frac{y}{x}$, quæ reducta ad communem denominatorem erit fractio $\frac{x^2+y^2}{yx}$, in quâ x & y ſimiliter involvuntur, &c.

Æquatio, cujus radix eſt x, ex æquationibus, in quibus ſimiliter involvuntur x & y, deducta; parem numerum radicum æqualium vel inæqualium continet; ni valor quantitatis x habeat correſpondentem valorem quantitatis y ei æqualem.

Eadem principia applicari poſſunt ad plures æquationes, in quibus plures incognitæ quantitates ſimiliter involvuntur.

3. Sint duæ prædictæ æquationes $A=0$ & $B=0$, ſit C quæcunque rationalis algebraica functio literarum $(x$ & $y)$; in fractione C pro x ſcribatur y & pro y ſcribatur x, & ſit quantitas reſultans D: aſſumatur $\frac{C}{D}=z$; tum erit æquatio, cujus radix eſt z, recurrens; & conſe-

quenter

quenter æquatio, cujus radix eſt $v = \frac{C}{D} + \frac{D}{C}$, dimidias ſolummodo habet dimenſiones æquationis, cujus radix eſt vel x vel y.

Conſtat ex ratiocinatione in prioribus exemplis uſitatâ.

THEOR. XXXVIII.

1. Erit $x^n + y^n = z^n - n z^{n-2} v + n \cdot \frac{n-3}{2} z^{n-4} v^2 - n \cdot \frac{n-4}{2} \cdot \frac{n-5}{3} z^{n-6} v^3 + n \cdot \frac{n-5}{2} \cdot \frac{n-6}{3} \cdot \frac{n-7}{4} z^{n-8} v^4 -$ &c. ubi $z = x + y$ & $v = x \times y$.

2. Sit quæcunque irrationalis quantitas, in quâ ſimiliter involvuntur incognitæ quantitates $(x, y, z,$ &c.$)$; ſi modo per prob. 26. reducatur hæc quantitas, ita ut exterminentur irrationales quantitates; ſemper reſultat rationalis quantitas, in quâ ſimiliter involvuntur prædictæ quantitates $(x, y, z,$ &c.$)$.

3. Omnis poteſtas quantitatis, in quâ literæ $(x, y, z,$ &c.$)$ ſimiliter involvuntur, etiam ſimiliter involvent prædictas quantitates $(x, y, z,$ &c.

THEOR. XXXIX.

Sit quicunque æquationum numerus (n) totidem (n) incognitas quantitates habentium, & ſint $(x, y, z, v,$ &c.$)$ quædam incognitæ quantitates ſimiliter inter ſe in ſingulâ æquatione involutæ, & quarum numerus ſit (m); ſint etiam $(X, Y, Z, V,$ &c.$)$ incognitæ quantitates ſimiliter involutæ inter ſe in ſingulâ æquatione, & quarum numerus ſit (r); ſint etiam $(\alpha, \beta, \gamma, \delta,$ &c.$)$ quantitates incognitæ ſimiliter in ſingulâ æquatione inter ſe involutæ, & quarum numerus ſit (s); & ſic deinceps: numerus dimenſionum æquationis, cujus radix ſit $x + y + z + v +$ &c. $+ X + Y + Z + V +$ &c. $+ \alpha + \beta + \gamma + \delta +$ &c. vel quæcunque rationalis functio, in quâ $(x, y, z,$ &c.$)$ & $(X, Y, Z,$ &c.$)$, etiamque $(\alpha, \beta, \gamma, \delta,$ &c.$)$ ſimiliter involvuntur; erit ad numerum

merum dimenſionum æquationis, cujus radix ſit $x+y+$ &c. $+X+Y+$ &c. $+\alpha+\beta+$ &c. $+$ &c. vel ad quamcunque rationalem functionem in quâ $(x, y,$ &c.), & $(X, Y,$ &c.), etiamque $(\alpha, \beta,$ &c.), &c., ſimiliter involvuntur: (in cujus ſumma tot ſint literæ $x, y,$ &c. quot unitates in $\overset{\prime}{m}$; tot ſint literæ $X, Y,$ &c. quot unitates in $\overset{\prime}{r}$; tot ſint literæ $\alpha, \beta,$ &c. quot unitates in $\overset{\prime}{s}$; &c.) $:: 1 : m \times \frac{m-1}{2} \times \frac{m-2}{3} \ldots \frac{m-\overset{\prime}{m}+1}{\overset{\prime}{m}} \times r \times \frac{r-1}{2} \times \frac{r-2}{3} \ldots \frac{r-\overset{\prime}{r}+1}{\overset{\prime}{r}} \times s \times \frac{s-1}{2} \times \frac{s-2}{3} \ldots \frac{s-\overset{\prime}{s}+1}{\overset{\prime}{s}}$ &c.; ſi vero $\overset{\prime}{m}$ ſit o, tum pro $m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \ldots \frac{m-\overset{\prime}{m}+1}{\overset{\prime}{m}}$ ſcribatur 1: & ſic de reliquis.

Ex. Sit numerus (n) incognitarum quantitatum $\alpha, \beta, \gamma, \delta, \epsilon,$ &c. & ſint (n) datæ æquationes $\alpha+\beta+\gamma+\delta+\epsilon+$ &c. $=a$, $\alpha^2+\beta^2+\gamma^2+\delta^2+$ &c. $=b$, $\alpha^3+\beta^3+\gamma^3+\delta^3+$ &c. $=c$, $\alpha^4+\beta^4+\gamma^4+\delta^4+$ &c. $=d$, &c. Et dimenſiones æquationis, cujus radix ſit $\alpha+\beta+\gamma+\delta+$ &c. $=a$ erunt ad dimenſiones æquationis, cujus radix ſit $\alpha :: 1 : n$. ergo æquatio, cujus radix ſit α, n habet dimenſiones; erit $\alpha^n - a\alpha^{n-1} + \frac{a^2-b}{2}\alpha^{n-2} - \frac{a^3-3ab+2c}{2}\alpha^{n-3} + \frac{a^4-6a^2b+8ac-6d+3b^2}{24}\alpha^{n-4} -$ &c. $=o$. Lex hujuſce feriei conſtat ab ex. 1. prob. 5.

Et ſic de infinitis hujuſmodi exemplis. Facile enim fingi poſſunt infinita hujuſcemodi exempla.

Idem etiam mutatis mutandis affirmari poteſt de rationalibus functionibus (P), in quarum quadratis ſimiliter involvuntur quantitates $x, y,$ &c.; & $X, Y,$ &c.; etiamque $\alpha, \beta,$ &c.; &c.; nam omnis affirmativa radix æquationis, cujus radix eſt P, habet negativam ei æqualem.

THEOR.

THEOR. XL.

1. Sit æquatio $A = o$, quæ continet duas incognitas quantitates x & y; in eâ pro x ſcribatur $\frac{1}{x}$, & reſultet æquatio $B = o$; tum ex duabus æquationibus $A = o$ & $B = o$ deducatur æquatio, cujus radix eſt x; æquatio ſic deducta habebit duas radices quantitatis (x) $+ 1$ & $- 1$; omnes vero reliquæ radices continebuntur in æquatione recurrente $x^n + ax^{n-1} + bx^{n-2} \ldots + bx^2 + ax + 1 = o$.

DEMONSTRATIO.

1. 1. Sit α radix vel valor radicis x in æquationibus $A = o$ & $B = o$, tum erit $\frac{1}{\alpha}$ radix ejuſdem quantitatis x: ſcribatur enim α pro x in æquationibus $(A = o$ & $B = o)$, & reſultent æquationes $\pi = o$ & $\sigma = o$; deinde ſcribatur $\frac{1}{\alpha}$ pro x in prædictis æquationibus $(A = o$ & $B = o)$, reſultabunt æquationes prædictæ $\sigma = o$ & $\pi = o$; conſequenter, ſi α ſit radix quantitatis x in datis æquationibus $(A = o$ & $B = o)$, tum erit $\frac{1}{\alpha}$ radix quantitatis x in iiſdem æquationibus contenta; & exinde æquatio, cujus radix eſt x, æqualis erit contento ſub ſubſequentibus factoribus $(x - \alpha) \times \left(x - \frac{1}{\alpha}\right) \times (x - \beta) \times \left(x - \frac{1}{\beta}\right) \times (x - \gamma) \times \left(x - \frac{1}{\gamma}\right)$ $\times$ &c. $= o$, quæ erit recurrens æquatio.

1. 2. Duæ radices quantitatis x erunt $+ 1$ & $- 1$; ſcribatur enim $+ 1$ pro x in æquationibus $A = o$ & $B = o$, ex utrâque eadem reſultabit æquatio $\sigma = o$; deinde e numero radicum quantitatis (y) in reſultante æquatione $\sigma = o$ contentarum dici poteſt numerus radicum quantitatis (x), quæ ſunt 1: ſingulus enim prædictus valor quantitatis y habet valorem (1) quantitatis x; conſequenter tot erunt valores

(1)

(1) quantitatis x, quot ſunt valores quantitatis y in æquatione $\sigma = 0$ contenti.

Eodem modo inveniri poteſt numerus radicum quantitatis x, quæ ſunt -1.

1.3. Sit $x^n + ax^{n-1} + bx^{n-2} + \ldots + bx^2 + ax + 1 = 0$ recurrens æquatio: & 1mo. ſit n par numerus; tum æquatio, cujus radix eſt $z = x + \frac{1}{x}$, ſolummodo dimidium habet numeri dimenſionum æquationis, cujus radix eſt x: ſit enim α unus valor quantitatis (x); tum erit $\left(\frac{1}{\alpha}\right)$ alter valor quantitatis x: at, ſi $x = \alpha$, tum erit $z = \alpha + \frac{1}{\alpha}$; etiamque, ſi $x = \frac{1}{\alpha}$, tum erit $z = \frac{1}{\alpha} + \frac{1}{\frac{1}{\alpha}} = \frac{1}{\alpha} + \alpha$; conſequenter idem valor $\left(\alpha + \frac{1}{\alpha}\right)$ quantitatis z correſpondet duobus valoribus α & $\frac{1}{\alpha}$ quantitatis x. 2do. Sit n impar numerus, tum erit unus valor quantitatis $(x) = -1$; & conſequenter ejus correſpondens valor quantitatis $(z) = x + \frac{1}{x} = -1 - \frac{1}{1} = -2$; & quoniam unus valor quantitatis $(z) = \alpha + \frac{1}{\alpha}$ ſemper correſpondet duobus valoribus $\left(\alpha \;\&\; \frac{1}{\alpha}\right)$ quantitatis (x), ergo numerus valorum quantitatis (z) erit $\frac{n-1}{2}$ ſimul cum prædicto valore -2 adjuncto.

Idem dici poteſt de dimenſionibus æquationis, cujus radix (z) eſt talis rationalis functio radicis (x) recurrentis æquationis; qualis, ſi modo in eâ pro x ſcribatur $\frac{1}{x}$, functio reſultans erit eadem ac prædicta.

2. In æquatione $A = 0$, quæ continet duas incognitas quantitates

x &

x & y, pro x ſcribatur $a-x$, & reſultet æquatio $B=o$; tum ex duabus æquationibus $A=o$ & $B=o$ inveniatur æquatio, cujus radix eſt x: æquationes ſic deductæ habent unum valorem quantitatis $(x)=\frac{a}{2}$; reliqui autem valores quantitatis x continentur in æquatione hujuſce formulæ $(x-\alpha)\times(x-a+\alpha)\times(x-\beta)\times(x-a+\beta)\times(x-\gamma)\times(x-a+\gamma)\times(x-\delta)\times(x-a+\delta)\times$ &c. $=o$.

Facile enim ſcribendo α & $a-\alpha$ pro x in datis æquationibus $A=o$ & $B=o$ probari poteſt quantitatem $a-\alpha$ eſſe radicem vel valorem quantitatis (x) in æquationibus prædictis contentæ, ſi modo (α) ſit radix ejuſdem quantitatis (x).

Unus valor quantitatis (x) erit $=\frac{a}{2}$; ſcribatur enim $\frac{a}{2}$ pro x in datis æquationibus $A=o$ & $B=o$, ex utriſque eadem reſultabit æquatio $\sigma=o$; & exinde tot erunt valores quantitatis $(x)=\frac{a}{2}$, quot ſunt valores quantitatis (y) in æquatione $\sigma=o$.

2. 2. Sit quæcunque rationalis functio (v) quantitatum x & $a-x$, in quâ x & $a-x$ ſimiliter involvuntur; tum æquatio, cujus radix eſt x, haud plures habet quam dimidium numeri radicum æquationis, cujus radix eſt x vel $a-x$.

Si enim in prædictâ functione (v) pro x ſcribatur α vel $a-\alpha$, eadem reſultabit quantitas; conſequenter unus valor functionis (v) reſpondet duobus valoribus α & $a-\alpha$ quantitatis (x).

Sit u rationalis functio quantitatum x & $a-x$; in quâ ſimiliter involvuntur x & $a-x$, & ſinguli duo correſpondentes termini, e. g. $x^m\times(a-x)^r$ & $x^r\times(a-x)^m$ diverſa habeant ſigna; i. e. ſi alter habeat ſignum $+$ affixum, tum alter habet ſignum $-$; æquatio, cujus radix eſt (u), habet ejus negativas & affirmativas radices reſpective inter ſe æquales.

Hoc facile conſtabit e ſcribendo pro x reſpective α & $a-\alpha$.

2.3. In functione v pro x ſcribatur $a-x$, & pro $a-x$ ſcribatur x;

&

& ſit quantitas reſultans (w); tum erit æquatio, cujus radix eſt $u = \frac{x}{w}$, recurrens; & æquatio, cujus radix eſt $\frac{x}{w} + \frac{w}{x}$, dimidias ſolummodo habet dimenſiones æquationis, cujus radix eſt vel x vel y.

3. Generaliter ſit x functio quantitatis (z) viz. $\phi : z$, ita ut z ſit eadem functio quantitatis (x): viz. $\phi : x$; tum in datâ æquatione $A = o$ pro (x) ſcribatur ejus valor $\phi : z$, & in æquatione reſultante pro z ſcribatur (x), & ſit æquatio exinde reſultans $B = o$; tum valores quantitatis (x) in aſſumptâ æquatione $x = \phi : x$ erunt etiam valores quantitatis (x) in aſſumptis æquationibus $A = o$ & $B = o$: ſi vero α ſit valor quantitatis x, tum $(\phi : \alpha)$ etiam erit valor quantitatis x.

Sit rationalis functio (v) quantitatum x & $\phi : x$, in quâ x & $\phi : x$ ſimiliter involvuntur; tum æquatio, cujus radix eſt v, haud plures habet quam dimidium numeri radicum æquationis, cujus radix eſt x vel $\phi : x$.

Si in prædictâ functione (v) quantitatbus $\alpha^m \times (\phi : a)^r$ & $a^r \times (\phi : a)^m$ diverſa ſigna ſemper ſunt affixa; tum æquatio, cujus radix eſt v, habet ejus affirmativas & negativas radices reſpective inter ſe æquales.

Si in quâlibet rationali functione (v) quantitatum x & $\phi : x$ pro x ſcribatur $\phi : x$ & pro $\phi : x$ ſcribatur (x), & reſultet quantitas w; tum erit æquatio, cujus radix eſt $u = \frac{w}{v}$, recurrens: &c.

4. Sit quæcunque æquatio $A = o$ relationem inter duas incognitas quantitates x & y deſignans. In æquatione $A = o$ pro x ſcribatur y & pro y ſcribatur x, & ſit æquatio reſultans $B = o$; tum æquatio, cujus radix ſit $x + y$ vel rationalis functio (π) quantitatum x & y, in quâ x & y ſimiliter involvuntur, haud continet plures quam dimidium numeri radicum æquationis, quas habet æquatio, cujus radix ſit incognita quantitas x vel y.

Si in prædictâ functione (π) quantitatibus $x^m y^r$ & $y^m x^r$ diverſa ſigna

ſint affixa; tum æquatio, cujus radix eſt π, habet ejus affirmativas & negativas radices reſpective inter ſe æquales.

Si pro x ſcribatur y, & pro y ſcribatur x in rationali functione (π) quantitatum x & y, & reſultet quantitas ρ; tum erit æquatio, cujus radix eſt $\frac{\pi}{\rho}$, recurrens.

5. Sit $A=o$, ubi A eſt functio quantitatum x & y; aſſumatur $x=\phi:(x \,\&\, y)$; in datâ æquatione $A=o$ pro x ſcribatur functio aſſumpta $\phi:(x \,\&\, y)$ quantitatum x & y, & ſit æquatio reſultans $B=o$; tum erunt valores incognitarum quantitatum x & y in æquationibus $A=o$ & $x=\phi:(x \,\&\, y)$ aliqui valores earundem quantitatum x & y in duabus æquationibus $A=o$ & $B=o$ contenti.

6. Sit $A=o$, & aſſumatur æquatio $P=o$ relationem inter duas incognitas quantitates (x & y) in æquatione $A=o$ contentas & novam quantitatem (z) exprimens: ita reducantur duæ æquationes $A=o$ & $P=o$, in unam, ut exterminetur incognita quantitas x; & reſultet æquatio $B=o$: in duabus æquationibus $P=o$ & $B=o$ pro z ſcribatur x, & reſultent duæ æquationes $Q=o$ & $C=o$: tum erunt radices incognitarum quantitatum x & y in æquationibus $A=o$ & $Q=o$, aliquæ radices earundem incognitarum quantitatum in æquationibus $A=o$ & $C=o$ contentæ.

7. Sit æquatio $A=o$, in quâ continentur incognitæ quantitates x & y; aſſumatur æquatio $Q=o$ relationem exprimens inter quantitates x & z, in quâ ſimiliter involvuntur prædictæ quantitates x & z; tum ita reducantur duæ æquationes $A=o$ & $Q=o$ in unam $L=o$, ut exterminetur incognita quantitas (x); & in æquatione $L=o$ pro z ſcribatur x, & ſit æquatio reſultans $B=o$; reducatur æquatio $B=o$, ita ut exterminentur irrationales quantitates, & ſit æquatio exinde reſultans $C=o$: deinde in æquatione $Q=o$ pro z ſcribatur (x); radices quantitatis (x) in æquatione reſultante erunt nonnullæ radices quantitatis (x), quæ inveniuntur in æquationibus $A=o$ & $C=o$: ſi in æquatione $Q=o$ (α) & (β) ſint correſpondentes valores quantita-

 tum

tum $(x$ & $z)$; & α ſit radix quantitatis x in æquationibus $A = o$ & $C = o$; tum (β) etiam erit radix quantitatis (x) in iiſdem æquationibus contenta: conſimiles etiam propoſitiones etiam deduci poſſunt de pluribus æquationibus plures incognitas quantitates habentibus.

PROB. XLIII.

1. *Sint quantitates* A & B *functiones literarum* x, y, &c.; *invenire, annon* A *ſit functio quantitatis* B.

Aſſumantur $A = v$ & $B = w$; ita reducantur duæ æquationes reſultantes in unam, ut exterminetur quantitas x, & ſi etiam ſimul exterminentur y, &c.; tum quantitas A erit functio quantitatis B: ſin aliter vero non.

2. Sint duæ quantitates A & B, & æquationes $P = o$, $Q = o$, $R = o$, &c.; ubi A, B; P, Q, R, &c. ſunt functiones incognitarum quantitatum $(x, y, z,$ &c.$)$ invenire, annon quantitas A ſit functio quantitatis B.

Aſſumantur $A = v$, $B = w$; $P = o$, $Q = o$, $R = o$, &c.; reducantur hæ æquationes in unam, ita ut evaneſcant incognitæ quantitates $(x, y, z,$ &c.$)$, & ſi ſimul evaneſcant omnes incognitæ quantitates præter v & w; tum erit v functio quantitatis w: ſin aliter vero non.

THEOR. XLI.

* Sit æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - tx^{n-5} +$ &c. $= o$; etiamque $p' = P$, $pP - q' = Q$, $pQ - qP + r' = R$, $pR - qQ + rP - s' = S$, $pS - qR + rQ - sP + t' = T$, & ſic deinceps: ſint a, b, c, d, e, &c. ſumma datæ æquationis radicum, quadratorum e ſingulis radicibus; cuborum; quadrato-quadratorum; &c.; tum erunt $\dot{a} = P$, $\frac{1}{2}\dot{b} = Q$, $\frac{1}{3}\dot{c} = R$, $\frac{1}{4}\dot{d} = S$, $\frac{1}{5}\dot{e} = T$, &c.; ſi modo in fluxionibus $\dot{a}$, $\dot{b}$, $\dot{c}$, $\dot{d}$, $\dot{e}$, &c. pro $\dot{p}$, $\dot{q}$, $\dot{r}$, $\dot{s}$, $\dot{t}$, &c. ſcribantur reſpective p', q', r', s', t', &c.

Hoc facile demonſtrari poteſt.

SCHO-

SCHOLIUM.

In radicibus duarum vel plurium æquationum duas vel plures incognitas quantitates $(x, y, \&c.)$ habentium inveſtigandis, frequentiſſime prædictis methodis fruſtra utimur; confugiendum eſt ad continuas approximationes.

Sint e. g. duæ datæ æquationes duas incognitas quantitates $(x \& y)$ habentes $y^n + \overline{a + bx}\, y^{n-1} + \&c. = 0$ & $y^m + \overline{A + Bx} \times y^{m-1} + \&c. = 0$, & ſint correſpondentes radices incognitarum quantitatum $(x \& y)$ reſpective $\alpha, \beta, \gamma, \delta, \&c.$ & $\pi, \rho, \sigma, \tau, \&c.$ ad inveniendos ex approximatione correſpondentes valores $(\alpha \& \pi)$ incognitarum quantitatum $(x \& y)$, neceſſe eſt, ut prius dentur quantitates $(k \& l)$ ad prædictos correſpondentes valores $(\alpha \& \pi)$ multo magis appropinquantes, quam ad ullas alias correſpondentes incognitarum quantitatum $(x \& y)$ radices $(\beta, \gamma, \delta, \&c.$ & $\rho, \sigma, \tau, \&c.)$: in datis æquationibus pro quantitatibus $(x \& y)$ ſcribantur reſpective $z - k$ & $v - l$, & ſint æquationes reſultantes $\Gamma + \Delta z + \mathrm{E}v + \mathrm{Z}z^2 + \mathrm{H}zv + \Theta x^2 + \&c. = 0$, & $\Pi + \mathrm{P}z + \Sigma v + \mathrm{T}z^2 + \Upsilon zv + \Phi v^2 + \&c. = 0$; & e conſtitutione coefficientium colligi poſſint $\Gamma + \Delta z + \mathrm{E}v$ prope $= 0$, & $\Pi + \mathrm{P}z + \Sigma v$ prope $= 0$, & exinde approximatio ad z erit $\frac{\mathrm{E}\Pi - \Sigma\Gamma}{\Delta\Sigma - \mathrm{EP}}$ & approximatio ad v erit $\frac{\Delta\Pi - \mathrm{P}\Gamma}{\mathrm{PE} - \Delta\Sigma}$.

Si vero k & l multo magis approximent ad duas correſpondentes radices incognitarum quantitatum $(x \& y)$ reſpective, quam ad ullas alias; & pro x & y ſubſtituantur reſpective $z - k$ & $v - l$ in datis æquationibus; & e conſtitutione coefficientium colligi poſſint vel $\Gamma + \Delta z + \mathrm{E}v$ prope $= 0$, & $\Pi + \mathrm{P}z + \Sigma v + \mathrm{T}z^2 + \Upsilon zv + \Phi v^2$ prope $= 0$; vel $\Pi + \mathrm{P}z + \Sigma v$ prope $= 0$, & $\Gamma + \Delta z + \mathrm{E}v + \mathrm{Z}z^2 + \mathrm{H}zv + \Theta v^2$ prope $= 0$; e quibus æquationibus inveniri poſſunt approximationes ad quantitates z & v: & ſic progredi licet, cum k & l ad tres vel plures correſpondentes incognitarum quantitatum

 radices

radices multo magis approximent, quam ad reliquas: & ſi demonſtrari poſſit has continuas approximationes perpetuo multo minores evadere; tum exinde conſtat ultimo eas propiores ad veras radices quæſitas quam pro quâvis datâ differentiâ accedere.

Etiamque facile conſtat methodus inveniendi rationem, quam habet approximatio ad veram radicem, a terminis diverſarum radicum incognitarum quantitatum expreſſam.

2. Ad inveniendas autem continuas approximationes ad impoſſibilem radicem $(a+b\sqrt{-1})$ in unâ æquatione unam ſolummodo incognitam quantitatem, vel ad impoſſibiles correſpondentes incognitarum quantitatum radices in duabus vel pluribus æquationibus duas vel plures incognitas quantitates habentibus; neceſſe eſt, ut haud ſolummodo dentur quantitates magis appropinquantes ad poſſibiles partes radicum quæſitarum reſpective, quam ad poſſibiles partes reliquarum incognitarum quantitatum radicum; ſed etiam dentur quantitates ad impoſſibiles partes quæſitarum radicum magis appropinquantes quam ad impoſſibiles partes reliquarum incognitarum quantitatum radicum: & ex eâdem prorſus methodo, quæ in poſſibilibus radicibus inveſtigandis prius docetur, erui poſſunt continuæ approximationes & ad poſſibiles & impoſſibiles quæſitarum radicum partes. Et omnia, quæ prius docentur de poſſibilibus radicibus; de impoſſibilibus etiam vere affirmari poſſunt: omnia præcedentia æque ad (n) æquationes (n) incognitas quantitates habentes applicari poſſunt.

Ex. Sit æquatio $x^n - px^{n-1} + qx^{n-2} - \&c. = 0$, & ſit $a+b\sqrt{-1}$ approximatio ad impoſſibilem datæ æquationis radicem: in datâ æquatione pro x ſcribatur $a+a'+\overline{b+b'}\sqrt{-1}$; reſultabit $(a^n - n \cdot \frac{n-1}{2} a^{n-2}b^2 + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} a^{n-4}b^4 - \&c. - pa^{n-1} + \overline{n-1} \cdot \frac{n-2}{2} pa^{n-3}b^2 - \overline{n-1} \cdot \frac{n-2}{2} \cdot \frac{n-3}{3} \cdot \frac{n-4}{4} pa^{n-5}b^4 + \&c. + qa^{n-2} - \overline{n-2} \cdot \frac{n-3}{2} qa^{n-4}b^2 + \&c.) = A + (na^{n-1}b - n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} a^{n-3}b^3 +$

n.

$$n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdot \frac{n-4}{5} a^{n-5} b^5 - \&c. - \overline{n-1}\, p a^{n-2} b + \overline{n-1} \cdot$$

$$\frac{n-2}{2} \cdot \frac{n-3}{3} p a^{n-4} b^3 - \overline{n-1} \cdot \frac{n-2}{2} \cdot \frac{n-3}{3} \cdot \frac{n-4}{4} \cdot \frac{n-5}{5} p a^{n-6} b^5 +$$

$$\&c. + \overline{n-2}\, q a^{n-3} b - \overline{n-2} \cdot \frac{n-3}{2} \cdot \frac{n-4}{3} q a^{n-5} b^3 + \&c. - \&c.)$$

$$\sqrt{-1} = B\sqrt{-1} + (n a^{n-1} - n \cdot \frac{n-1}{2} \times \overline{n-2}\, a^{n-3} b^2 + n \cdot \frac{n-1}{2} \cdot$$

$$\frac{n-2}{3} \cdot \frac{n-3}{4} \times \overline{n-4}\, a^{n-5} b^4 - \&c. - \overline{n-1}\, p a^{n-2} + \overline{n-1} \cdot \frac{n-2}{2} \times$$

$$\overline{n-3}\, p a^{n-4} b^2 - \overline{n-1} \cdot \frac{n-2}{2} \cdot \frac{n-3}{3} \cdot \frac{n-4}{4} \cdot \overline{n-5}\, p a^{n-6} b^4 - \&c.$$

$$+ \overline{n-2}\, q a^{n-3} - \overline{n-2} \cdot \frac{n-3}{2} \cdot \overline{n-4}\, q a^{n-5} b^2 + \&c. - \&c.)\, a' = C a'$$

$$-(n \cdot \overline{n-1}\, a^{n-2} b - n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \overline{n-3}\, a^{n-4} b^3 + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4}$$

$$\cdot \frac{n-4}{5} \cdot \overline{n-5}\, a^{n-6} b^5 - \&c. - \overline{n-1} \cdot \overline{n-2}\, p a^{n-3} b + \overline{n-1} \cdot \frac{n-2}{2} \cdot \frac{n-3}{3} \cdot$$

$$\overline{n-4}\, p a^{n-5} b^3 - \&c. + \overline{n-2} \cdot \overline{n-3}\, q a^{n-4} b - \overline{n-2} \cdot \frac{n-3}{2} \cdot \frac{n-4}{3} \cdot \overline{n-5}$$

$$\times q a^{n-6} b^3 + \&c.\ \&c.)\, b' = -E b' + (n \cdot \overline{n-1}\, a^{n-2} b - n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \overline{n-3}$$

$$a^{n-4} b^3 + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdot \frac{n-4}{5} \cdot \overline{n-5}\, a^{n-6} b^5 - \&c. - \overline{n-1} \cdot \overline{n-2}$$

$$p a^{n-3} b + \overline{n-1} \cdot \frac{n-2}{2} \cdot \frac{n-3}{3} \cdot \overline{n-4}\, p a^{n-5} b^3 - \&c. + \overline{n-2} \cdot \overline{n-3}\, q a^{n-4} b$$

$$- \overline{n-2} \cdot \frac{n-3}{2} \cdot \frac{n-4}{3} \cdot \overline{n-5}\, a^{n-6} b^3 + \&c.\ \&c.)\, a' \sqrt{-1} = F a' \sqrt{-1}$$

$$+ (n a^{n-1} - n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \times 3 a^{n-3} b^2 + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdot$$

$$\frac{n-4}{5} \times 5 a^{n-5} b^4 - \&c. - \overline{n-1}\, p a^{n-2} + \overline{n-1} \cdot \frac{n-2}{2} \times \frac{n-3}{3} \times 3 p a^{n-4} b^2$$

— n — 1.

$- \overline{n-1} \times \frac{n-2}{2} \times \frac{n-3}{3} \cdot \frac{n-4}{4} \cdot \frac{n-5}{5} \times 5pa^{n-6}b^4 +$ &c. $+ \overline{n-2}\, qa^{n-3}$ $- \overline{n-2} \cdot \frac{n-3}{2} \times \frac{n-4}{3} \times 3qa^{n-5}b^2 +$ &c. $-$ &c.) $b'\sqrt{-1} = Db'$ $\sqrt{-1}$, &c. $= o$: nunc fiat $A + Ca' - Eb' = o$ & $(B + Db' + Fa')$ $\sqrt{-1} = o$, & conſequenter approximatio $a' = -\frac{EB + DA}{FE + CD}$, $b' = \frac{FA - CB}{FE + CD}$.

Eâdem methodo perpetuo novæ approximationes inveniri poſſunt.

Si duo vel plures valores quantitatum a vel b ſint prope inter ſe æquales, tum e principiis prius poſitis facile erui poſſunt earum approximationes.

Eadem principia etiam applicari poſſunt ad inveniendas approximationes conſimiles in duabus vel pluribus æquationibus duas vel plures incognitas quantitates habentibus.

3. Sint duæ æquationes $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - tx^{n-5}$ + &c. $= o$, & $x^n - \overline{p + p'}\,x^{n-1} + \overline{q + q'}\,x^{n-2} - \overline{r + r'}\,x^{n-3} + \overline{s + s'}\,x^{n-4}$ $- \overline{t + t'}\,x^{n-5}$ $-$ &c. $= o$, quarum radices ſint a, b, c, d, e, &c.; & $a + a'$, $b + b'$, $c + c'$, $d + d'$, $e + e'$, &c. reſpective: tum erit $x^n - \overline{p + p'}\,x^{n-1}$ $+ \overline{q + q'}\,x^{n-2} - \overline{r + r'}\,x^{n-3} +$ &c. $= (x - a - a') \times (x - b - b') \times (x - c - c') \times (x - d - d') \times$ &c. prope $= x^n - (a + b + c + d +$ &c. $+ a' + b' + c' + d' +$ &c.$)\, x^{n-1} + (ab + ac + bc +$ &c. $+ a \times \overline{b' + c' + d' +}$ &c. $+ b \times \overline{a' + c' + d' +}$ &c. $+ c \times \overline{a' + b' + d' +}$ &c. $+$ &c.$)\, x^{n-2} - (abc + abd + acd + bcd +$ &c. $+ ab\,\overline{c' + d' +}$ &c. $+ ac\,\overline{b' + d' +}$ &c. $+ bc\,\overline{a' + d' +}$ &c. $+ ad\,\overline{b' + c' +}$ &c. $+$ &c.$)$ $x^{n-3} +$ &c. $= o$, ſi modo a', b', c', d', &c. ſint multo minores quam quantitates a, b, c, d, &c.

Hinc prope

$a' + b' + c' + d' +$ &c. $= p' = P$,

$aa' + bb' + cc' + dd' +$ &c. $= pp' - q' = Q$,

$a^2a' + b^2b' + c^2c' + d^2d' +$ &c. $= pQ - qP + r' = R$,

$a^3a' + b^3b' + c^3c' + d^3d' +$ &c. $= pR - qQ + rP - s' = S$,

a^{n-5}

$a^{n-5}d' + b^{n-5}b' + c^{n-5}c' + \&c. = N,$
$a^{n-4}d' + b^{n-4}b' + c^{n-4}c' + \&c. = M,$
$a^{n-3}d' + b^{n-3}b' + c^{n-3}c' + \&c. = L,$
$a^{n-2}d' + b^{n-2}b' + c^{n-2}c' + \&c. = K,$
$a^{n-1}d' + b^{n-1}b' + c^{n-1}c' + \&c. = I$; unde

$$d' = \frac{I - \overline{b+c+d+\&c}\times K + \overline{bc+bd+cd+be+\&c}\times L - \overline{bcd+bce+\&c}\times M + \&c}{(a-b)\times(a-c)\times(a-d)\times(a-e)\times\&c.}$$
$$= \frac{a^{n-1}p' - a^{n-2}q' + a^{n-3}r' - a^{n-4}s' + \&c.}{na^{n-1} - \overline{n-1}\,pa^{n-2} + \overline{n-2}\,qa^{n-3} - \overline{n-3}\,ra^{n-4} + \&c.}.$$
$$b' = \frac{I - \overline{a+c+d+\&c}\times K + \overline{ac+ad+cd+ce+\&c}\times L - \overline{acd+ace+\&c}\times M + \&c}{(b-a)\times(b-c)\times(b-d)\times(b-e)\times\&c.}$$
$$= \frac{b^{n-1}p' - b^{n-2}q' + b^{n-3}r' - b^{n-4}s' + \&c.}{nb^{n-1} - \overline{n-1}\,pb^{n-2} + \overline{n-2}\,qb^{n-3} - \overline{n-3}\,rb^{n-4} + \&c.}.$$
&c. &c.

Si vero duæ radices $(a$ & $b)$ æquationis $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = o$ ſint prope inter ſe æquales; tum ex quadraticâ æquatione $(n\cdot\frac{n-1}{2}a^{n-2} - \overline{n-1}\cdot\frac{n-2}{2}pa^{n-3} + \overline{n-2}\times\frac{n-3}{2}qa^{n-4} - \&c.)\times d'^2 + (na^{n-1} - \overline{n-1}\,\overline{p+p'}\,a^{n-2} + \overline{n-2}\,\overline{q+q'}\,a^{n-3} - \&c.)\,d' - p'a^{n-1} + q'a^{n-2} - r'a^{n-3} + \&c. = o$ inveniri poſſunt quam proxime duo valores quantitatis d'; i. e. incrementa vel decrementa radicum $(a$ & $b)$ reſpectìve.

Si vero tres vel plures radices $(a, b, c, \&c.)$ æquationis $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \&c. = o$ ſint prope inter ſe æquales; ex conſimilibus principiis erui poſſunt quam proxime incrementa vel decrementa radicum prædictarum $(a, b, c, \&c.)$.

4. Sint $a, b, c, d,$ &c. radices æquationis $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \&c. = o$, quarum a major ſit quam b, b quam c, c quam d, &c.; etiamque ſint $A, B, C, D,$ &c. radices æquationis $x^n - Px^{n-1} + Qx^{n-2} - Rx^{n-3} + \&c. = o$, quarum A major ſit quam B, B quam C, C quam D, &c. harum duarum æquationum ſint omnes radices poſſibiles; tum æquatio $x^n - px^{n-1} + qx^{n-2} - \&c. = o$ in æquationem

tur; ea mutari poteſt in aliam, quæ nullam continet fractionalem quantitatem, e. g. ſit æquatio $Nx^n + px^{n-1} + qx^{n-2} + rx^{n-3} + \&c. \ldots Qx^2 + Rx + S = 0$: ubi N, p, q, r, &c. Q, R, S integros denotant numeros; ſcribatur $\frac{z}{N}$ pro x, reſultabit æquatio $z^n + pNz^{n-1} + qN^2z^{n-2} + rN^3z^{n-3} + \&c. \ldots QN^{n-2}z^2 + RN^{n-1}z + SN^n = 0$, cujus radix ſit ($a$), & $Nx - a$ erit diviſor quæſitus.

Hinc omnis radix, quæ eſt rationalis fractio, pro ſuo denominatore habebit aliquem numerum, qui eſt diviſor numeri N. Sit autem (a) diviſor quantitatis S, & m diviſor quantitatis N; ſubſtituantur pro x in datâ æquatione ($Nx^n + px^{n-1} + qx^{n-2} + rx^{n-3} \ldots Qx^2 + Rx + S = 0$) 0, 1, -1, 10, -10, &c. & quantitatum reſultantium ſint a, $a - m$, $a + m$, $a - 10m$, $a + 10m$, &c. reſpectivi diviſores: vel ſi modo ſubſtituantur pro x ſucceſſive 0, 1, -1, 2, -2, &c. in datâ æquatione; & quantitatum reſultantium ſint a, $a - m$, $a + m$, $a - 2m$, $a + 2m$, &c. reſpectivi diviſores; $mx - a$ erit quantitas pro diviſore datæ æquationis tentanda.

Eodem modo transformetur data æquatio in aliam, cujus radices ſint quæcunque rationalis functio datæ æquationis radicum; π erit ea rationalis functio radicis (a); ſi modo a & π ſint rationales & correſpondentes datæ & reſultantis æquationis radices.

2. 1. Eſt etiam alia methodus inveniendi diviſores æquationis $S + Rx + Qx^2 + Px^3 + Ox^4 \ldots x^n = 0$; in quâ S, R, Q, P, &c. integri ſunt numeri.

Sit a radix quæſita, qui ſit integer numerus; $\frac{S}{a}$, $\frac{\frac{S}{a} + R}{a}$, $\frac{\frac{\frac{S}{a} + R}{a} + Q}{a}$, &c. integri erunt numeri; i. e. ſi modo A, B, C, D, E,

&c.

CAP. V.

DE RATIONALIBUS ET INTEGRIS QUANTITATIBUS.

PROB. XLIV.

DATA æquatione unam ſolummodo incognitam quantitatem (x) *habente* $x^n + px^{n-1} + qx^{n-2} + rx^{n-3} \dots Px^3 + Qx^2 + Rx + S = 0$, *invenire ejus rationales diviſores.*

1. 1mo. Sint p, q, r, &c. P, $\mathcal{Q}$, R, S integri numeri; & quoniam $\overline{x^{n-1} + px^{n-2} + qx^{n-3} + rx^{n-4} \dots Px^2 + \mathcal{Q}x + R} \times x = -S$; radix (x), ſi modo ſit integer numerus, erit diviſor ultimæ datæ æquationis coefficientis (S).

Transformetur data æquatio in alteram, cujus radices vel majores vel minores ſunt quam radices datæ æquationis per datum numerum (a); & $(x \pm a)$ erit diviſor ultimi reſultantis æquationis termini $a^n \mp pa^{n-1} + qa^{n-2} \mp ra^{n-3} + \&c.$

E. G. Sint a ſucceſſive 0, 1, −1, 10, −10, 100, −100, &c. & z radix datæ æquationis; & ſubſtituantur pro radice x in datâ æquatione 0, 1, −1, 10, −10, &c. numerorum reſultantium z, $z-1$, $z+1$, $z-10$, $z+10$, &c. erunt reſpective diviſores.

Sint a reſpective 0, 1, −1, 2, −2, 3, −3, &c. & z radix datæ æquationis; z, $z-1$, $z+1$, $z-2$, $z+2$, $z-3$, $z+3$, &c. erunt reſpective diviſores numerorum reſultantium e ſubſtitutione quantitatum 0, 1, −1, 2, −2, 3, −3, &c. pro radice (x) in datâ æquatione.

E prædictis diviſoribus igitur inveniri poſſunt datæ æquationis rationales radices.

Hoc problema invenit Jacobus a Waſſenaer.

2do. Si vero fractionales quantitates in datâ æquatione contineantur,

æquationem haud admittere ſimplicem diviſorem; poteſt tamen habere quadraticum diviſorem, qui ſit $mx^2 - ax + b$: in quo caſu b erit diviſor ultimi datæ æquationis termini (S); & m diviſor coefficientis (N) datæ æquationis termini (x^n); & $m - a + b$ diviſor quantitatis $S + R + Q + P + O +$ &c. & $m + a + b$ diviſor quantitatis $S - R + Q - P + O -$ &c. etiamque

$$\frac{R+\frac{aS}{b}}{b}\left(\frac{Rb+aS}{b^2}\right),\ \frac{Q-\frac{mS}{b}+a\times\frac{R+\frac{aS}{b}}{b}}{b}\left(\frac{Qb^2-mSb+bRa+a^2S}{b^3}\right),$$

&c. integri erunt numeri: vel, ſi modo $A, B, C, D, E,$ &c. reſpective denotent præcedentes numeros; tum $\frac{S}{b}(A)$, $\frac{R+aA}{b}(B)$, $\frac{Q+aB-mA}{b}$ (C), $\frac{P+aC-mB}{b}(D)$, &c. integri erunt numeri.

Et ſic lex facile conſtat, e quâ detegi poſſunt trium vel plurium dimenſionum diviſores.

Aliter vero inveniri poſſunt rationales quadratici datæ æquationis diviſores $(mx^2 - ax + b)$. In datâ æquatione pro radice (x) ſubſtitue tres vel quatuor vel plures terminos hujus progreſſionis 3, 2, 1, 0, — 1, — 2, — 3, &c. diviſores omnes numerorum reſultantium ſigillatim adde & ſubduc quadratis correſpondentium terminorum progreſſionis illius ductis in aliquem numeralem diviſorem (m) quantitatis (N), & ſummas differentiaſque e regione progreſſionis colloca. Dein progreſſiones omnes collaterales nota, quæ per iſtas ſummas differentiaſque percurrunt: erit $\pm b$ terminus iſtiuſmodi progreſſionis primæ, qui oritur e ſcribendo o pro x; $\pm a$ differentia, quæ oritur ſubducendo $m \pm b$ de termino maxime ſuperiori, qui ſtat e regione termini (1) progreſſionis primæ.

Si vero nullum habeat quadraticum diviſorem data æquatio; poteſt tamen habere cubicum, quadrato-quadraticum, &c. qui inveniri poſſunt, quærendo in prædictis ſummis differentiiſque progreſſiones non arith-

arithmeticas quidem ſed alias quaſdem; quarum terminorum differentiæ ſecundæ, tertiæ, &c. ſunt in arithmeticâ progreſſione.

Eodem modo transformetur data æquatio in aliam, cujus radices ſint quæcunque rationalis functio datæ æquationis radicum; exinde inveniri poſſunt rationales quadratici, cubici, &c. diviſores quæſiti.

Inveniendi diviſorem quantitatis, quæ unam ſolummodo continet literam, eadem erit methodus; ac æquationis unam ſolummodo incognitam quantitatem habentis.

4. Datâ quantitate duas vel plures incognitas quantitates involvente; invenire utrum ea ullum recipiat diviſorem, necne.

1mo. Supponantur omnes quantitates præter unam nihilo æquales, & inveniantur diviſores quantitatis reſultantis; deinde ſupponantur omnes præter duas incognitæ quantitates nihilo æquales, & inveniantur diviſores quantitatis reſultantis; & ſic de reliquis: vel aliter, collocentur ſinguli datæ quantitatis termini juxta dimenſiones cujuſlibet literæ (y), & inveniantur diviſores ejus ultimi termini; & ſubſtituantur pro (y) quilibet numeri $-2, -1, 0, 1, 2,$ &c. & exinde per præcedentes methodos inveniri poſſunt diviſores quæſiti.

Si plures ſint quam duæ incognitæ quantitates, tum ſolummodo pluries repetenda eſt operatio: ſingula enim operatio unam deſtruet incognitam quantitatem.

2do. Si vero fractiones contineantur in datâ quantitate; tum ſinguli termini datæ quantitatis reducendi ſunt in communem denominatorem; & diviſores & numeratoris & denominatoris quærendi ſunt: ſi vero irrationales termini in datâ quantitate contineantur; pro ſingulis diverſis irrationalibus terminis ſubſtitue diverſas literas, & per præcedentem methodum inveniendi diviſores quantitatis duas vel plures incognitas quantitates involventis ſæpe erui poſſunt diviſores quæſiti.

5. Ex formulis fractionum vel irrationalium quantitatum in datâ vel datis æquationibus contentarum ſæpe acquiri poſſunt formulæ integrorum numerorum vel rationalium fractionum in iis contentarum. e. g. Sit æquatio $x^n + a x^{n-1} + \sqrt[m]{c + dx^r} + \&c. = 0$; in hoc caſu,

caſu, cum x ſit fractio rationalis, erit etiam $\sqrt[m]{c + dx^r}$ fractio rationalis; ni in datâ æquatione aliæ dentur irrationales quantitates, quæ eam $\sqrt[m]{c + dx^r}$ deſtruent, vel ad rationalem fractionem reducent: ſi $\sqrt[m]{c + dx^r}$ ſit integer numerus vel rationalis fractio, facile erui poteſt formula quantitatis (x).

6. In quibuſdam caſibus facile conſequitur ex aliis obſervationibus æquationes haud admittere rationales diviſores. E. g. Sit æquatio $x^{2n} + 3ax^m + 3bx^r +$ &c. $= 3d + 2$; ubi a, b, &c. d, integri ſunt numeri; tum, quoniam nullus quadratus (x^{2n}) numerus per 3 diviſus relinquit 2; æquatio haud admittet rationalem diviſorem: & ſic e reſiduis poſt diviſionem per quoſcunque numeros, quæ haud recipiat rei natura; vel per extractionem; vel e limitibus inter quos neceſſario ponitur radix quæſita, &c. infinitæ conſimiles propoſitiones inveniri poſſunt.

7. E differentiis inter quaſcunque quantitates; vel rationibus vel relationibus, quas inter ſe habent; ſæpe demonſtrari poteſt nullas dari dati generis quantitates; vel limites inter quos dantur numeri dati generis.

Ex. Differentia inter duos proxime ſucceſſivos quadratos (x^2 & $\overline{x+1}^2$) i. e. quorum differentia inter radices ſit 1. erit $\overline{x+1}^2 - x^2 = 2x + 1$: hinc quadratus ($x^2$), auctus numero minore quam $2x + 1$, non poteſt eſſe quadratus; i. e. ſit quantitas $x^2 + x + b$, aſſumatur $2x + 1 = x + b$, erit $x = b - 1$; conſequenter $x^2 + x + b$ haud poteſt eſſe quadratus, ſi x major ſit quam $b - 1$.

Generaliter ſit quantitas $x^2 + ax + b = x^2 + 2ex + e^2$; unde $x = \frac{b - e^2}{2e - a}$, & exinde e haud poteſt eſſe major quam $\frac{a}{2}$ & $\sqrt{b}$.

Ex. 2. Quantitas $x^3 + ax^2 + bx + c$, ubi a, b & c ſunt integri numeri, non poteſt eſſe cubus, ſi $ax^2 + bx + c$ minor ſit quam $3x^2 + 3x + 1$; hinc inveniri poſſunt limites, inter quos ponitur quantitas x: & ſic argumentari liceat de ſuperioribus poteſtatibus, &c.; earum limitibus, &c.

Ex. 3.

Ex. 3. Sit $x^2 + 2ax = 4z^{2n}$, tum erit $z = -a + \sqrt{4z^{2n} + a^2}$; ſed $\sqrt{4z^{2n} + a^2}$ haud poteſt eſſe minor numerus quam $2z^n + 1$, exinde $4z^n + 1$ haud major erit quam a^2: &c.

8. Sit quæcunque quantitas, in quâ continetur diviſor ſimplex, qui nullum habet in eâ æqualem; tum data quantitas haud erit quadratus vel cubus vel biquadratus, &c.

Ex. 1. Si quantitas (x) habeat ſimplicem diviſorem a, cujus quadratus (a^2) haud etiam ſit diviſor quantitatis (x); tum (a) etiam erit diviſor quantitatis $ex + fx^2 + gx^3 +$ &c. ſi $e, f, g,$ &c. integri ſint numeri; ergo $x \times (\pm d + ex + fx^2 + gx^3 +$ &c.$)$, ſi d haud diviſibilis ſit per a; haud poteſt eſſe quadratus, cubus, biquadratus, &c.

Ex. 2. Si a^m ſit maxima poteſtas quantitatis a, quæ eſt diviſor quantitatis x; & a^r haud ſit diviſor quantitatis d, ubi r haud major ſit quam m; tum $x \times (d + ex + fx^2 +$ &c.) haud erit numerus, qui ſit $(r + m)$ poteſtas cujuſcunque integri numeri.

Si $x = \alpha^{r+m}$; tum, ut fiat rectangulum $x \times (d + ex + fx^2 +$ &c.$)$ poteſtas prædicti generis; neceſſe eſt, ut $d + ex + fx^2 +$ &c. $= \beta^{r+m}$, ubi α & β ſunt integri numeri.

THEOR. XLII.

Sint duæ æquationes duas incognitas quantitates $(x$ & $y)$ habentes; & ſit x rationalis quantitas, y etiam erit rationalis quantitas; & quantitates $(x$ & $y)$ ſemper eandem habebunt irrationalitatem; i. e. quantitas x per eaſdem irrationales quantitates exprimetur, ac quantitas (y), ni duo vel plures valores quantitatum $(x$ vel $y)$ ſint inter ſe æquales: ſi vero duo (α & β) ſint diverſi valores quantitatis (x), qui correſpondent uni valori (π) incognitæ quantitatis (y); tum quadratica æquatio, cujus radices ſunt α & β, nullas habet irrationales quantitates, quæ non inveniuntur in correſpondente valore (π) incognitæ quantitatis (y): ſi vero tres valores (α, β, γ) incognitæ quantitatis (x)

correſpondeant uni valori (π) incognitæ quantitatis (y); tum cubica æquatio, cujus radices ſunt α, β, γ; nullas habet irrationales quantitates, quæ haud inveniuntur in (π): & ſic deinceps.

Si modo in datis æquationibus pro incognitâ quantitate y ſcribatur π; & duarum æquationum reſultantium, in quibus una ſolummodo continetur incognita quantitas x, inveniatur communis diviſor; exinde facile conſtabit hujuſce theorematis veritas, nam per prædictam operationem haud novæ exorientur irrationales quantitates.

Eadem affirmari poſſunt de pluribus æquationibus plures incognitas quantitates habentibus.

Cor. Sint duæ æquationes $Nx^n + (a + by)\,x^{n-1} + (c + dy + ey^2)\,x^{n-2} + \&c. = 0$ & $Hx^m + (A + By)\,x^{m-1} + \&c. = 0$, quarum coefficientes N, a, b, &c.; H, A, B, &c. ſint omnes integri numeri vel affirmativi vel negativi, duas incognitas quantitates (x & y) habentes; & ſint coefficientes terminorum, in quibus maximæ inveniuntur dimenſiones incognitæ quantitatis (x) in utrâque æquatione, reſpective N & H; coefficientes vero terminorum, in quibus maximæ inveniuntur dimenſiones incognitæ quantitatis (y) in utrâque æquatione, reſpective S & s; & ſi modo y & x ſint rationales quantitates & y integer numerus; x erit fractio, cujus denominator erit communis diviſor coefficientium N & H: & vice versâ ſi x ſit integer numerus, y erit fractio, cujus denominator erit communis diviſor coefficientium S & s: ſi vero quantitas (x) ſit quæcunque fractio, cujus denominator ſit λ; & maximæ dimenſiones quantitatis (x) in utrâque æquatione ſint reſpective q & r; tum quantitas (y) erit fractio, cujus denominator eſt diviſor communis quantitatum $\lambda^q S$ & $\lambda^r s$. Sint $Sy^n + A'x\,y^{n-1} + B'x^2y^{n-2} + \&c.$ & $sy^m + a'xy^{m-1} + b'x^2y^{m-2} + \&c.$ termini reſpectivi duarum æquationum, in quibus maximæ inveniuntur dimenſiones incognitarum quantitatum (x & y); prioris quantitatis ſint $y - \alpha x$, $y - \beta x$, $y - \gamma x$, $y - \delta x$, &c. ſimplices diviſores; poſterioris vero $y - \pi x$, $y - \varrho x$, $y - \sigma x$, &c. quantitas x erit fractio, cujus denominator eſt diviſor contenti $S^m \times s^n \times (\alpha - \pi) \times (\alpha - \varrho) \times (\alpha - \sigma) \times \&c.$

&c. $\times (\beta - \pi) \times (\beta - \varrho) \times (\beta - \sigma) \times$ &c. $\times (\gamma - \pi) \times (\gamma - \varrho) \times (\gamma - \sigma)$ $\times$ &c.

Eadem etiam facile applicari poſſunt; cum nihilo ſint æquales quicunque termini, vel plures ſint æquationes plures incognitas quantitates habentes.

PROB. XLV.

Datis duabus æquationibus duas incognitas quautitates habentibus; invevire integrales correſpondentes, ſi quas forte habeant, incognitarum quantitatum (x & y) *radices.*

1. In datis æquationibus pro incognitâ quantitate (y) ſubſtitue tres vel plures terminos arithmeticæ ſeriei $-1, 0, 1, 2,$ &c. vel ſeriei $-1, 0, 1, 10,$ &c. duas quantitates reſultantes ex ſingulâ ſubſtitutione colloca juxta dimenſiones literæ (x), ita ut illi termini primum locum occupent, in quibus litera iſta (x) eſt plurimarum dimenſionum, &c. & perpetuâ ablatione minoris de majori, & reliqui de ablato, exterminetur incognita quantitas (x); & numerorum reſultantium e priori ſubſtitutione ſint termini arithmeticæ ſeriei $z+1$, z, $z-1$, $z-2$, &c. diviſores reſpectivi: e poſteriori ſint termini $z+1$, z, $z-1$, $z-10$, &c. diviſores reſpective: tum diviſor z erit integralis incognitæ quantitatis (y) quæſita radix.

Ex. Sint duæ æquationes $y^2 + (1+3x) \times y + 2x^2 + 4x - 18 = 0$, & $y^2 + 3xy + 3x^2 + 4x - 17 = 0$; pro y ſubſtituatur (0), reſultant quantitates $2x^2 + 4x - 18$, & $3x^2 + 4x - 17$; tranſformentur hi termini, ita ut exterminetur (x), & operatio eſt

$\overline{2x^2+4x-18} \times 3 = 6x^2+12x-54$

$\overline{3x^2+4x-17} \times 2 = 6x^2+8x-34$ auferatur inferior de ſuperiori,

reſiduum erit $4x-20$, tum $\overline{2x^2+4x-18} \times 2 = 4x^2+8x-36$

$\overline{4x-20} \times x = 4x^2-20x$

reſiduum $28x-36$

$\overline{4x-20} \times 7 = 28x-140$

reſiduum 104

 etiam-

etiamque de $\overline{3x^2+4x-17}\times 4=12x^2+16x-68$
auferatur $\overline{4x-20}\times 3x=12x^2-60x$
reſiduum erit $76x-68$
de quo reſiduo auferatur $\overline{4x-20}\times 19=76x-380$
reſiduum erit $+312$, quorum numerorum (104 & 312) integralis radix quantitatis y erit diviſor, i. e. erit diviſor numeri (104): & ſic ſubſtituantur 1 & -1 pro quantitate (y), & quantitates reſultantes erunt $2x^2+7x-16$ & $3x^2+7x-16$; $2x^2+x-18$ & $3x^2+x-16$ reſpective, & exinde numeri reſultantes erunt 512 & 768; 972 & 1458; quorum maximi communes diviſores ſunt 256 & 486 reſpective: numerorum 256, 104, 486, &c. inveniantur diviſores, & unica ſolummodo eſt arithmetica progreſſio (1, 2, 3, &c.), cujus communis differentia eſt (1); ergo numerus (2), qui fuit diviſor numeri 104 reſultantis e ſubſtitutione (0) pro incognitâ quantitate (y), pro integrali quantitatis (y) radice tentandus eſt.

E problematibus præcedentibus facile deduci poteſt methodus inveniendi fractionales correſpondentes radices incognitarum quantitatum (x & y).

Ex principiis in problem. præcedentibus datis quadraticæ, cubicæ, &c. rationales æquationes; quarum radices ſunt radices incognitæ quantitatis (y) duarum æquationum duas vel plures incognitas quantitates (x, y, &c.) habentium, inveniri poſſunt; ſi modo earum duo, tres, &c. correſpondentes valores incognitæ quantitatis (x) ſint æquales; ſin aliter ad reductionem æquationum neceſſarium eſt confugere.

Et ſic in pluribus ſimiliter ac in unâ æquatione unam incognitam quantitatem habente; transformentur datæ æquationes in alias, quarum radices ſunt quæcunque rationales datarum æquationum radicum functiones; & exinde deduci poteſt methodus eruendi rationales datarum æquationum radices.

2. Eſt etiam alia methodus hoc problema reſolvendi: ſint duæ æquationes

tiones duas incognitas quantitates habentes $S - Ry + Qy^2 - Py^3 + \&c. \ldots y^n = 0$, & $s - ry + qy^2 - py^3 + \&c. \ldots y^m = 0$; in quibus literæ *S*, *R*, *Q*, *P*, &c. *s*, *r*, *q*, *p*, &c. denotant reſpective integras functiones literæ (x); & ſit *z* rationalis integra radix quantitatis (y) quæſita; tum $\frac{S}{z}$, $\frac{Rz - S}{z^2}\left(\frac{R - A}{z}\right)$, $\frac{Qz^2 - Rz + S}{z^3}\left(\frac{Q - B}{z}\right)$, &c. $\frac{s}{z}$, $\frac{rz - s}{z^2}\left(\frac{r - a}{z}\right)$, $\frac{qz^2 - rz + s}{z^3}\left(\frac{q - b}{z}\right)$, &c. (ubi *A*, *B*, *C*, &c. *a*, *b*, *c*, &c. præcedentes denotant fractiones) integri erunt numeri.

Ea, quæ prius dicta fuerunt de fractionalibus radicibus, & quadraticis, cubicis, &c. æquationibus inveniendis; quarum radices ſunt datarum æquationum incognitæ quantitatis radices, huc etiam proferre liceat.

Cum vero requiratur, ut radices ſint integri numeri: ſæpe e limitibus, inter quos facile conſtat quaſdam conſiſtere radices; vel e diviſoribus; vel e datis formulis numerorum facile deduci poſſunt incognitarum quantitatum integri valores. Et ſic detegi poſſunt rationales radices.

Omnia hæc etiam mutatis mutandis applicari poſſunt ad tres vel plures æquationes tres vel plures incognitas quantitates habentes.

PROB. XLVI.

Datâ æquatione; invenire, utrum ea reſolvi poteſt extractionum radicum quadraticarum, cubicarum, &c. ope.

Datâ æquatione, dantur etiam ejus dimenſiones; aſſumatur generaliter reſolutio, quæ ſolummodo habet radices quadraticas, cubicas &c. ita ut, ſi modo exterminentur ſurdæ quantitates, reſultet æquatio datarum dimenſionum: fiant reſultans & data æquatio inter ſe æquales, & conſit problema.

Ex. Invenire, utrum data æquatio duo vel quatuor vel octo &c. dimenſionum reſolvi poteſt extractionum radicum quadraticarum ope, necne.

Aſſumatur generaliter reſolutio; quæ radices habet quadraticas, ita ut reſultans æquatio duo vel quatuor vel octo, &c. habeat dimenſiones; erit reſolutio $x=\sqrt{a}+b$, vel $x=\sqrt{a+m\sqrt[2]{b}+n\sqrt[2]{b}}+c$, vel $x=\sqrt[2]{a+m\sqrt{b+n\sqrt{c}}+o\sqrt{c}}+r\sqrt{b+n\sqrt{c}}+s\sqrt{c}+d$, & reducantur hæ æquationes, ita ut exterminentur irrationales quantitates, reſultant æquationes $x^2-2bx+b^2-a=0$ & $x^4-4cx^3-\overline{2n^2b-6c^2}+2ax^2-\overline{4c^3-4ac+4mnb}-4n^2cbx+\overline{n^2b+c^2-a-m-2nc}^2\times b=0$, & x^8-8dx^7+ &c. $=0$. Fiant reſultantes & datæ æquationes inter ſe æquales; exinde tot reſultant æquationes, quot incognitæ quantitates; unde e præcedentibus problematibus inveniri poſſunt, utrum quantitates a, m, b, n, c, vel $a, m, b, n, c, o, r, s, d$ integros vel rationales habeant valores, necne; & conficitur problema.

Et methodo haud diſſimili de æquationibus plures dimenſiones habentibus, & extractionibus ope quadraticarum, cubicarum, &c. radicum agere liceat; i. e. aſſumantur generales quantitates, quæ neceſſario continent omnes quantitates dati generis, & ex earum reductione detegi poteſt, utrum datæ æquationes tales recipiant extractiones, necne.

Lemma 1. Sint duæ fractiones $\frac{A}{a}$ & $\frac{B}{b}$ ad minimos terminos reductæ, & earum denominatores haud ſint iidem; tum $\frac{A}{a}\pm\frac{B}{b}$ haud poteſt eſſe integer numerus.

2. Sint $\frac{A}{a}, \frac{B}{b}, \frac{C}{c}, \frac{D}{d}$, &c. fractiones ad terminos minimos reductæ; ubi A, B, C, D, &c.; a, b, c, d, &c. ſint integri numeri; tum, ſi quicunque primus numerus π vel ejus poteſtas π^n ſit diviſor denominatoris a, & nullum alium dividat; $\frac{A}{a}\pm\frac{B}{b}\pm\frac{C}{c}\pm\frac{D}{d}\pm$ &c. haud poteſt eſſe integer numerus.

PROB.

PROB. XLVII.

1. *Datis quibuſcunque* (n) *quantitatibus rationalibus & integralibus functionibus quantitatis* x; *invenire quantitatem vel numerum, in quo univerſi communes diviſores datarum quantitatum continentur.*

Ita reducantur datæ quantitates per vulgarem operationem communes diviſores inveniendi, ut exterminetur incognita quantitas x; reſultabunt $(n-1)$ quantitates, in quarum maximo communi diviſore continentur univerſi communes diviſores quantitatum datarum. In hâc operatione ſemper evitandæ ſunt fractiones.

Ex. 1. Sint duæ datæ quantitates $5x+1$ & $7x+4$; tum primo, ut evitentur fractiones, ducatur $7x+4$ in 5, & reſultat $35x+20$; operatio erit $5x+1)\ 35x+20)\ 7$

$\underline{35x+\ 7}$

13; hinc 13 erit unicus diviſor, quem recipiunt datæ quantitates $5x+1$ & $7x+4$, ubi x eſt integer numerus.

Ex. 2. Sint duæ quantitates $3x^2+2x+8$ & $4x^2+7x+20$, inveniatur earum maximus communis diviſor, i. e. primum ducatur $3x^2+2x+8$ in 4 evitandi fractiones gratiâ, & reſultat $12x^2+8x+32$; tum erit $4x^2+7x+20)\ 12x^2+8x+32)\ 3$

$\underline{12x^2+21x+60}$

$13x+28$, nunc dividatur diviſor $4x^2+7x+20$ per reſiduum $13x+28$; evitandi fractiones cauſâ, ducatur $4x^2+7x+20$ in 13, & reſultat $52x^2+91x+260$; ergo $13x+28)\ 52x^2+91x+260)\ 4$

$\underline{52x^2+112x}$

$-21x+260$; nunc evitandi fractiones gratiâ ducatur $21x-260$ in 13, & reſultat $273x-3380$; unde $13x+28)\ 273x-3380)\ 21$

$\underline{273x+\ 588}$

reſiduum $=\ -3968$; ergo $3968=31\times128$ eſt numerus, in quo

quo continentur univerſi communes diviſores quantitatum datarum $4x^2+7x+20$ & $3x^2+2x+8$, ubi x eſt integer numerus; etiamque quantitatum $4x^2+7x+20$ & $13x+28$; & denique quantitatum $13x+28$ & $21x-260$.

Quantitates $4x^2+7x+20$ & $3x^2+2x+8$ nunquam recipiunt communem diviſorem 31×128; recipiunt ſolummodo diviſores rectanguli 16×31; reliqui diviſores oriuntur e quantitatibus multiplicandis fractiones evitandi gratiâ.

Ex. 3. Sint duæ quantitates x^2-5x+6 & x^3+x^2-x-10; operatio erit $x^2-5x+6)\,x^3+x^2-x-10\,(x+6$

$$\begin{array}{r}x^3-5x^2+6x\\ \hline +6x^2-7x\\ +6x^2-30x+36\\ \hline \end{array}$$

reſiduum $23x-46$; ut evitentur fractiones, ducatur x^2-5x+6 in 23, & productum dividatur per $23x-46$, i.e. operatio erit $23x-46)\,23x^2-115x+138=23\times(x^2-5x+6)\,(x-3$

$$\begin{array}{r}23x^2-46x\\ \hline -69x+138\\ -69x+138\\ \hline \end{array}$$

Hinc $23x-46=23\times(x-2)$ erit quantitas, in quâ continentur univerſi communes diviſores: dividantur x^2-5x+6 & x^3+x^2-x-10 per $x-2$, reſultant $x-3$ & x^2+3x+5: harum duarum qnantitatum unicus diviſor erit 23; ergo diviſores datarum quantitatum invenientur in earum communi diviſore $x-2$ & in diviſoribus quantitatum $x-3$ & x^2+3x+5 ex earum diviſione per communem diviſorem $x-2$ reſultantium. Et ſic de quocunque alio conſimili caſu.

Forſan quantitas reſultans plures recipiat diviſores quam exigat problema; hi vero exoriuntur & includuntur in multiplicatoribus, qui fractiones evitandi gratiâ in datas quantitates vel reſidua ducuntur.

2. Si aliquæ ex his quantitatibus A, B, C, &c. recipiant diviſorem, i.e. $A=a$

$A = a \times \alpha$; tum inveniantur quantitates, in quibus continentur univerſi diviſores quantitatum a, B, C, &c., etiamque quantitatum α, B, C, &c.

Si duæ vel plures A, B, &c. communem diviſorem (α) recipiant, i. e. $A = a \times \alpha^m$, $B = b \times \alpha^n$, &c.; tum inveniantur univerſi diviſores quantitatum α, C, &c.: & a, α, C, &c.; & b, α, C, &c.; & a, b, C, &c.; & perficitur prob.

3. Eadem principia etiam applicari poſſunt ad tres vel plures datas quantitates; etiamque ad (m) quantitates, in quibus duæ vel plures continentur incognitæ quantitates, quarum numerus minor eſt quam (m); etiamque ad negativas propoſitiones.

Ex. Sint duæ datæ quantitates $a^r - b^r$ & $a^s - b^s$, in quibus a & b, r & s ſint inter ſe primi numeri; per methodum prius traditam inveniatur earum diviſor, in quo continentur univerſi earum communes diviſores; facile conſtat eum eſſe hujuſce formulæ $a^t \times (a - b)$; ſed quoniam a & b ſunt inter ſe primi, nec a nec a^t erit diviſor quantitatum b^r vel b^s, nec conſequenter vel quantitatum $a^r - b^r$ vel $a^s - b^s$; ergo $a - b$ erit unicus communis diviſor quantitatum $a^r - b^r$ & $a^s - b^s$: dividantur hæ quantitates per earum communem diviſorem $a - b$, reſultant quantitates $a^{r-1} + a^{r-2}b + a^{r-3}b^2 + \ldots + b^{r-1}$ & $a^{s-1} + s^{s-2}b + a^{s-3}b^2 + \ldots + b^{s-1}$ inter ſe primi numeri.

Si modo dentur quantitatum prædictarum communes diviſores, facile deduci poſſunt formulæ incognitarum quantitatum, quæ prædictos communes diviſores recipiunt.

THEOR. XLIII.

Duæ quantitates $a^n + n \cdot \frac{n-1}{2} a^{n-2}b^2 + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} a^{n-4}b^4 + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdot \frac{n-4}{5} \cdot \frac{n-5}{6} a^{n-6}b^6 + \text{\&c.} = \frac{\overline{a+b}^n + \overline{a-b}^n}{2} = P$, & $n a^{n-1} + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} a^{n-3}b^2 + n \cdot \frac{n-1}{2} \cdot$

$n-2$

$\frac{n-2}{3} \cdot \frac{n-3}{4} \cdot \frac{n-4}{5} a^{n-5}b^4 + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \cdot \frac{n-3}{4} \cdot \frac{n-4}{5} \cdot \frac{n-5}{6} \cdot \frac{n-6}{7} a^{n-7}b^6 + \&c. = \frac{\overline{a+b}^n - \overline{a-b}^n}{2} = Q$; ubi a & b ſunt primi inter ſe numeri, non poſſunt habere communem diviſorem; niſi 2, 4, 8, 16, 32, ... 2^n.

DEMONSTRATIO.

Dividatur $2P = \overline{a+b}^n + \overline{a-b}^n$ per $\overline{a+b}^n - \overline{a-b}^n$; operatio erit $\overline{a+b}^n - \overline{a-b}^n \,)\, \overline{a+b}^n + \overline{a-b}^n \,(\, 1$

$\overline{a+b}^n - \overline{a-b}^n$

$2 \times \overline{a-b}^n$; ergo, ſi quantitates P & Q habeant communem diviſorem, quantitates $\overline{a+b}^n \pm \overline{a-b}^n$ & $2 \times \overline{a-b}^n$ eundem habent communem diviſorem; dividatur $\overline{a+b}^n + \overline{a-b}^n$ per $2 \times \overline{a-b}^n$, operatio erit $2 \times \overline{a-b}^n \,)\, 2 \times \overline{a+b}^n + 2 \times \overline{a-b}^n \,(\, 1$

$2 \times \overline{a-b}^n$

reſiduum $2 \times \overline{a+b}^n$; unde, ſi prædictæ quantitates habeant communem diviſorem, $\overline{a+b}^n$ & $\overline{a-b}^n$ habent eundem communem diviſorem: dividatur $a+b$ per $a-b$ &c. operatio erit

$a-b \,)\, a+b \,($

$a-b$

reſiduum $2b$, ergo $2b$ & $a \pm b$ habent eundem communem diviſorem ac quantitates $a-b$ & $a+b$; & conſequenter, quoniam a & b ſunt inter ſe primi, erit 2 unicus communis diviſor quantitatum $a+b$ & $a-b$; & exinde 2, 4, 8, 16, .. 2^n erunt univerſi communes diviſores quantitatum $\overline{a+b}^n$ & $\overline{a-b}^n$; etiamque quantitatum $\frac{\overline{a+b}^n + \overline{a-b}^n}{2}$ & $\frac{\overline{a+b}^n - \overline{a-b}^n}{2}$. q. e. d.

2. Sint

2. Sint a & c inter ſe primi numeri, & in duabus prædictis quantitatibus $(P$ & $Q)$ pro b^2, b^4, b^6, &c. ſcribantur reſpective c, c^2, c^3, &c.; duæ reſultantes quantitates nullum habere poſſunt communem diviſorem, præter prædictos $2, 4, 8, 16, \ldots 2^n$.

E præcedente caſu & operatione in prob 47. uſitatâ facile conſtat.

Conſimiles propoſitiones etiam affirmari poſſunt de ſummis e ſingulis duobus, tribus, quatuor, &c. terminis ſeriei; quæ oritur ab expanſione binomialis, trinomialis, &c. quantitatis in terminos ſecundum dimenſiones literæ x progredientes.

Facile etiam detegi poſſunt caſus, in quibus prædictæ quantitates dividi poſſunt per 2^n vel 2^{n-1} vel 2^{n-2} vel $2^{n-3} \ldots 8, 4$, & 2. E. g. Sit $n = 3$ & data quantitas $a + \sqrt{c}$, ubi a & c ſunt inter ſe primi numeri; tum duæ quantitates per præcedentem methodum deductæ, viz. $a^3 + 3ac$ & $3a^2 + c$ habent communem diviſorem $2^3 = 8$, cum $a = 4x + 1$ & $c = 4z + 1$, ubi x eſt integer & z impar numerus; habent diviſorem $2^2 = 4$, cum $a = 2x + 1$ & $c = 4v + 1$, ubi x & v ſunt integri numeri; habent diviſorem 2, cum $a = 2x + 1$ & $b = 2v + 1$; ſin aliter: i.e. ſi a vel c ſit par numerus, tum nullum recipient communem diviſorem; &c.

PROB. XLVIII.

Reducere æquationem $x^{2n} + px^{2n-1} + qx^{2n-2} + rx^{2n-3} + sx^{2n-4} + tx^{2n-5} + \&c. = 0$ *in duas alias* (P) $x^n + ax^{n-1} + bx^{n-2} + cx^{n-3} + dx^{n-4} + ex^{n-5} + fx^{n-6} + \&c. = \sqrt{l} \times (\alpha x^{n-1} + \beta x^{n-2} + \gamma x^{n-3} + \delta x^{n-4} + \&c.)$ *extractionis quadraticæ radicis ope.*

Reducatur æquatio *(P)*, ita ut exterminetur irrationalis quantitas $\sqrt{l}$, reſultat $x^{2n} + 2ax^{2n-1} + (2b + a^2 - l\alpha^2) \times x^{2n-2} + \overline{2c + 2ba - 2l\alpha\beta}\, x^{2n-3} + \overline{2d + 2ac + b^2 - l \times (2\alpha\gamma + \beta^2)}\, x^{2n-4} + \overline{2e + 2da + 2bc - l \times (2\alpha\delta + 2\beta\gamma)}\, x^{2n-5} + \&c. = 0$: fiant reſpectivi datæ & reſultantis æquationis termini inter ſe æquales, & reſultant $(2n)$ æquationes $2a = p$, $2b + a^2 - l\alpha^2 = q$, $2c + 2ab - 2l\alpha\beta = r$, $2d + 2ac + b^2 - l \times (2\alpha\gamma + \beta^2) = s$, $2e + 2ad + 2bc - l \times (2\alpha\delta + 2\beta\gamma) = t$, &c. Hæ autem $2n$ æquationes habent $(2n + 1)$ incognitas

cognitas quantitates, harum incognitarum una (l) ſemper ducta fuit in (n) incognitas quantitates $(\alpha, \beta, \gamma, \delta,$ &c.$)$; reducantur igitur hæ $(2n)$ æquationes in (n) æquationes, ita ut exterminentur (n) incognitæ quantitates $(a, b, c,$ &c.$)$ reſultant (n) æquationes nullas incognitas quantitates habentes, præter eas $(\alpha, \beta, \gamma, \delta, \varepsilon,$ &c.$)$, in quas incognita quantitas (l) ſemper ducta fuit: ergo reſultant (n) numeri, quorum omnium l eſt diviſor: i. e. e primâ æquatione $a = \frac{p}{2}$; e ſecundâ (ſi modo $\pi = q - \frac{1}{4}p^2$) $2b - l\alpha^2 = \pi$; e tertiâ (ſi modo $\varrho = r - \frac{1}{2}p\pi$) $2c = \varrho + l \times \left(2\alpha\beta - \frac{p\alpha^2}{2}\right)$; e quartâ (ſi modo $\sigma = s - \frac{1}{2}p\varrho - \frac{1}{4}\pi^2$) $2d = \sigma - l \times \left(p\alpha\beta - \frac{p^2\alpha^2 - 2\pi\alpha^2 - l\alpha^4}{4} - 2\alpha\gamma - \beta^2\right)$; & ſic deinceps: quærendus eſt communis diviſor integer & non quadratus (n) numerorum reſultantium, qui per 4 diviſus relinquat unitatem, ſi modo terminorum alternorum $p, r, t,$ &c. aliquis ſit impar: per diviſores intelligo tales fractiones, quales habent poteſtates numeri 2 pro ſuis denominatoribus: ſi a ſit dimidium imparis numeri, tum (α) etiam erit dimidium imparis numeri; ſi b, tum β; ſi c, tum γ; & ſic deinceps: ſi vero communem diviſorem (l) admittant prædicti (n) numeri; quære numerum quadratum (Γ^2), cui per (l) multiplicato ultimus æquationis terminus ſub ſigno proprio annexus quadratum numerum efficit: ſit enim data æquatio $P + Qx + Rx^2 +$ &c. $= 0$; aſſumatur æquatio $A + Bx + Cx^2 +$ &c. $= \sqrt{l} \times (\Gamma + \Delta x + Ex^2 +$ &c.$)$ reſultat data æquatio $A^2 - l \times \Gamma^2 + 2\overline{AB - l\Gamma\Delta}x +$ &c. $= 0$, unde $P = A^2 - l\Gamma^2$, $Q = 2AB - 2l\Gamma\Delta$, &c.

Hæc limitationes admittant ex hâc obſervatione, quod $A \pm \sqrt{l}\Gamma$, $B \pm \sqrt{l}\Delta$, $C \pm \sqrt{l}E$, &c. haud majores erunt quantitates quam quadruplus maximus terminus datæ æquationis; vel ex infinitis aliis.

Cor.

Cor. Sit $x^4+px^3+qx^2+rx+s=0$, & ſupponatur $x^2+\frac{px}{2}+b=\sqrt{l}\times(\alpha x+\beta)$: ſit $q-\frac{1}{4}p^2=\pi$, & $r-\frac{1}{2}p\pi=\varrho$, $s-\frac{1}{4}\pi^2=\sigma$; e prædictâ regulâ conſtat l eſſe communem diviſorem terminorum ϱ & 2σ, imparem & non quadratum, & per (4) diviſum unitatem relinquere, ſi terminorum p & r alteruter ſit impar: pone pro α diviſorem aliquem quantitatis $\frac{\varrho}{l}$, ſi p ſit par; vel imparis diviſoris dimidium, ſi p ſit impar; aufer quotum de $\frac{1}{2}p\alpha$, & reliqui dimidium dic β; deinde pro b pone $\frac{\pi+l\alpha^2}{2}$; & tenta, ſi l dividat b^2-s; quoti radix ſit rationalis, & æqualis l; ſi hoc contigerit, reducitur data æquatio; ſin aliter reductionem haud admittet.

Omnia hæc e præcedentibus æquationibus, viz. ($a=\frac{p}{2}$, $2b=\pi+l\alpha^2$, $l\times\overline{\frac{p\alpha^2}{2}-2\alpha\beta}=\varrho$, & $s=b^2-l\beta^2$) conjunctim conſideratis facile ſequuntur.

2. Alia deduci poteſt methodus hoc corollarium reſolvendi; deleatur ſecundus datæ æquationis terminus, & reſultet æquatio $x^4+Qx^2+Rx+S=0$: inveniatur, utrum æquatio $P^6+2qP^4+\overline{q^2-4s}\,P^2-r^2=0$ admittat rationalem radicem, & haud quadratum, necne; exinde conſtabit corollarium: eadem & multæ conſimiles methodi in æquationes majorum dimenſionum facile extendi poſſunt.

3. Datis n æquationibus numerum (m), qui minor ſit quam n, incognitarum quantitatum habentibus, quæ haud continuo ducantur in incognitam quantitatem (l); & quoque numerum (r) incognitarum quantitatum, quæ continuo ducantur in incognitam quantitatem (l); ita reducantur (n) datæ æquationes, ut exterminentur (m) incognitæ quantitates, in quas haud continuo ducta fuit incognita

quantitas *(l)*; exinde deduci poſſunt diverſi numeri, quorum *(l)* erit diviſor communis; ex quibuſque $m+1$ æquationibus inveſtigari poteſt numerus, cujus l eſt diviſor.

PROB. XLIX.

Sit data æquatio $x^{4n}+px^{4n-1}+qx^{4n-2}+rx^{4n-3}+sx^{4n-4}+tx^{4n-5}+ = 0$ *dimenſionum* (4n); *invenire utrum ea deprimi poteſt in æquationem* (n) *dimenſionum radicum quadraticarum extractionum ope, necne.*

Aſſumatur æquatio $x^n+ax^{n-1}+bx^{n-2}+$ &c. $+\sqrt{l}\times(\alpha x^{n-1}+\beta x^{n-2}+$ &c.$)=\sqrt{r+s\sqrt{l}}\times(\pi x^{n-1}+\rho x^{n-2}+$ &c.$)$ reducatur ea, ita ut exterminentur irrationales quantitates, reſultat $x^{4n}+4ax^{4n-1}+\overline{4b-2r\pi^2+6a^2-2l\alpha^2}\,x^{4n-2}+$ &c. $=0$.

Hinc colliguntur æquationes $4a=p$, $4b-2r\pi^2+6a^2-2l\alpha^2=q$, &c. reſultant $(4n)$ æquationes $(3n+2)$ incognitas & rationales quantitates habentes, quarum $(n+1)$ incognitæ quantitates α, β, &c. s ducuntur in incognitam quantitatem *(l)*; exterminentur omnes incognitæ quantitates præter prædictas (α, β, &c. s); reſultabunt diverſi numeri, quorum quantitas *(l)* erit communis diviſor.

Omnis incognita quantitas vel erit integer numerus, vel impar numerus per duo vel quatuor, &c. diviſus: denominatores, quos recipiunt quantitates incognitæ, e principiis in prob. 47. & theor. 43. traditis, deduci poſſunt.

Hæ incognitæ quantitates inveniri poſſunt e methodo communes diviſores inveniendi, quoniam plures dantur æquationes quam incognitæ quantitates: vel aliter e principiis in priori problemate contentis.

Et ſic in genere invenire liceat, utrum æquatio reduci poteſt extractionum radicum quadraticarum, vel cubicarum, &c. ope, necne.

Eadem etiam applicari poſſunt ad plures æquationes plures incognitas quantitates habentes.

THEOR.

THEOR. XLIV.

1. Sint n & m integri numeri; & e, f, g & k rationales; & $\sqrt{f}$, $\sqrt{k}$ & $\sqrt[2n+1]{g}$ irrationales quantitates.

1. 1. Erit $(e+\sqrt{f})^m = e^m + m.\frac{m-1}{2}e^{m-2}f + m.\frac{m-1}{2}.\frac{m-2}{3}.\frac{m-3}{4}e^{m-4}f^2 + \&c. + (me^{m-1} + m.\frac{m-1}{2}.\frac{m-2}{3}e^{m-3}f + m.\frac{m-1}{2}.\frac{m-2}{3}.\frac{m-3}{4}.\frac{m-4}{5}e^{m-5}f^2 + \&c.)\sqrt{f}$; ubi rationalis pars $e^m + m.\frac{m-1}{2}e^{m-2}f + m.\frac{m-1}{2}.\frac{m-2}{3}.\frac{m-3}{4}e^{m-4}f^2 + \&c. = A$; & irrationalis pars $(me^{m-1} + m.\frac{m-1}{2}.\frac{m-2}{3}e^{m-3}f + m.\frac{m-1}{2}.\frac{m-2}{3}.\frac{m-3}{4}.\frac{m-4}{5}e^{m-5}f^2 + \&c.)\sqrt{f} = B$; unde $A^2 - B^2 = e^{2m} - me^{2m-2}f + m.\frac{m-1}{2}e^{2m-4}f^2 - m.\frac{m-1}{2}.\frac{m-2}{3}e^{2m-6}f^3 + \&c. = \overline{e^2-f}^m$.

1. 2. Sit $\sqrt{k}\times\overline{e+\sqrt{f}} = x$; & exinde $x^m = (e^m + m.\frac{m-1}{2}e^{m-2}f + \&c.)k^{\frac{m}{2}} + (me^{m-1} + m.\frac{m-1}{2}.\frac{m-2}{3}e^{m-3}f + \&c.)k^{\frac{m}{2}}f^{\frac{1}{2}}$; ſint etiam $(e^m + m.\frac{m-1}{2}e^{m-2}f + \&c.)\times k^{\frac{m}{2}} = A$ & $(me^{m-1} + m.\frac{m-1}{2}.\frac{m-2}{3}e^{m-3}f + \&c.)k^{\frac{m}{2}}\times\sqrt{f} = B$; tum erit $A^2 - B^2 = \overline{e^2-f}^m\times k^m$.

1. 3. Sit $x = (e+\sqrt{f})\times\sqrt[2n+1]{g}$, vel $= \sqrt{k}\times(e+\sqrt{f})\times\sqrt[2n+1]{g}$; tum $x^{2n+1} = (e^{2n+1} + \overline{2n+1}.\frac{2n}{2}e^{2n-1}f + \overline{2n+1}.\frac{2n}{2}.\frac{2n-1}{3}.\frac{2n-2}{4}e^{2n-3}f^2 + \&c.)g + (\overline{2n+1}\,e^{2n} + \overline{2n+1}.\frac{2n}{2}.\frac{2n-1}{3}e^{2n-2} + \&c.)$

$\times g$

$\times g \times \sqrt{f}$ vel $= (e^{2n+1} + \overline{2n+1} \,.\, \frac{2n}{2} e^{2n-1} f + \&c.) \times g \times k^{\frac{2n+1}{2}} + (\overline{2n+1}$ $e^{2n} + \overline{2n+1} \,.\, \frac{2n}{2} \,.\, \frac{2n-1}{3} e^{2n-2} f + \&c.) \times g \times k^{\frac{2n+1}{2}} \times \sqrt{f}$: aſſumatur rationalis quantitas $(e^{2n+1} + \overline{2n+1} \,.\, \frac{2n}{2} e^{2n-1} f + \&c.)\, g = A$, & irrationalis $(\overline{2n+1}\, e^{2n} + \overline{2n+1} \,.\, \frac{2n}{2} \,.\, \frac{2n-1}{3} e^{2n-2} f + \&c.) \times g \times \sqrt{f} = B$; tum erit in priori caſu $A^2 - B^2 = \overline{e^2 - f}^{\,2n+1} \times g^2$, & in poſteriori $(A \times k^{\frac{2n+1}{2}})^2 - (B \times k^{\frac{2n+1}{2}})^2 = \overline{e^2 - f}^{\,2n+1} \times k^{2n+1} \times g^2$: & exinde $\sqrt[2n+1]{A^2 - B^2} = (e^2 - f) \times \sqrt[2n+1]{g^2}$ in priori caſu, & $\sqrt{A^2 - B^2} \times k$ $= (e^2 - f) \times k \times \sqrt[2n+1]{g^2}$ in poſteriori; ſed $\sqrt[2n+1]{g^2}$ eſt irrationalis quantitas; ergo in utroque caſu $\sqrt[2n+1]{A^2 - B^2}$ eſt irrationalis quantitas, ſi modo $\sqrt[2n+1]{g}$ ſit irrationalis quantitas.

Cor. 1. Hinc, ſi literæ A & B vel ſint integri numeri vel quadraticæ radices integrorum numerorum, quorum A major ſit quam B; & $\sqrt[2n+1]{A^2 - B^2}$ ſit integer numerus; radix $\sqrt[2n+1]{A + B}$ in formulâ $(e + \sqrt{f}) \times \sqrt[2n+1]{g}$ minime, at in formulâ $e + \sqrt{f}$ vel in formulâ $(e + \sqrt{f}) \times \sqrt{k}$ forſan contineri poteſt.

1, 4. Sit $(a^2 - b) \times N^2 = (A^2 - B^2) \times N^2 = \pi^{2n+1}$, ubi literæ a, b, N & π integros denotant numeros; ducatur N in $a + \sqrt{b}$, reſultat $Na + N\sqrt{b}$; ejus radix $(2n+1)$ poteſtatis, i. e. $\sqrt[2n+1]{Na + N\sqrt{b}}$ forſan contineri poteſt in formulâ $e + \sqrt{f}$; minime vero in formulâ $(e + \sqrt{f}) \times \sqrt[2n+1]{g}$.

2. Sit $(a^2 - b) \times N = (A^2 - B^2) \times N = \pi^{2n+1}$; tum $\sqrt[2n+1]{N^{\frac{1}{2}} a + N^{\frac{1}{2}} \sqrt{b}}$ forſan

forſan contineri poteſt in formulâ $(e+\sqrt{f})\times\sqrt{k}$; minime vero in formulâ $(e+\sqrt{f})\times\sqrt[2n+1]{g}\times\sqrt{k}$.

Erunt enim in priori caſu $Na=A$ & $N\sqrt{b}=B$; in poſteriori caſu $N^{\frac{1}{2}}a=A$ & $N^{\frac{1}{2}}\times\sqrt{b}=B$; unde differentia quadratorum erit $A^2-B^2=N^2\times(a^2-b)$ in priori caſu, & $=N\times(a^2-b)$ in poſteriori caſu; & conſequenter $\sqrt[2n+1]{Na+N\sqrt{b}}$ & $\sqrt[2n+1]{N^{\frac{1}{2}}a+\sqrt{N}\times\sqrt{b}}$ forſan contineri poſſunt reſpective in formulis $e+\sqrt{f}$ & $(e+\sqrt{f})\times\sqrt{k}$; minime vero in formulis prædictis, quæ continent $\sqrt[2n+1]{g}$.

Cor. Hinc, ſit data quantitas $a+\sqrt{b}$; & multiplicetur hæc quantitas in N vel $N^{\frac{1}{2}}$; ubi ita aſſumatur numerus N, ut evadat $N^2\times(a^2-b)=\pi^{2n+1}$ vel $N(a^2-b)=\pi^{2n+1}$; quantitas reſultans $N\times(a+\sqrt{b})$ vel $N^{\frac{1}{2}}\times(a+\sqrt{b})$ forſan habet ejus radicem $(2n+1)$ poteſtatis hujuſce formulæ $e+\sqrt{f}$ vel hujuſce $\sqrt{k}+\sqrt{f}$, &c.

2. Sit radix $\sqrt[n]{e}+\sqrt[n]{f}=\pi$; inveniatur $\left(\sqrt[n]{e}+\sqrt[n]{f}\right)^m=e^{\frac{m}{n}}+f^{\frac{m}{n}}+me^{\frac{m-1}{n}}f^{\frac{1}{n}}+m\cdot\frac{m-1}{2}e^{\frac{m-2}{n}}f^{\frac{2}{n}}+\&c.=P$; tum e prob. 26. erui poteſt quantitas, quæ ducta in P facit rationale rectangulum Q, cujus dimenſiones ſunt m; erit $Q=\overline{e\pm f}^{\,m}$, viz. erit $\overline{e+f}^{\,m}$, ſi n ſit impar numerus; ſin aliter $\overline{e-f}^{\,m}$. e. g. ſit $n=3$; tum $\overline{e^{\frac{1}{3}}+f^{\frac{1}{3}}}^{\,m}=(e^{\frac{m}{3}}+m\cdot\frac{m-1}{2}\cdot\frac{m-2}{3}e^{\frac{m-3}{3}}f+m\cdot\frac{m-1}{2}\cdot\frac{m-2}{3}\cdot\frac{m-3}{4}\cdot\frac{m-4}{5}\cdot\frac{m-5}{6}e^{\frac{m-6}{3}}f^2+\&c.=A)+(me^{\frac{m-1}{3}}f^{\frac{1}{3}}+m\cdot\frac{m-1}{2}\cdot\frac{m-2}{3}\cdot\frac{m-3}{4}e^{\frac{m-4}{3}}f^{\frac{4}{3}}+\&c.=B)+(m\cdot\frac{m-1}{2}e^{\frac{m-2}{3}}f^{\frac{2}{3}}+m\cdot\frac{m-1}{2}\cdot\frac{m-2}{3}\cdot\frac{m-3}{4}\cdot\frac{m-4}{5}e^{\frac{m-5}{3}}f^{\frac{5}{3}}+\&c.=C)$; & exinde $(A+\alpha B+\alpha^2C)\times(A+\beta B+\beta^2C)\times(A+\gamma B+\gamma^2C)$

$+\gamma^2 C) = A^3 - 3ABC + B^3 + C^3 = \overline{e+f}^3$; hic α, β & γ ſunt tres radices æquationis $z^3 - 1 = 0$.

Sit π trinomialis, &c. quantitas; ex eâdem methodo deduci poteſt π^m, &c.

3. Sint n & m numeri inter ſe primi, & $A + B\sqrt[n]{p} + C\sqrt[n]{p^2} + D\sqrt[n]{p^3} \ldots + H\sqrt[n]{p^{n-2}} + L\sqrt[n]{p^{n-1}} = \sigma$ quantitas, cujus radicis (m) poteſtatis formula requiratur.

Inveniatur quantitas P, quæ in σ ducta facit rectangulum $P \times \sigma = \pi^m$; tum erit formula radicis quæſitæ $(\lambda) = \dfrac{a + b\sqrt[n]{p} + c\sqrt[n]{p^2} \ldots + k\sqrt[n]{p^{n-2}} + b\sqrt[n]{p^{n-1}}}{Q\sqrt[m]{P}}$; ubi $A, B, C, D, \ldots H, L, p$; $a, b, c, \ldots k, b, Q$; P, σ & π ſunt integri numeri: numerus Q ex prob. 47. & theor. 43. facile deduci poteſt.

Inveniatur λ^n, & ex æquatis inter ſe correſpondentibus quantitatum λ^n & σ terminis, erui poſſunt valores integri quantitatum a, b, c, d, &c.

Et ſic de infinitis hujuſcemodi propoſitionibus.

PROB. L.

Datâ æquatione, invenire utrum ea ullas habeat radices datæ formulæ, in cujus formulâ literæ contentæ ſemper rationales quantitates denotent, necne.

Per methodum in capite tertio traditam inveniatur, utrum ullæ ſint radices, quæ datam habeant formulam, necne: & per problem. præcedentia inveniatur, utrum literæ contentæ ſint integrales vel rationales, necne.

In quam plurimis caſibus multo facilior erui poteſt ſolutio e terminis correſpondentibus inter ſe comparandis.

Ex. 1. Sit formula $\sqrt{x} + \sqrt{y}$, ubi x & y rationales denotent quantitates, & x major ſit quam y: & ſit data æquatio $w^2 = a + \sqrt{b} = x + y + 2\sqrt{xy}$, unde $x + y = a$ & $2\sqrt{xy} = b$, & conſequenter

$x^2 -$

$x^2 - 2xy + y^2 = a^2 - b$ & $x - y = \sqrt{a^2 - b}$; hinc erit $x = \frac{a + \sqrt{a^2 - b}}{2}$, & $y = \frac{a - \sqrt{a^2 - b}}{2}$; & exinde conſtat ejus radices (w) haud datam habere formulam, ni $a^2 - b$ vel ſit quadratum vel o.

Sit $w^2 = \sqrt{a} + \sqrt{b}$; facile reduci poteſt in formulam $w^2 = \sqrt{a} \times \left(1 + \sqrt{\frac{b}{a}}\right)$, vel in formulam $w^2 = \sqrt{b} \times \left(1 + \sqrt{\frac{a}{b}}\right)$.

Eadem methodus etiam applicari poteſt ad extrahendam radicem quatuor vel octo vel ſedecim &c. dimenſionum; vel ad extrahendam radicem quadraticam, &c. e tribus vel pluribus hujuſcemodi quantitatibus $a + \sqrt{b} + \sqrt{c} + \sqrt{d} +$ &c. vel citius inveniri poteſt radix e pluribus quam duabus quantitatibus dividendo factum quarumvis duarum radicalium per tertiam aliquam radicalem, quæ producit quotum rationalem & integrum; nam quoti iſtius radix erit duplum partis radicis quæſitæ.

Ex. 2. Datâ quantitate $N \times (a + \sqrt{b})$, ubi $N^2a^2 - N^2b = \pi^{2n+1}$; invenire ejus radicem $\sqrt[2n+1]{Na + N\sqrt{b}}$ ſub formulâ $e + \sqrt{f}$; ſi modo talem recipiat; ubi N, a, b, n, e & f integros denotant numeros.

1. Sit $x + \sqrt{y} = \sqrt[2n+1]{N \times (a + \sqrt{b})}$, tum $x - \sqrt{y} = \sqrt[2n+1]{N \times (a - \sqrt{b})}$, & conſequenter $(x + \sqrt{y}) \times (x - \sqrt{y}) = x^2 - y = \sqrt[2n+1]{N^2a^2 - N^2b} = \pi$; ſit a major quam $\sqrt{b}$, inveniatur ſumma $\frac{1}{2}\sqrt[2n+1]{N \times (a + \sqrt{b})} + \frac{1}{2}\sqrt[2n+1]{N \times (a - \sqrt{b})}$, ad primum decimarum locum vera, cui ſummæ ſit z proximus integer numerus; tum $z = x$; & $y = z^2 - \pi$: ſit a minor quam $\sqrt{b}$, & inveniatur ſumma $\frac{1}{2}\sqrt[2n+1]{N \times (a + \sqrt{b})} - \frac{1}{2}\sqrt[2n+1]{N \times (a - \sqrt{b})}$, ad primum decimarum locum vera, cui ſummæ ſit z proximus integer numerus; tum erit $z = x$ & $y = z^2 - \pi$. Facile conſtat e prædictis.

2. Sit $a^2 - b = \pi^{2n+1}$, inveniatur quantitas rationalis (v) radice quæſita $\sqrt[2n+1]{a+\sqrt{b}}$ paulo major, ita ut differentia non major ſit quam $\frac{1}{2}$; tum, ſi a^2 major ſit quam b & z ſit proximus integer numerus ad ſummam $v+\frac{\pi}{v}$, eâ vero haud major; erit ejus ſemiſſis ($\frac{1}{2}z$) pars rationalis radicis quæſitæ, i. e. $\frac{1}{2}z = x$ & $y = \frac{1}{4}z^2 - \pi$.

Si vero a^2 minor ſit quam b, & z ſit proximus integer numerus ad differentiam $v - \frac{\pi}{v}$, eâ vero minor; erit $\frac{1}{2}z = x$ & $y = \frac{1}{4}z^2 - \pi$. *demonſtratio.*

Sit $v = \sqrt[2n+1]{a+\sqrt{b}} + \varrho$, tum ϱ non major eſt quam $\frac{1}{2}$; ſed $\sqrt[2n+1]{a+\sqrt{b}}$ ſemper major eſt quam $\sqrt[2n+1]{a-\sqrt{b}}$, etiamque quam unitas: erit $\frac{\pi}{v} = \frac{\sqrt[2n+1]{a^2-b}}{\sqrt[2n+1]{a+\sqrt{b}}+\varrho} = \sqrt[2n+1]{a-\sqrt{b}} - \frac{\varrho\times\sqrt[2n+1]{a-\sqrt{b}}}{\sqrt[2n+1]{a+\sqrt{b}}} + \frac{\varrho^2\times\sqrt[2n+1]{a-\sqrt{b}}}{\left(\sqrt[2n+1]{a+\sqrt{b}}\right)^2} -$ &c.; unde $v+\frac{\pi}{v} = \sqrt[2n+1]{a+\sqrt{b}} + \sqrt[2n+1]{a-\sqrt{b}} + \varrho - \frac{\varrho\times\sqrt[2n+1]{a-\sqrt{b}}}{\sqrt[2n+1]{a+\sqrt{b}}} +$ &c.; ſed $\sqrt[2n+1]{a+\sqrt{b}} + \sqrt[2n+1]{a-\sqrt{b}} = 2x$; ergo $v+\frac{\pi}{v}$ major eſt quam $2x$ per quantitatem $\varrho - \frac{\varrho\sqrt[2n+1]{a-\sqrt{b}}}{\sqrt[2n+1]{a+\sqrt{b}}} +$ &c.; quæ ſemper minor erit quam $\frac{1}{2}$:

Erit

Erit $v - \frac{\pi}{v} = \sqrt[2n+1]{a+\sqrt{b}} - \sqrt[2n+1]{a-\sqrt{b}} + \varrho + \frac{\varrho \times \sqrt[2n+1]{a-\sqrt{b}}}{\sqrt[2n+1]{a+\sqrt{b}}}$ — &c.; ſi vero a^2 minor ſit quam b, tum $\sqrt[2n+1]{a+\sqrt{b}} - \sqrt[2n+1]{a-\sqrt{b}} = 2x$; ergo $2x$ minor erit quam $v - \frac{\pi}{v}$ per quantitatem minorem quam $\varrho + \frac{\varrho \times \sqrt[2n+1]{a-\sqrt{b}}}{\sqrt[2n+1]{a+\sqrt{b}}}$; & quoniam ϱ haud major eſt quam $\frac{1}{2}$, per quantitatem minorem quam 1.

3. Datâ quantitate $N^{\frac{1}{2}} \times (a+\sqrt{b}) = N^{\frac{1}{2}}(A+B)$; ubi $N \times (a^2 - b) = \pi^{2n+1}$, & A major ſit quam B; invenire ejus radicem $\sqrt[2n+1]{N^{\frac{1}{2}}(a+\sqrt{b})}$ in formulâ $\sqrt{x}+\sqrt{y}$, ubi $\sqrt{x}$ major eſt quam $\sqrt{y}$; computa $\sqrt[2n+1]{N^{\frac{1}{2}}(A+B)}$ in numeris integris proximis, ſit illud r; i. e. (r) erit $\sqrt[2n+1]{N^{\frac{1}{2}}(a+\sqrt{b})} \pm \varrho$, ubi ϱ minor eſt quam $\frac{1}{2}$; conſequenter $r + \frac{\pi}{r} = \sqrt[2n+1]{N^{\frac{1}{2}}(a+\sqrt{b})} + \sqrt[2n+1]{N^{\frac{1}{2}}(a-\sqrt{b})} \pm \varrho \mp \varrho \times \sqrt[2n+1]{\frac{a-\sqrt{b}}{a+\sqrt{b}}} + \varrho^2 \times \sqrt[2n+1]{\frac{a-\sqrt{b}}{(a+\sqrt{b})^2}} \mp \varrho^3 \times \sqrt[2n+1]{\frac{a-\sqrt{b}}{(a+\sqrt{b})^3}} +$ &c. $= \sqrt{x} \pm$ quantitate minore quam ϱ, & conſequenter quam $\frac{1}{2}$: divide $A\sqrt{N}$ per maximum rationalem & integrum diviſorem, ſit quotus s; & (s) erit irrationalis factor majoris partis $\sqrt{x}$, & haud minor quam 1; dividatur ſumma $r + \frac{\pi}{r}$ (quæ haud differt a radice quæſitâ per $\frac{1}{2}$) per $2s$ bis prædictam irrationalem quantitatem; tum quotiens haud differt

quadratornm $x^2 - y^2 = r$; quære etiam ſummam quadratorum $x^2 + y^2 = \frac{1}{2}\sqrt[n]{\overline{A+B}^2 \times p} + \frac{1}{2}\sqrt[n]{\overline{A-B}^2 \times p}$; ſi vero radix exhiberi poſſit, neceſſe eſt ut ſumma binarum irrationalium quantitatum fiat integer numerus. Hic itaque prodibit, ſi numeri integri quærantur proxime accedentes ad valores irrationales $\sqrt[n]{\overline{A+B}^2 \times p}$ & $\sqrt[n]{\overline{A-B}^2 \times p}$, ſumendo alterum juſto majorem, alterum juſto minorem: ſit igitur proxime $\sqrt[n]{\overline{A+B}^2 \times p} = s \pm$ fractione, & $\sqrt[n]{\overline{A-B}^2 \times p} = t \mp$ fractione: hinc $x^2 + y^2 = \frac{s+t}{2}$, & $x^2 = \frac{2r+s+t}{4}$, & $y^2 = \frac{s+t-2r}{4}$, unde radix quæſita erit $\frac{\sqrt{2r+s+t} + \sqrt{s+t-2r}}{2\sqrt[2n]{p}}$. Hæc regula ſemper radices quæſitas, ſi modo ullæ ſint, deteget; diviſorem enim 2 recipiet, quæ in prædictis methodis haud continetur. Si vero data quantitas $A + B$ multiplicetur in 2^n, tum quantitatis reſultantis per prædictas regulas ſemper extrahi poteſt radix quæſita, ſi modo ea radix per hanc regulam inveniri poſſit, e. g. ſit data quantitas $2 + \sqrt{5}$, cujus radix cubica erit $\frac{1+\sqrt{5}}{2}$, quæ per priores methodos haud inveniri poteſt; ducatur $2 + \sqrt{5}$ in 2^3, & radix cubica quantitatis $16 + 8\sqrt{5}$ per prædictas methodos extrahi poteſt.

1. Hæ autem regulæ fallunt, cum una quantitas A vel B ſit impoſſibilis: hoc in caſu pone $(A^2 - B^2) \times p = r^n$; & ſit radix quæſita $\frac{x+y}{2\sqrt[2n]{p}}$: aſſumatur æquatio $z^n - nr^2 z^{n-2} + n \cdot \frac{n-3}{2} r^4 z^{n-4} - n \times \frac{n-4}{2} \times \frac{n-5}{3} r^6 z^{n-6} +$ &c. $= 2p \times (A^2 + B^2)$, ex quâ valor ipſius z innoteſcet, & erit $x^2 + y^2 = \frac{1}{2}z$, & $x^2 - y^2 = r$; unde $x =$

$\sqrt{z+r}$

$\frac{\sqrt{z+2r}}{2}$ & $y = \frac{\sqrt{z-2r}}{2}$ & radix poteſtatis (n) binomialis $A+B$ erit $\frac{\sqrt{z+2r} \pm \sqrt{z-2r}}{2\sqrt[n]{p}}$.

In prædictis æquationibus pro z ſcribatur ru, reſultat æquatio $u^n - nu^{n-2} + n \cdot \frac{n-3}{2} u^{n-4} - \&c. = 2 \times \frac{A^2+B^2}{A^2-B^2}$; exinde radix quæſita erit $\frac{\sqrt{u+2} \pm \sqrt{u-2}}{2} \times \sqrt[2n]{A^2-B^2}$.

Aliter: primo ita reducatur data quantitas, ut evadat $A^2 - B^2 = \pi^n$, inveniantur radices æquationis $z^n - n\pi z^{n-2} + n \cdot \frac{n-3}{2} \pi^2 z^{n-4} - \&c. = 2A$; erit $z = 2x$ & $y = \sqrt{\frac{z^2-4\pi}{4}}$; unde radix quæſita $x+y = \frac{z+\sqrt{z^2-4\pi}}{2}$.

PROB. LI.

1. *Datâ quantitate ex infinitis partibus conflatâ; invenire utrum ea per rationalem fractionem cujuſvis aſſumptæ quantitatis exprimi poteſt, necne.*

Dividatur data quantitas per aſſumptam quantitatem, ſi modo data quantitas major ſit quam aſſumpta; ſin aliter aſſumpta per datam; & per reliquum dividatur diviſor, & ſic per reliquum continuo dividatur diviſor; & ſi reliquum tandem evaneſcat, i. e. operatio haud in infinitum progrediatur; tum data quantitas erit rationalis fractio quantitatis aſſumptæ; ſin aliter vero non.

Ex. Invenire, utrum 1, 12 12 12 12 12 &c. in infinitum exprimi poteſt per rationalem fractionem integralium numerorum, necne.

Aſſumo numerum integralem (1), & divido datam majorem quantitatem 1, 12 12 12 &c. per minorem aſſumptam; operatio erit

1 | 1,

$$1 \,|\, 1,\ 12\ 12\ 12\ 12\ \&c.)\ 1$$
$$\quad 1$$
divisor

$$\text{reliquum},\ 12\ 12\ 12\ 12\ \&c.)\ 1\ \ (8$$
$$,96\ 96\ 96\ 96\ \&c.$$
2^dus divisor

$$2^{\text{dum}}\ \text{reliquum}\ ,03\ 03\ 03\ 03\ \&c.)\ ,12\ 12\ 12\ 12\ \&c.)\ 4$$
$$,12\ 12\ 12\ 12\ \&c.$$
$$0\quad 0\quad 0$$

unde ultimum reliquum evaneſcit, conſequenter data quantitas erit rationalis fractio.

Eadem methodus etiam applicari poteſt ad literales quantitates, &c. e. g. ſit ſeries $2 - \frac{5x}{3} - \frac{35x^2}{9} + \frac{40x^3}{27} + \frac{565x^4}{81} - \frac{35x^5}{243}$ &c.; invenire, annon ſit algebraica rationalis functio literæ x; dividatur unitas per datam ſeriem, & ſeries ipſa per proxime ſubſequens reſiduum; & ſic deinceps: i. e. ſecundum vulgarem methodum communes diviſores detegendi operatio erit, viz. $2 - \frac{5x}{3} - \frac{35x^2}{9} + \frac{40x^3}{27} + \frac{565x^4}{81} - \frac{35x^5}{243}$ &c.) 1 $\left(\frac{1}{2}\right.$

$$1 - \frac{5}{6}x - \frac{35}{18}x^2 + \frac{40}{54}x^3 + \frac{565}{162}x^4 - \&c.$$

$$\text{reſiduum}\quad \frac{5}{6}x + \frac{35}{18}x^2 - \frac{40}{54}x^3 - \frac{565}{162}x^4 + \&c.;\ \frac{5}{6}x + \frac{35}{18}x^2 - \frac{40}{54}x^3 - \frac{565}{162}x^4\ \&c.)\ 2 - \frac{5x}{3} - \frac{35x^2}{9} + \frac{40x^3}{27} + \frac{565x^4}{81} - \&c.\ \left(\frac{12}{5x} - \frac{38}{5}\right.$$

$$2 + \frac{14x}{3} - \frac{16x^2}{9} - \frac{226x^3}{27} + \&c.$$

$$-\frac{19}{3}x - \frac{19}{9}x^2 + \frac{266}{27}x^3 + \&c.$$

$$-\frac{19}{3}x - \frac{133}{9}x^2 + \frac{152}{27}x^3 + \&c.$$

$$\frac{114}{9}x^2 + \frac{114}{27}x^3\ \&c.)\ \frac{5}{6}x + \frac{35}{18}x^2\ \&c.\ \left(\frac{5}{76}x + \frac{5}{38}\right.$$

5

$$\frac{5}{6}x + \frac{5}{18}x^2 + \&c.$$

$$\frac{5}{3}x^2 + \&c.\ \&c.\ \&$$

exinde per coroll. 2. ſubſequentis theorematis erit $2 - \frac{5x}{3} - \frac{35x^2}{9}$ &c. $= \cfrac{1}{\frac{1}{2} + \cfrac{1}{\frac{12}{5x} - \frac{38}{5} + \cfrac{1}{\frac{5}{76x} + \frac{5}{38}}}} = \cfrac{1}{\frac{1}{2} + \cfrac{1}{\frac{12}{5x} - \frac{38}{5} + \frac{76x}{5 + 10x}}} =$ $\cfrac{1}{\frac{1}{2} + \frac{5x + 10x^2}{12 - 14x}} = \frac{12 - 14x}{6 - 2x + 10x^2} = \frac{6 - 7x}{3 - x + 5x^2}$.

Ex. 2. Sit fractio $\frac{a}{c} + \frac{cb - da}{c^2}x + \frac{d^2a - dcb}{c^3}x^2 + \frac{d^2cb - d^3a}{c^4}x^3 + \&c.$, tum operatio erit hujuſmodi $\frac{a}{c} + \frac{cb - da}{c^2}x + \frac{d^2a - dcb}{c^3}x^2 + \&c.)\ 1\ (\frac{c}{a}$

$$1 + \frac{cb - da}{ca}x + \frac{d^2a - dcb}{c^2a}x^2\ \&c.$$

$$- \frac{da - cb}{ca}x + \frac{dcb - d^2a}{c^2a}x^2 + \frac{d^3a - d^2cb}{c^3a}x^3\ \&c.: \frac{da - cb}{ca}x + \frac{dcb - d^2a}{c^2a}x^2 + \frac{d^3a - d^2cb}{c^3a}x^3\ \&c.)\ \frac{a}{c} + \frac{cb - da}{c^2}x + \frac{d^2a - dcb}{c^3}x^3\ \&c.\ (\frac{a^2}{da - cbx} +$$

$$\frac{a}{c} - \frac{da}{c^2}x + \frac{d^2a}{c^3}x^2 + \&c.$$

$$\frac{b}{c}x - \frac{db}{c^2}x^2\ \&c.$$

$$\frac{b}{c}x - \frac{db}{c^2}x^2\ \&c.$$

Hinc

Hinc quoniam reſiduum evaneſcit, i. e. terminat operatio; ſeries exprimi poteſt per rationalem fractionem quantitatis x; per corol. 2. theorem. ſubſequentis erit $\frac{a}{c}+\frac{cb-da}{c^2}x+\frac{d^2a-dcb}{c^3}x^2+$ &c. $=$

$$\frac{1}{\frac{c}{a}+\frac{1}{\frac{a^2}{(da-cb)x}+\frac{ab}{da-cb}}}=\frac{1}{\frac{c}{a}+\frac{da-cbx}{a^2+abx}}=\frac{a+bx}{c+dx}.$$

Si vero operatio non terminet, tum ſeries non exprimi poteſt per rationalem functionem quantitatis x.

2. Si modo data quantitas ſit numeralis fractio; tum ejus figuræ, quod facile conſtat ex diviſione, in infinitum ſemper redeunt eodem ordine; i. e. in primo exemplo fuerunt 12, 12, 12, 12, 12, &c. in infinitum; & vice versâ, ſi figuræ ſemper redeant eodem ordine, tum data quantitas numeralis erit fractio: ſit enim prima ſeries (A) quantitatum in prædicto exemplo (,12); & prima ſeries ,12 ducta in α (,01) in prædicto exemplo ſecundæ ſeriei (,0 0 12) æqualis; ſecunda ducta in α tertiæ æqualis; & ſic deinceps: ubi α in decimalibus quantitatibus vel ,1, vel ,01, vel 001, &c. ſemper denotat: data quantitas erit $1+A+\alpha A+\alpha^2 A+\alpha^3 A+\alpha^4 A+$ &c. in infinit. $=1+\frac{A}{1-\alpha}=1+\frac{,12}{1-,01}=\frac{111}{99}$.

Et ſic de omnibus hujuſce generis ſeriebus.

Cor. 2. Sit 47619 ordo in quo redit fractio decimalis; i. e. ſit fractio 47619 47619 47619 &c. in infinitum; dividatur unitas per 47619, & quotiens erit 210000 210000 21 &c. dividatur reciproce unitas per 21, & quotiens erit data fractio 476190476190 &c. Et ſic de quâcunque aliâ fractione.

Cor. 3. E quotientibus methodo in problemate acquiſitis inveniri poſſunt fractiones, quæ proxime eandem fractionem ita exhibeant; ut nulla fractio, cujus termini ipſis haud ſunt majores, propius accedat: ſit enim ſeries 1, 1546 1546 1546 &c. & operatio erit per problem.

1 | 1 ,1546 1546 &c.) 1
 | 1
,1546 1546 &c.) 1 (6
,9276 9276 &c.
,0723 0723 &c. | ,1546 1546 &c.) 2
 | ,1446 1446 &c.
 ,01 0001 0001 &c.

& ſic deinceps: & reſpectivæ quotientes invenientur (1, 6, 2, 7, 4, 2, 1, 7): tum quoniam numerator per denominatorem diviſus dedit primam quotientem 1, erit $\frac{1}{1}$ minor quam fractio quæſita; ſed quoniam denominator per primum reliquum diviſus dedit ſecundam quotientem 6, & conſequenter reliquum erit minus quam ſexta pars denominatoris, & fractio $\left(1+\frac{1}{6}=\frac{7}{6}\right)$ major quam vera: ſed quoniam primum reliquum per ſecundum diviſum dedit quotientem 2, & duæ proxime præcedentes fractiones fuerunt $\frac{1}{1}$ & $\frac{7}{6}$, erit $\frac{7\times 2+1}{6\times 2+1} = \frac{15}{13}$ proxime minor quam vera fractio; & quoniam 7 eſt ſubſequens quotiens $\frac{15\times 7+7}{13\times 7+6} = \frac{112}{97}$ erit fractio proxime major quam vera; & quoniam quotientes reſpective fuerunt 1, 6, 2, 7, 4, 2, 1, 7, fractiones erunt

$$\left(\frac{1}{1},\quad \frac{7\times2+1}{6\times2+1}\left(\frac{15}{13}\right)\quad \frac{112\times4+15}{97\times4+13}\left(\frac{463}{401}\right)\right.$$

$$\frac{6\times1+1}{6}\left(\frac{7}{6}\right),\quad \frac{15\times7+7}{13\times7+6}\left(\frac{112}{97}\right)\quad \frac{463\times2\cdot}{401\times2\cdot}$$

$$\left(\frac{1038\times1+463}{899\times1+401}\left(\frac{1501}{1300}\right)\right.$$

$$\frac{1501\times7+1038}{1300\times7+899}\left(\frac{11545}{9999}\right).$$

Superior columna fractiones verâ minores denotat, inferior vero verâ

verâ majores; ni ultima, quæ eſt vera fractio: inter has reſpectivas facile eſſet alias interponere: e. g. inter fractiones $\frac{c}{d}$ & $\frac{na+c}{nb+d}$ interponentur fractiones $\frac{a+c}{b+d}$, $\frac{2a+c}{2b+d}$, $\frac{3a+c}{3b+d}$, $\frac{4a+c}{4b+d}$, &c. . . . $\frac{\overline{n-1}a+c}{\overline{n-1}b+d}$.

Hanc methodum approximationum primus invenit Walliſius.

Sint a, b, c, d, e, f, g, &c. prædictæ ſucceſſivæ quotientes, & reſpectivæ fractiones erunt $\frac{a}{1}$, $\frac{ab+1}{b}$, $\frac{abc+a+c}{bc+1}$, $\frac{abcd+ab+cd+ad+1}{bcd+b+d}$, $\frac{abcde+abc+abe+ade+cde+a+c+e}{bcde+bc+de+be+1}$: & ſic deinceps. Numerator & denominator hujuſce ſeriei ſemper eandem obſervant legem; alter vero unam literam ſemper obtinet, qui non invenitur in altero, ea litera eſt prima quotiens a: hæc autem litera invenitur in numeratore vel denominatore, prout prima quotiens ſit quotiens e diviſione numeratoris per denominatorem, vel denominatoris per numeratorem.

In hâc ſerie primus terminus erit $abcdefg$ &c. & omnes reliqui ex hoc cognoſci poſſunt; 1mo auferendo continuo duas ſucceſſivas quotientes; e. g. a & b ſunt ſucceſſivæ quotientes, auferantur e primo termino a & b, reſtabit alter terminus quæſitus $cdefg$ &c. b & c ſunt ſucceſſivæ quotientes, auferantur hæ quotientes e primo termino, & reſtabit $adefg$ &c. tertius terminus quæſitus: c & d ſunt ſucceſſivæ quotientes, auferantur e primo termino c & d, & reſtabit quartus terminus quæſitus $abefg$ &c. & ſic deinceps: deinde auferantur e primo termino duæ ſucceſſivæ quotientes, & duæ aliæ ſucceſſivæ quotientes & reſtabunt alii termini; deinde auferantur duæ ſucceſſivæ quotientes ter, quater, &c. uſque donec vel ſolummodo una quotiens vel nulla (in quo caſu terminus erit 1) reſtat: aggregatum ex omnibus terminis, qui e prædictâ methodo formari poſſunt, erit ſeries quæſita.

 4. Sint

3. Sint duæ fractiones, quarum una habet α figuras in denominatore, β vero in numeratore; altera vero n in denominatore, m vero in numeratore; dividantur reſpectivi numeratores per ſuos denominatores, & ſit $n+m$ haud minor quam $\alpha+\beta$; haud poſſunt reſultare plures quam $n+m$ figuræ in quotiente eædem, ni utriuſque fractionis idem ſit valor: hinc fractio, quæ habet $n+m+1$ figuras in quotiente e numeratore per denominatorem diviſo reſultante eaſdem, quas habet data ſeries, propior vel æquidiſtans erit valori ſeriei, quam ulla alia fractio, quæ ſolummodo habet $n+m$ figuras in ejus numeratore & denominatore conjunctim contentas.

Sint numeri $(N$ & $M)$, in quibus contineantur (n) figuræ, quarum ſint m primæ eædem; & in r contineantur (s) figuræ; tum in (r) poteſtate numerorum $(N$ & $M)$ ad minimum erunt $m-s-1$ primæ figuræ eædem.

Methodo haud multum diſſimili inveniri poſſunt; annon duæ vel plures rationales fractiones ſunt, quæ in datas quantitates in infinitum progredientes ductæ, ſummam rectangulorum æqualem rationali fractioni producunt; & exinde inveſtigare licet, utrum data quantitas in infinitum progrediens exprimi poteſt per radices fractionum, necne: ſed hæc ad ſummationem ſerierum pertinent.

THEOR. XLV.

Sit fractio $\cfrac{\alpha}{a+\cfrac{\beta}{b+\cfrac{\gamma}{c+\cfrac{\delta}{d+\cfrac{\epsilon}{e+\cfrac{\zeta}{f+\eta, \&c.}}}}}}$

Reducatur hæc

Fractio in vulgarem

Fractionem; reſultant

reſpectivæ fractiones $\frac{\alpha}{a}$ major quam vera, $\cfrac{\alpha}{a+\cfrac{\beta}{b}}=\frac{b\alpha}{ba+\beta}$ minor

quam

quam vera, & ſic alternatim fractiones reſultantes majores & minores erunt quam vera: tertia vero invenitur $\frac{\alpha}{a+\frac{\beta}{b+\frac{\gamma}{c}}} = \alpha \times \frac{bc+\gamma}{abc+a\gamma+c\beta}$;

quarta vero $\alpha \times \frac{bcd+b\times\delta+d\times\gamma}{abcd+ab\times\delta+ad\times\gamma+dc\beta+\delta\beta}$; quinta vero $\alpha \times \frac{bcde+bc\times\varepsilon+be\times\delta+ed\gamma+\gamma\varepsilon}{abcde+abc\times\varepsilon+abe\times\delta+ade\times\gamma+cde\times\beta+a\times\gamma\varepsilon+e\times\beta\delta+c\beta\varepsilon}$, &c, Pro $\alpha, \beta, \gamma, \delta, \varepsilon$, &c. ſcribatur reſpective unitas, & eadem erit ſeries ac ea in coroll. 3. problem. præced. tradita. Lex autem, quam obſervant literæ $\alpha, \beta, \gamma, \delta, \varepsilon$, &c. ſic enunciari poteſt. Scribantur termini duarum ſerierum $a, b, c, d, e, f, \ldots l, m$, &c.
$\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \ldots \lambda, \mu$, &c. infra ſe invicem reſpective; & duæ ſucceſſivæ literæ prædictæ ſeriei (ut conſtat e præcedente problem.) ſemper ſimul evaneſcent; i. e. vel a & b, vel b & c, vel c & d, vel d & e, &c. pro ſingulis duabus ſucceſſivis literis deſtructis (l & m) prædictæ ſeriei ſcribatur unica litera (μ) inferioris ordinis infra poſteriorem literam (m) deſtructam poſita: e. g. in quintâ fractione primus terminus numeratoris eſt $bcde$, ſecundus vero $bc \times \varepsilon$; in hoc termino deſtruuntur duæ ſucceſſivæ literæ d & e; ergo in hoc termino pro literis deſtructis d & e ſcribenda eſt litera ε, quæ infra poſteriorem literam (e) ſuperioris ſeriei (a, b, c, d, &c.) deſtructam in poſteriori ſerie ($\alpha, \beta, \gamma, \delta$, &c.) ponitur.

Hæ fractiones etiam ex hâc methodo inveniri poſſunt: ſcribantur reſpective infra ſe

$$\begin{array}{llllll} a,\ b, & c, & d, & e, & & \text{\&c.} \\ 0,\ \frac{\alpha}{a}, & \frac{b\alpha}{ab+\beta}, & \frac{bc\alpha+\gamma\alpha}{abc+a\gamma+c\beta}, & \frac{bcd\alpha+b\delta\alpha+d\gamma\alpha}{abcd+ab\delta+ad\gamma+cd\beta+\beta\delta}, & & \text{\&c.} \\ \alpha,\ \beta,\ \gamma, & \delta, & \varepsilon, & \zeta, & & \text{\&c.} \end{array}$$

& cujuſque fractionis numerator per literam ſupra ſcriptam multiplicatus una cum numeratore præcedentis fractionis per ſuam infra ſcriptam literam multiplicato præbet numeratorem ſequentis fractionis:

nis: atque eodem modo cujuſque fractionis denominator per literam ſuam ſupra poſitam multiplicatus una cum denominatore præcedentis fractionis per literam ſuam infra ſcriptam multiplicato præbet denominatorem fractionis ſequentis.

Cor. 1. Inveniantur prædictarum fractionum differentiæ; i. e. ſubtrahatur quæque fractio e præcedente; reſultabunt reſpectivæ fractiones $\frac{\alpha}{a}$, $\frac{\beta\alpha}{a\times(ab+\beta)}$, $\frac{\alpha\beta\gamma}{(ab+\beta)\times(abc+a\gamma+c\beta)}$, &c. e quibus facile conſtabit lex, quam obſervant ſeries, quæ has differentias exprimunt.

Cor. 2. Sit fractio $\frac{A}{B}$ in continuam fractionem tranſmutanda: dividatur A per B, ſitque quotus a & reſiduum C; per hoc reſiduum (C) dividatur præcedens diviſor, & ſit quotus b & reſiduum D; i. e. ſit operatio

$$\begin{array}{l} B|A|a \\ \quad C|B|b \\ \qquad D|C|c \\ \qquad\quad E|D|d \\ \qquad\qquad \&c. \end{array}$$

tum erit $a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e+1\ \&c.}}}}=\frac{A}{B}$, ejus vero reciproca $\frac{B}{A}=\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\ \&c.}}}}$

Hanc methodum primum invenit Vice comes Brounker: cum vero hæ ſeries in infinitum excurrant, ad tractatum de hiſce ſeriebus ſunt referendæ.

PROB. LII.

Datâ æquatione $ax+by+cz+dv+\&c.+f=0$, *in quâ incognitæ quantitates* x, y, z, v, &c. *unam ſolummodo habent dimenſionem*; & a, b, c, d, &c. f *integri ſunt numeri*; *invenire integros earum correſpondentes valores.*

1. Sit coefficiens (a) termini (ax) minima, quæ invenitur in datâ æquatione; dividantur reliqui termini by, cz, dv, &c. f per hanc coefficientem

efficientem a; & ſit reſiduum $b'y$ e diviſione termini (by) per coefficientem a; reſiduum $c'z$ e diviſione termini (cz) per coefficientem a; & ſic de reliquis; i. e. ſit aggregatum reſiduorum $b'y + c'z + d'v +$ &c. $+ f'$: aſſumatur nova æquatio $au + b'y + c'z + d'v +$ &c. $+ f' = 0$; ſit b' minima coefficiens, quæ invenitur in novâ æquatione; dividantur reliqui au, $c'z$, $d'v$, &c. f' per hanc coefficientem b', & ſint reſidua e diviſione terminorum au, $c'z$, $d'v$, &c. per hanc coefficientem reſpective $a'u$, $c''z$, $d''v$, &c. & eorum aggregatum $a'u + c''z + d''v +$ &c. $+ f''$; aſſumatur nova æquatio $b'\phi + a'u + c''z + d''v +$ &c. $+ f'' = 0$: & ita repetatur hæc operatio, uſque donec reſiduum nihilo ſit æquale: deinde pro reſpectivis ultimæ æquationis quantitatibus aſſumantur quicunque integri numeri, & ope æquationum hinc deductarum invenientur incognitarum quantitatum (x, y, z, v) correſpondentes integri valores.

Hæc reſolutio melius intelligi poteſt per exempla.

Ex. 1. Sit $16y + 20 = 7x$, invenire integros quantitatum $(x$ & $v)$ correſpondentes valores.

dividatur $16y + 20$ per 7 minorem coefficientem, & operatio erit

$$7 \,|\, 16y + 20 \,|\, 2y + 2$$
$$\underline{14y + 14}$$

reſiduum erit $2y + 6$: ſupponatur $2y + 6 = 7v$: dividatur hæc nova æquatio per minorem coefficientem 2, & operatio eſt $2 \,|\, 7v - 6$

$$\underline{6v - 6}$$

reſiduum $= v$:

aſſumatur nova æquatio $v = 2w$; & e ſecundâ æquatione $2y + 6 = 7v = 14w$, & exinde $y = 7w - 3$; e vero datâ æquatione $16y + 20 = 112w - 28 = 7x$ ſequitur $x = 16w - 4$; unde correſpondentes valores quantitatum $(x$ & $y)$ erunt $16w - 4$ & $7w - 3$ reſpective, ubi w eſt quicunque integer numerus.

Ex. 2.

Ex. 2. Sit $8x + 12y + 20 = 15v$: dividatur ea per minimam coefficientem 8, & operatio erit $8|12y + 20 - 15v|y + 2 - v$

$$\underline{8y + 16 - 8v}$$

reſiduum $4y + 4 - 7v$; aſſumatur $8z + 4y + 4 = 7v$, quæ eſt ſecunda æquatio: dividatur ea per 4 minimam coefficientem, & operatio erit $4|8z + 4 - 7v|2z + 1 - v$

$$\underline{8z + 4 - 4v}$$

reſiduum $-3v$: aſſumatur $4w = 3v$, quæ eſt tertia æquatio: dividatur ea per 3 minorem coefficientem, & reſultat $3|4w|w$:

$$\underline{3w}$$
$$1w$$

aſſumatur $1w = 3u$ quarta æquatio; & exinde e tertiâ æquatione $4w = 12u = 3v$, & $v = 4u$; e ſecundâ vero $8z + 4y + 4 = 7v = 28u$, & exinde $y = 7u - 1 - 2z$; e primâ i. e. datâ æquatione $8x + 12y + 20 = 8x + 84u - 24z - 12 + 20 = 15v = 60u$; unde $x = 3z - 1 - 3u$, & correſpondens valor incognitæ quantitatis $y = 7u - 2z - 1$, & quantitatis $v = 4u$; & pro z & u aſſumi poſſunt quicunque integri numeri.

2. Sit $px' + qy' + rz' +$ &c. $= l$; & ſi $\frac{l}{p} = \alpha$ & α, p, q, r, &c. & l, m, n, &c. ſint integri numeri; tum erunt $\alpha - mq - nr -$ &c., mp, np, &c. integri correſpondentes valores quantitatum x', y', z', &c. reſpective: hinc ſi in reductione prius traditâ perventum ſit ad æquationem, in quâ l dividi poteſt per p; tum unus integer correſpondens valor e ſingulis incognitis quantitatibus $(x, y, z,$ &c.$)$ ſine ulteriori proceſſu exinde erui poteſt.

Cor. 1. Sit (m) numerus incognitarum quantitatum in datâ æquatione contentarum; $(m - 1)$ quantitates plerumque reſultant, pro quibus aſſumi poſſunt quicunque integri numeri.

Cor. 2. Sit $ax + by + cz + dv +$ &c. $= A$; & ſi datæ coefficientes a, b, c, d, &c. communem habeant diviſorem, qui haud etiam

dividit numerum A, tum nullos integros valores incognitarum quantitatum x, y, z, v, &c. correſpondentes admittet data æquatio.

Cor. 3. Datis limitibus, inter quos conſiſtunt quæcunque incognitæ quantitates; ſubſtituantur hi limites pro ſuis quantitatibus in datâ & reſultantibus æquationibus; exinde conſtabunt omnes reſpectivi limites, qui ex datis ſequuntur: e. g. ſi requiratur, ut quæcunque quantitas ſemper ſit affirmativus & integer numerus; minimus autem integer numerus eſt 1; ergo ſcribatur 1 pro eâ quantitate & in datâ & reſultante æquatione; & exinde conſtabunt limites, quos præbet prædictus limes 1. Hinc etiam conſtabit numerus valorum inter quoſcunque duos limites contentorum.

Cor. 4. Sit æquatio $ax + by = P$, & coefficientium a & b ſit π maximus diviſor, & ſint p & q correſpondentes valores incognitarum quantitatum x & y: correſpondentes valores incognitarum quantitatum x & y erunt reſpective

$$\left.\begin{matrix} p,\ p \pm \frac{b}{\pi},\ p \pm \frac{2b}{\pi},\ p \pm \frac{3b}{\pi},\ p \pm \frac{4b}{\pi}, \\ q,\ q \mp \frac{a}{\pi},\ q \mp \frac{2a}{\pi},\ q \mp \frac{3a}{\pi},\ q \mp \frac{4a}{\pi}, \end{matrix}\right\} \text{\&c.}$$

Dato igitur correſpondente valore e ſingulis incognitis quantitatibus x & y, facile deduci poſſunt omnes ejuſdem generis valores.

Unus valor vel e priori methodo inveniri poteſt; vel ſi modo a & b nullum accipiant communem diviſorem, & maxima coefficiens (a) per minimam b dividatur, & diviſor per reſiduum, & ſic deinceps; uſque donec ad aliquod reſiduum (π) perveniatur, quod dividit numerum P: & ſint $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$, &c. reſpectivæ quotientes ex hujuſmodi diviſione, & ſcribantur

$$\begin{matrix} \alpha, & \beta, & \gamma, & \delta, & \varepsilon, & \zeta, & \text{\&c.} \\ +1, & -\alpha, & \beta A + 1, & -\gamma B - A, & \delta C + B, & -\varepsilon D - C, & \zeta E + D, \text{ \&c.} \end{matrix}$$

ubi A, B, C, D, E, &c. præcedentes terminos affirmative ſumptos reſpective denotant: multiplicetur ultimus terminus inferioris columnæ per $\frac{P}{\pi}$, productum erit valor quantitatis y quæſitus: ſi vero minor

 valor

valor quæſitus ſit, dividatur prædictum productum per quantitatem a & reſiduum erit æqualis vel minor valor quantitatis y.

Ex. Sit $256x - 87y = -50$; & dividatur 256 per 87, & diviſor 87 per reſiduum, & ſic deinceps; primum reſiduum, quod dividit P (50), erit 5; & duæ invenientur correſpondentes quotientes 2 & 1; hinc $\overset{2,}{1}, \overset{1,}{-2}, 2\times 1+1 = 3$; unde valor quantitatis (y) ſic inventus erit $3\times\frac{P}{\pi} = \pm 3\times\frac{50}{5} = \pm 30$; erit $+30$, ſi P ſit -50; ſin aliter, erit -30: ſecundum reſiduum, quod dividit P, erit 2; tres invenientur ejus correſpondentes quotientes, viz. 2, 1 & 16; hinc $\overset{2,}{1}, \overset{1,}{-2}, \overset{16,}{2\times 1+1=3}, -(16\times 3+2) = -50$; unde ſecundus valor quantitatis y per hanc methodum inventus erit $50\times\frac{P}{2} = 50\times\frac{50}{2} = 1250$; qui erit negativus, ſi præcedens valor inventus (30) ſit affirmativus (nam quantitates inventæ 3 & -50 diverſa habent ſigna); & vice verſâ erit affirmativus, ſi præcedens valor (30) ſit negativus: tertium vel ultimum reſiduum, quod dividit 50, erit 1; & reſpectivæ quotientes invenientur 2, 1, 16 & 2; hinc $\overset{2,}{1}, \overset{1,}{-2}, 2\times 1+1 = \overset{16}{3}, \overset{2,}{-(16\times 3+2) = -50}, 50\times 2+3 = 103$; & exinde ducatur 103 in $P = -50$, reſultat valor incognitæ quantitatis $y = 50\times 103 = 5150$, & ejus correſpondens valor quantitatis $x = 1750$: minimi vero affirmativi integri valores quantitatum y & x invenientur 30 & 10 reſpective.

4. 2. Quantitates A, B, C, D, E, &c. erunt reſpective α, $\alpha\beta+1$, $\alpha\beta\gamma+\alpha+\gamma$, $\alpha\beta\gamma\delta+\alpha\beta+\alpha\delta+\gamma\delta+1$, $\alpha\beta\gamma\delta\epsilon+\alpha\beta\gamma+\alpha\beta\epsilon+\alpha\delta\epsilon+\gamma\delta\epsilon+\alpha+\gamma+\epsilon$, &c. cujus lex eadem erit ac ea quæ in cor. 3. prob. 51. docetur.

4. 3. Sint $\overset{\beta,}{1}, \overset{\gamma,}{-\beta}, \overset{\delta,}{\gamma H+1}, \overset{\epsilon,}{-\delta I-H}, \overset{\zeta,}{\epsilon K+I}, \overset{\&c.}{\zeta L+K}$, &c. ubi

$H, I, K,$

H, I, K, L, &c. præcedentes terminos affirmative ſumptos reſpective denotant; multiplicetur ultimus terminus inferioris columnæ per $\frac{P}{\pi}$, productum erit valor quantitatis x quæſitus, correſpondens valori quantitatis y prius invento; e. g. in præcedente exemplo fuerunt β, γ & δ reſpective 1, 16 & 2; unde

$$\begin{matrix} 1, & 16, & & 2 \\ 1, & -1, & 16\times 1+1=17, & -2\times 17-1=-35 \end{matrix}$$

& conſequenter $-1\times\frac{-50}{5}=10$, $17\times\frac{-50}{2}=-425$, $-35\times-50=1750$ erunt tres valores quantitatis (x), qui reſpondent tribus valoribus 30, 1250 & 5150 quantitatis (y) prius inventis.

Cor. 5. Sint c, d, e, f, g, &c. reſpectiva reſidua e diviſione coefficientis a per b, diviſoris b per primum reſiduum, primi reſidui per ſecundum, & ſic deinceps; uſque ad reſiduum 1; tum unus valor quantitatis y erit $=\pm aP\times\left(\frac{1}{ab}-\frac{1}{bc}+\frac{1}{cd}-\frac{1}{de}+\frac{1}{ef}-\&c.\right)$ eouſque hâc ſerie continuatâ, donec quantitas P per factorem aliquem denominatoris dividi poteſt.

Ejus correſpondens valor quantitatis (x) erit $=\pm bP\times\left(\frac{1}{bc}-\frac{1}{cd}+\frac{1}{de}-\frac{1}{ef}+\&c.\right)$ hâc ſerie ad eundem terminum ac in præcedente valore quantitatis (y) continuatâ.

Erit enim $ax+by=abP\times\left(\frac{1}{ab}-\frac{1}{bc}+\frac{1}{cd}-\frac{1}{de}+\frac{1}{ef}-\&c.\right)$ $+abP\times\left(\frac{1}{bc}-\frac{1}{cd}+\frac{1}{de}-\frac{1}{ef}+\&c.\right)=abP\times\left(\frac{1}{ab}\right)=P$; unde, ſi prima ſeries ſit valor quantitatis (y), ſecunda ſeries erit valor quantitatis x: prædictum valorem quantitatis (y) eſſe integrum numerum ſic probari poteſt; in prædictâ quantitate $ab\left(\frac{1}{ab}-\frac{1}{bc}+\right.$

$\frac{1}{cd} - \frac{1}{de} + \frac{1}{ef} -$ &c.$)$ pro $\frac{a-c}{b}$, $\frac{b-d}{c}$, $\frac{c-e}{d}$, $\frac{d-f}{e}$, &c. ſcribantur reſpective quotientes α, β, γ, δ, &c.; unde $ab\left(\frac{1}{ab}\right)$, $ac \times \left(\frac{1}{ab} - \frac{1}{bc}\right)$, $ad \times \left(\frac{1}{ab} - \frac{1}{bc} + \frac{1}{cd}\right)$, $ae \times \left(\frac{1}{ab} - \frac{1}{bc} + \frac{1}{cd} - \frac{1}{de}\right)$, $af \times \left(\frac{1}{ab} - \frac{1}{bc} + \frac{1}{cd} - \frac{1}{de} + \frac{1}{ef}\right)$, &c. erunt reſpective prædictæ quantitates 1, $-\alpha$, $\beta A + 1$, $-\gamma B - A$, $\delta C + B$, &c.: & ſic de valoribus quantitatis (x) inveſtigandis.

6. Sint $\alpha, \beta, \gamma, \delta, \varepsilon$, &c. quotientes prædictæ e diviſione quantitatis a per b, & diviſoris b per reſiduum c, & ſic deinceps: tum erunt reſpectiva reſidua $(c, d, e, f, g,$ &c.$)$ $c = a - b\alpha$, $d = b(\alpha\beta + 1) - a(\beta)$, $e = a(\beta\gamma + 1) - b(\alpha\beta\gamma + \alpha + \gamma)$, $f = b(\alpha\beta\gamma\delta + \alpha\beta + \alpha\delta + \gamma\delta + 1) - a(\beta\gamma\delta + \beta + \delta)$; in genere erit $\pm b(\alpha\beta\gamma\delta\varepsilon\zeta$ &c. $+ \gamma\delta\varepsilon\zeta$ &c. $+ \alpha\delta\varepsilon\zeta$ &c. $+ \alpha\beta\varepsilon\zeta$ &c. $+ \alpha\beta\gamma\zeta$ &c. $+ \alpha\beta\gamma\delta$ &c. $+$ &c. $+ \alpha\beta$ &c. $+ \gamma\delta$ &c. $+ \varepsilon\zeta$ &c. $+$ &c. $+$ &c.$)$ $\mp a(\beta\gamma\delta\varepsilon\zeta$ &c. $+ \delta\varepsilon\zeta$ &c. $+ \beta\varepsilon\zeta$ &c. $+ \beta\gamma\zeta$ &c. $+ \beta\gamma\delta$ &c. $+ \beta$ &c. $+ \delta$ &c. $+ \zeta$ &c. $+$ &c.$)$: hi duo termini obſervant legem in cor. 3. prob. 51. datam; at poſterior terminus in a ductus haud involvit primam quotientem α.

P R O B. LIII.

Sint duæ æquationes vel tres vel plures (m) *incognitas quantitates habentes, quæ unam ſolummodo habent dimenſionem; eas in unam transformare, ita ut exterminetur una incognita quantitas* (x); *&, ſi modo incognitæ quantitates reſultantis æquationis ſint integri numeri, exterminata quantitas* (x) *etiam ſit integer numerus.*

1. Sint duæ æquationes $\alpha x + \beta y +$ &c. $= A$

& $\pi x + \varrho y +$ &c. $= B$

aſſumatur æquatio $\alpha m - \pi n = 1$, & per problema præcedens inveniantur

niantur integri valores quantitatum m & n, qui ſint r & s reſpective; ducatur prior data æquatio in r, poſterior vero in s, reſultant $r\alpha x + r\beta y +$ &c. $= rA$, & $s\pi x + s\varrho y +$ &c. $= sB$, quarum differentia erit $x + \overline{r\beta - s\varrho}\, y +$ &c. $= rA - sB$, exinde $x = rA - sB + (s\varrho - r\beta) \times y$; ſubſtituatur hæc quantitas pro ſuo valore (x), & conficietur problema.

Ex. Sint duæ æquationes $7x + 6y + 4z = 12$ & $5x + 10y + 12z = 20$; eas transformare in unam, ita ut exterminetur incognita quantitas (x); &c.: aſſumatur æquatio $7m - 5n = 1$; 3 & 4 erunt integri correſpondentes valores incognitarum quantitatum m & n; ducatur prior data æquatio in 3, poſterior vero in 4; reſultant duæ æquationes $21x + 18y + 12z = 36$ & $20x + 40y + 48z = 80$, quarum differentia erit $x - 22y - 36z = -44$, & exinde $x = 22y + 36z - 44$; quo valore quantitatis (x) in primâ æquatione ſubſtituto, reſultat $160y + 256z - 308 = 12$, i. e. $5y + 8z = 10$.

2. Si quantitates α & π communem habeant diviſorem; tum forſan e præcedenti methodo minime reſolvi poteſt problema.

In hoc caſu inveniantur ex alterâ æquatione integri valores ſingulæ incognitæ quantitatis per præcedens problema; quibus valoribus pro ſuis quantitatibus in alterâ æquatione ſubſtitutis, reſultat æquatio quæſita; e quâ inveniri poſſunt correſpondentes valores ſingularum incognitarum quantitatum.

Ex. Sint duæ æquationes $6x + 20y - 8z + 9v = 33$
& $9x + 14y + 12z - 21v = 99$
eas in unam transformare, ita ut exterminetur quantitas x; &c.

E priori æquatione per methodum prius traditam inveniantur integri correſpondentes valores quantitatum x, y, z, v; hi autem erunt $v = 1 - 2w$; $y = 3w - 3u + z$, $x = 10u - 7w - 2z + 4$; ubi u, z & w quoſcunque integros numeros aſſumptos reſpective denotant; quibus valoribus pro ſuis reſpectivis quantitatibus in ſecundâ æquatione ſubſtitutis, reſultat $21w + 48u + 8z = 84$; & transformantur duæ æquationes in unam, ita ut exterminetur incognita quantitas (x); &c.

Cor.

Cor. E reſultanti æquatione inveniatur $w = -4 + 8a$; & $z = 21 - 21.a - 6u$, ubi a & u quoſlibet numeros denotant; & exinde $v = 9 - 16a$, $y = 3a - 9u + 9$ & $x = 22u - 14a - 10$.

PROB. LIV.

1. *Invenire æquationem duas incognitas quantitates* (x & y) *habentem, cujus correſpondentes valores incognitarum quantitatum* (x & y) *ſunt reſpective* $x = Az^n + Bz^{n-1} + Cz^{n-2} + \&c.$ & $y = az^\alpha + bz^{\alpha-1} + cz^{\alpha-2} + \&c.$ *ubi* n *æqualis vel major eſt quam* α.

Aſſumatur æquatio $Py^m + \overline{p + qx}\,y^{m-1} + \overline{r + sx + tx^2}\,y^{m-2} + \&c. = o$, ita ut $\left(m \,.\, \frac{m+3}{2}\right)$ æqualis vel major ſit quam $(nm + 1)$: in aſſumptâ æquatione pro incognitis quantitatibus $(x$ & $y)$ ſubſtituantur earum reſpectivi valores $Az^n + Bz^{n-1} + Cz^{n-2} + \&c.$ & $az^\alpha + bz^{\alpha-1} + cz^{\alpha-2} + \&c.$ æquationis $(n \times m)$ dimenſionum reſultantis terminorum ſingulorum coefficientes nihilo æquales fiant: & ſit $\left(m \times \frac{m+3}{2}\right)$ major quam $nm + 1$, tum infinitæ deduci poſſunt hujuſce generis æquationes; ſit æqualis, tum una ſolummodo deduci poteſt æquatio, in quâ incognitarum quantitatum $(x$ & $y)$ correſpondentes valores ſunt datæ quantitates.

Cor. Aſſumi poſſunt infinitæ diverſæ æquationes; quæ hoc problema reſolvunt: tot vel plures incognitas coefficientes ſolummodo habere debet aſſumpta æquatio, quot dimenſiones ſunt æquationis reſultantis e ſubſtitutione correſpondentium valorum incognitarum quantitatum $(x$ & $y)$ in aſſumptâ æquatione pro quantitatibus ipſis unitate auctæ.

2. Eodem plane modo inveniri poſſunt æquationes duas incognitas quantitates $(x$ & $y)$ habentes, quarum correſpondentes valores incognitarum quantitatum $x = \frac{az^n + bz^{n-1} + cz^{n-2} + \&c.}{\acute{a}z^r + \acute{b}z^{r-1} + \acute{c}z^{r-2} + \&c.}$ & $y =$

$Az^m +$

$\dfrac{Az^m + Bz^{m-1} + Cz^{m-2} + \&c.}{\overset{1}{A}z^s + \overset{1}{B}z^{s-1} + \overset{1}{C}z^{s-2} + \&c.}$. Et ſic de infinitis conſimilibus.

Cor. 1. Hinc facile inveniri poſſunt infinitæ æquationes duas incognitas quantitates $(x \,\&\, y)$ habentes; quarum $(x \,\&\, y)$ correſpondentes valores generaliter per quantitates quarumlibet formularum, in quibus ſolummodo una continetur incognita quantitas (z), exprimi poſſunt.

Cor. 2. Sint $(n - 1)$ incognitæ quantitates in datis correſpondentibus valoribus e ſingulis incognitis quantitatibus $(x, y, \&c.)$ contentæ: aſſumatur æquatio (n) incognitas quantitates $(x, y, \&c.)$ involvens; ita ut, ſi modo ſubſtituantur in aſſumptâ æquatione pro ſingulis incognitis quantitatibus earum dati correſpondentes valores, evaneſcant ſinguli reſultantis æquationis termini: tales autem æquationes plerumque aſſumi poſſunt, ſi modo numerus incognitarum coefficientium aſſumptæ æquationis æqualis vel major ſit quam numerus terminorum reſultantis æquationis.

Et ſic de pluribus æquationibus deducendis, in quibus incognitæ quantitates contentæ ſunt datæ quantitates.

PROB. LV.

Datâ æquatione duas incognitas quantitates (x & y) *habente; invenire rationales correſpondentes valores incognitarum quantitatum* (x & y), *ſi modo hi correſpondentes valores per algebraicam rationalem functionem aſſumptæ incognitæ quantitatis* (z) *generaliter exprimi poſſint.*

Sit æquatio $Py^n + \overline{p + qx}\, y^{n-1} + \overline{r + sx + tx^2}\, y^{n-2} + \&c. = 0$; pro incognitis quantitatibus $(x \,\&\, y)$ ſubſtituantur reſpective $x = Az^n + Bz^{n-1} + Cz^{n-2} + \&c.$ & $y = az^n + bz^{n-1} + cz^{n-2} + \&c.$ ſi modo integrales valores incognitarum quantitatum $(x \,\&\, y)$ requirantur; vel $x = \dfrac{Az^n + Bz^{n-1} + Cz^{n-2} + \&c.}{Pz^n + Qz^{n-1} + Rz^{n-2} + \&c.}$, & $y =$

$az^n +$

$\frac{az^n + bz^{n-1} + cz^{n-2} + \&c.}{Pz^s + \mathfrak{Q}z^{s-1} + Rz^{s-2} + \&c.}$ ſi ſolummodo rationales incognitarum quantitatum $(x \,\&\, y)$ valores quærantur, in datâ æquatione pro ſuis valoribus; æquationis reſultantis fiant ſinguli termini nihilo reſpective æquales, & ex æquationibus reſultantibus inveniri poſſunt valores incognitarum coefficientium quæſiti.

Ex. Sit æquatio $4y^2 - 4xy + x^2 + 7y - 4x - 4 = 0$; ſcribantur pro incognitis quantitatibus $(x \,\&\, y)$ reſpective $y = z^2 + Bz + C$, & $x = az^2 + bz + c$; reſultat æquatio $(4 - 4a + a^2) \times z^4 + (8B - 4aB - 4b + 2ab) \times z^3 + (8C - 4c - 4aC + 2ac + 4B^2 - 4Bb + b^2 + 7 - 4a) \times z^2 + (8BC - 4bC - 4Bc + 2cb + 7B - 4b) \times z + 4C^2 - 4Cc + c^2 + 7C - 4c - 4 = 0$. Fiant coefficientes reſultantis æquationis nihilo reſpective æquales, & invenientur coefficientium (a, b, c, B, C) valores reſpective (2, 11, 1, 6, 1).

In hoc caſu cum evaneſcat primus terminus z^4, evaneſcet etiam ſecundus z^3; & exinde plures reſultant incognitæ quantitates quam æquationes; & infinitas diverſas hujuſmodi reſolutiones admittit data æquatio.

Et ſimiliter ratiocinari licet de rationalibus valoribus incognitarum quantitatum inveniendis.

Et ſic aſſumi poſſunt $x = Az^r + Bz^{r-1} + \&c.$, & $y = Az^s + Bx^{s-1} + \&c.$ vel $x = \frac{Az^r + Bz^{r-1} + \&c.}{Pz^s + \mathfrak{Q}z^{s-1} + \&c.}$, & $y = \frac{az^l + bz^{l-1} + \&c.}{pz^m + qz^{m-1} + \&c.}$

Cor. Sit data æquatio $py^n + \overline{a + bx}\, y^{n-1} + \overline{c + dx + ex^2}\, y^{n-2} + \overline{kx^3 + \&c.}\, y^{n-3} + \&c. = 0$; ſi termini $py^n + bxy^{n-1} + ex^2y^{n-2} + \&c.$ in quibus maximæ inveniuntur dimenſiones, nullum ſimplicem rationalem admittant diviſorem; tum minime correſpondentes incognitarum quantitatum $(x \,\&\, y)$ valores generaliter exprimi poſſunt per algebraicam integralem vel rationalem functionem aſſumptæ quantitatis (z).

Si admittant ſimplicem rationalem diviſorem prædicti termini, qui

ſit $y - \alpha x$; & aſſumatur $x = az^n + bz^{n-1} +$ &c. tum erit $y = a\alpha z^n$ + &c. Si vero pro x & y aſſumantur reſpective $\frac{az^r + bz^{r-1} + \&c.}{pz^s + qz^{s-1} + \&c.}$, & $\frac{Az^r + Bz^{r-1} + \&c.}{Pz^s + Qz^{s-1} + \&c.}$; tum erit $\frac{A}{P} = \frac{a}{p} \times \alpha$.

Cum vero unus ſolummodo inveniatur terminus in datâ æquatione, in quo maximæ inveniuntur dimenſiones; tum e methodo convergentes ſeries inveniendi deduci poteſt relatio inter terminos quantitatis (z), in quibus maximæ inveniuntur ejus dimenſiones, e. g. ſit $x = az^n + bz^{n-1} +$ &c. & $y = Az^m + Bz^{m-1} +$ &c. e methodo convergentes ſeries inveniendi conſtabit relatio inter A & a, n & m.

Et ſic de relatione inter terminos, in quibus maximæ inveniuntur dimenſiones quantitatis (z), detegendâ, cum rationales ſolummodo requirantur valores incognitarum quantitatum $(x$ & $y)$.

Eadem etiam applicari poſſunt ad (m) æquationes $(m + 1$ vel plures) incognitas quantitates habentes.

PROB. LVI.

Datis duabus æquationibus tres vel plures incognitas quantitates (x, y, v, z, &c.) *habentibus; eas in unam reducere, ita ut exterminetur incognita quantitas* (x); *&, ſi incognitæ quantitates in reſultanti æquatione ſint rationales, exterminata etiam ſit rationalis quantitas.*

Reducantur datæ æquationes per methodum primam in prob. 32. traditam; &, ſi tandem perventum ſit ad æquationem $Ax = B$, ubi A & B reſpective denotant quaſcunque rationales functiones ſingularum incognitarum quantitatum $(y, z, v,$ &c.$)$ præter exterminandam (x), ſubſtituatur $\frac{B}{A}$ pro x in alterâ datâ æquatione; reducentur duæ æquationes in unam, ita ut exterminetur incognita quantitas (x); &, ſi reliquæ incognitæ quantitates $(y, z, v,$ &c.$)$ ſint rationales, exterminata x etiam ſit rationalis quantitas.

2. Si vero æquationes per prædictam methodum reductæ nunquam

 per-

perveniant ad formulam $Ax = B$; ſed tandem perveniant ad duas reſultantes æquationes $Ax^2 + Bx = C$, & $Ax^2 + Bx = D$; ubi A, B, C, D reſpective denotant quaſcunque rationales functiones incognitarum quantitatum $(y, v, z,$ &c.$)$ in datâ æquatione contentarum: in hoc caſu omnis valor ſingularum incognitarum quantitatum $(y, v, z,$ &c.$)$ habet duos valores incognitæ quantitatis (x) ſibi ipſi reſpondentes: ſi vero radix quadratica quantitatis $B^2 + 4AC$ vel $B^2 + 4AD$ haud extrahi poſſit, tum exterminetur quantitas $(y$ vel v, vel z, &c.$)$, quæ non habet duos valores omni reliquarum incognitarum quantitatum valori reſpondentes; ſi talis in datis æquationibus inveniatur quantitas (v), tum tandem ad æquationem $Av = B$ pervenire liceat: ſin haud talis incognita quantitas in datis æquationibus inveniatur; tum nunquam per prædictam methodum reduci poſſunt duæ æquationes in unam, ita ut exterminetur incognita quantitas; &, ſi reſultantis æquationis incognitæ quantitates ſint rationales, exterminata etiam ſit rationalis.

3. Si vero per præcedentem methodum haud reduci poſſint $(n+1)$ æquationes in $(n$ vel pauciores$)$ æquationes: inveniantur una vel plures incognitæ quantitates in rationalibus terminis reliquarum, ſi modo poſſibile ſit; ſubſtituantur hic valor vel hi valores inventi pro ſuis incognitis quantitatibus, & reducuntur $(n+1)$ æquationes in n vel pauciores, &c.

Si vero duæ vel (n) incognitæ quantitates inveniri poſſint rationalibus terminis reliquarum & novæ incognitæ quantitatis aſſumptæ; vel etiam $(n-1)$ aſſumptarum incognitarum quantitatum; tum exinde reduci poſſunt datæ æquationes in pauciores, &c.

Cor. Sint $n+1$ datæ æquationes (n) incognitas quantitates habentes, quæ ſolummodo in (n) æquationibus unam habent dimenſionem: inveniantur ex (n) æquationibus valores (n) incognitarum quantitatum, & in unâ reliquâ æquatione pro ſuis valoribus reſpective ſubſtitutantur; reducentur hæ $n+1$ æquationes in unam, ita ut exterminentur (n) incognitæ quantitates; &, ſi incognitæ quantitates reſultantis æquationis ſint rationales, exterminatæ etiam ſint rationales.

Viſum

Viſum eſt hoc in loco quaſdam regulas adjicere de quibuſdam diverſis methodis rationales valores incognitarum quantitatum in datâ æquatione contentarum correſpondentes detegendi.

1. Cum detur æquatio incognitas quantitates $(x, y, z, v$, &c.$)$ involvens; & in omni datæ æquationis termino vel inveniantur n vel $n-1$ dimenſiones quarumcunque incognitarum quantitatum $(x, y, z$, &c.$)$: ſi modo omnes dimenſiones prædictæ ſint n, ſubſtituantur pro his incognitis quantitatibus $(y, z$, &c.$)$ in datâ æquatione reſpective αx, βx, &c. tum e datâ æquatione exterminabitur incognita quantitas (x): ſi modo dimenſiones ſint n & $n-1$, tum reſultabit æquatio unam dimenſionem incognitæ quantitatis x ſolummodo continens; ex hâc igitur æquatione inveniri poteſt valor incognitæ quantitatis (x) in terminis rationalibus incognitarum quantitatum $(y, z, v$, &c.$)$

Cor. Sæpe uſui inſervit hæc ſubſtitutionis methodus, cum duæ vel plures ſint datæ hujuſmodi æquationes.

Ex. 1. Sint $zy + az + by = v^2$ & $zy + ez + \frac{be}{a}y = w^2$; pro y, v, w per regulam ſcribantur reſpective αz, βz, γz: reſultant e priore æquatione $z = \frac{a + b\alpha}{\beta^2 - \alpha}$, e poſteriori vero $z = \frac{e + \frac{be}{a} \times \alpha}{\gamma^2 - \alpha}$; unde $\frac{a}{\beta^2 - \alpha} = \frac{e}{\gamma^2 - \alpha}$: exinde cognoſci poteſt quantitas (α) terminis quantitatum β & γ: & ſic de infinitis hujuſcemodi propoſitionibus.

2. Sit æquatio $ax^2 + by^2 + cz^2 +$ &c. $+ A = pv^2 + qw^2 + ru^2 +$ &c. $+ B$; ubi A & B quaſcunque rationales functiones incognitarum quantitatum denotant, in quarum terminis vel una vel nulla continetur dimenſio incognitarum quantitatum x, y, z, v, w, u, &c. ſit etiam $a + b \times \beta^2 + c \times \gamma^2 +$ &c. $= p \times \pi^2 + q \times \varrho^2 + r \times \sigma^2 +$ &c.

Subſtituantur $\beta x + \dot{y} = y$, $\gamma x + \dot{z} = z$, &c. $\pi x + \dot{v} = v$, $\varrho x + w = w$

$\dot{w} = w$, $\sigma x + \dot{u} = u$, &c. & ſcribantur hæ quantitates pro ſuis reſpectivis valoribus in datâ æquatione; reſultat æquatio, in quâ una ſolummodo continetur dimenſio incognitæ quantitatis x; unde inveniri poſſunt incognitæ quantitates x, y, z, v, &c. in functionibus incognitarum $\dot{y}, \dot{z}$, &c. $\dot{v}, \dot{u}, \dot{w}$, &c.

Ex. 1. Sit $x^2 + y^2 = z^2$; pro z ſcribatur $x + \dot{z}$, & reſultat æquatio $y^2 - \dot{z}^2 = 2\dot{z}x$; unde $x = \frac{y^2 - \dot{z}^2}{2\dot{z}}$ & $z = \frac{y^2 + \dot{z}^2}{2\dot{z}}$.

Ex. 2. Sit $a^2 + b^2 = x^2 + y^2$: aſſumatur $x = a + \dot{x}$ & $y = b - \dot{y}$, quibus ſubſtitutis, reſultat æquatio $2a\dot{x} + \dot{x}^2 - 2b\dot{y} + \dot{y}^2 = o$, quæ ad præcedentem methodum pertinet. Termini enim ultra duas dimenſiones non aſcendunt.

Aliter ducatur data æquatio in z^2, & reſultat $(a^2 + b^2) \times z^2 = z^2x^2 (v^2) + z^2y^2 (w^2)$; ſcribantur pro v & w reſpective $az + \dot{v}$ & $bz - \dot{w}$, & reſultat æquatio $2az\dot{v} + \dot{v}^2 - 2bz\dot{w} + \dot{w}^2 = o$, unde $z = \frac{\dot{v}^2 + \dot{w}^2}{2 \cdot \overline{b\dot{w} - a\dot{v}}}$. Et ſic de æquationibus ſuperiorum dimenſionum.

Cor. Cum duæ vel plures ſint datæ æquationes; hæc methodus ſubſtitutionis, quæ ita transformat quantitates, ut quoſdam terminos deſtruant, frequenter uſui inſervit; i. e. vel in ſimplici vel in duplicatâ, &c. æqualitate maxime præſtat; ut termini, in quibus maximæ inveniuntur dimenſiones quarumcunque incognitarum quantitatum, pro primâ ſubſtitutione fiant inter ſe æquales; & exinde ſæpe deduci poteſt reſolutio quæſita.

Ex. 1. Sit $ax^3 + bx^2 + cx + d^2 = v^2$; invenire v & x in rationalibus quantitatibus: fingatur $v = d + \frac{cx}{2d}$ (ut deſtruantur duo termini $d^2 + cx$); tum erit $d^2 + cx + \frac{c^2x^2}{4d^2} = d^2 + cx + bx^2 + ax^3$;

ax^3; unde $b + ax = \frac{c^2}{4d^2}$, & exinde conſtat rationalis valor quantitatis x.

Ex. Sit $x^3 + y^3 = v^3 - z^3$; aſſumantur $x = p + q$, $y = p - q$, $v = r + s$ & $z = r - s$; unde $x^3 + y^3 = 2p^3 + 6pq^2 = v^3 - z^3 = 2s^3 + 6r^2s$; & exinde $p(p^2 + 3q^2) = s(s^2 + 3r^2)$: in hâc æquatione pro literis p, q, r & s ſcribantur reſpective $ax + 3by$, $bx - ay$, $dy + cx$ & $3cy - dx$; reſultat $(ax + 3by)(a^2 + 3b^2) = (3cy - dx)(d^2 + 3c^2)$, unde $x = \frac{3c(d^2 + 3c^2) - 3b(a^2 + 3b^2)}{a(a^2 + 3b^2) + d(d^2 + 3c^2)}y$.

3. Sæpe vero præſtat, ut inveniatur per vulgares æquationum reſolutionum methodos cujuſcunque quantitatis valor; & valoris reſultantis fiat irrationalis pars æqualis rationali quantitati: & ſic de ſingulâ irrationali quantitatis reſultantis parte ſeparatim vel conjunctim; exinde in ſimplici & duplicatâ æqualitate ſæpe acquiri poſſunt reſolutiones.

Ex. 1. Sint $a^2z^2 + bz + c = v^2$ & $a^2z^2 + dz + e = w^2$: ſupponantur $az + \alpha$ & $az + \beta$ reſpective v & w per præcedentem caſum, reſultant duæ æquationes $bz + c = 2a \times \alpha z + \alpha^2$ & $dz + e = 2a\beta z + \beta^2$, unde $z = \frac{\alpha^2 - c}{b - 2a\alpha} = \frac{\beta^2 - e}{d - 2a\beta}$; inveniatur radix incognitæ quantitatis (α), hujus radicis fiat irrationalis pars æqualis rationali quantitati aſſumptæ per quamcunque datam methodum; exinde deducentur valores quantitatum α & β quæſiti.

4. Datis (n) æquationibus $(n + m)$ incognitas quantitates habentibus, quæ omnes eſſe rationales requiruntur; pro $(m - 1)$ incognitis quantitatibus aſſumantur quicunque rationales valores, qui reddent ſolutionem eo magis facilem; & exinde ſæpe deduci poſſunt rationales valores $(m + n)$ incognitarum quantitatum.

Ex. 1. Sit $x^4 + y^4 + z^4 = v^2$. Fingatur $v = \alpha - x^2$, unde $x^2 = \frac{\alpha^2 - y^4 - z^4}{2\alpha}$; ut deſtruantur z^4 & y^4, & ſolutio eo magis facilis reddatur,

datur, fingatur $\alpha = z^2 + y^2$; unde reſultat $x^2 = \frac{z^2 y^2}{y^2 + z^2}$: ſupponatur $y^2 + z^2 = w^2$; & exinde inveniantur valores incognitarum quantitatum y, z & w; & conficitur exemplum.

Ex. 2. Sint $z^2 + a = v^2 = \overline{z + \alpha}^2$, $zy + a = \beta^2$, $y^2 + a = w^2 = \overline{y + \gamma}^2$: reſultat $y = \frac{a - \gamma^2}{2\gamma} = -2\alpha$, ſi modo aſſumatur $\beta = \alpha$; unde deduci poſſunt valores correſpondentes quantitatum α, γ, y, &c.

Ex. 3. Sint $zy + a^2 = w^2$, $zv + a^2 = u^2$ & $yv + a^2 = x^2$: fingatur $u = a + tz$; $x = vyt + s$; &, ut ſolutio eo magis facilis reddatur, aſſumatur $w = a + vy$; exinde reſultabit $y = \frac{s^2 - a^2}{2avt^2 + 2at - 2svt}$.

Cor. Ut reſolutio (n) æquationum, in quibus continentur $n + m$ incognitæ quantitates $(x, y, z$, &c.) generalis evadat; plerumque valores ſingularum incognitarum quantitatum $(x, y, z$, &c.) reſpective exprimi debent per rationales functiones, quæ (m) independentes quantitates ad libitum aſſumendas involvunt.

Hæ reſolutiones haud generales dici poſſunt reſolutiones dati problematis; at potius reſolutiones problematis, quod in dato problemate continetur, i. e. particularis caſus dati problematis, cenſendæ ſunt: potius ex cognitis quibuſdam caſibus quam ex generali ratiocinatione erui poſſunt hujuſmodi reſolutiones.

5. Invento e quâcunque methodo operandi uno valore ſingularum incognitarum quantitatum: ſubſtituatur hic valor aſſumptâ incognitâ quantitate auctus vel diminutus, &c. pro eâdem incognitâ quantitate in ſingulis æquationibus; exorientur novæ æquationes, e quibus præcendenti operandi methodo inveſtigari poſſunt novi valores ſingularum incognitarum quantitatum; & ſic deinceps.

Hìc haud abs re ſit obſervare; quod æquationes, quæ videntur eaſdem involvere incognitas quantitates, ſæpe vero tanquam independentes æquationes haberi debent: e. g. ſint $2x^2 = y^3$ & $\overline{z + x}^2 = w^3$: hìc

hic utraque æquatio eandem habet incognitam quantitatem x; ſi vero pro $z+x$ ſcribatur v, exorientur duæ independentes æquationes $2x^2=y^3$ & $v^2=w^3$: quarum reſolutiones ſunt reſolutiones duorum independentium problematum.

6. Facile inveniri poſſunt æquationes plures incognitas quantitates habentes, quarum rationales vel integrales valores ſingularum incognitarum quantitatum innoteſcunt.

Aſſumantur reſolutiones, e quibus detegi poſſunt rationales valores ſingularum incognitarum quantitatum; & exinde formentur æquationes, quibus aſſumptæ reſolutiones reſpondent; & quod requiritur, factum eſt.

Ex. 1. Sint $xy+x+y=v^2$ & $z=1+x+y\pm 2v$ reſolutiones aſſumptæ, quarum rationales valores cognoſcuntur; tum exinde conſequuntur $xz+x+z=(v\pm x\pm 1)^2$; $yz+y+z=(v\pm y\pm 1)^2$; $xy+z$, $xz+y$, $yz+x$; $xy+xz+yz$ & $xy+xx+yz+x+y+z$ eſſe quadrata.

Ex. 2. Sit $P=0$; cujus incognitarum quantitatum rationales valores aſſumi poſſunt, tum $q^2+\alpha P+\beta P^2+\gamma P^3+$ &c. erit quadratus; $r^3+\pi P+\varrho P^2+\sigma P^3+$ &c. erit cubus, &c. quæcunque ſint rationales quantitates pro q, α, β, γ, &c. r, π, ϱ, σ, &c. aſſumptæ: exinde formari poſſunt quotcunque æquationes, quarum correſpondentes rationales incognitarum quantitatum valores facile deduci poſſunt: vel aſſumantur duæ vel plures æquationes $P=0$, $Q=0$, &c.; tum erunt $p^2+\alpha P+\beta Q+$ &c. $+\gamma P^2+\delta PQ+\epsilon Q^2+$ &c. quadratus; & $q^3+\alpha P+\beta Q+$ &c. $+\gamma P^2+\delta PQ+\epsilon Q^2+$ &c. cubus; & ſic deinceps de diverſis poteſtatibus.

Ex. 3. Si modo requiratur, ut data quantitas ſit quadratus, cubus, &c.; addantur vel ſubtrahantur de eâ quæcunque quantitates, ita ut fiat quadratus, cubus, &c.; fiant quantitates addendæ vel ſubtrahendæ nihilo (ſi modo poſſibile ſit) æquales; reſultant æquationes, quæ reddunt datam quantitatem quadratum, cubum, &c. Et ſic de pluribus quantitatibus: e. g. ſit quantitas data $x^2+xy+y^2+ax+by+e$,

a

$+ e$, addatur quantitas $c^2 - e + \overline{2c - b}y + \overline{2c - a}x + xy$, reſultat quadratus $\overline{x + y + c}^2$; fiat igitur $c^2 - e + \overline{2c - b}y + \overline{2c - a}x + xy = 0$; & erit data quantitas quadratus.

7. Haud nunquam reſolvi poſſunt hujuſmodi propoſitiones juxta ſubſequentem methodum.

Reducantur datæ quantitates, ſi modo fieri poſſint, in alias formulas, quarum relationes cognoſcuntur; exinde deduci poſſunt earum reſolutiones.

Ex. Sit $x^3 + y^3 = v^2$, ubi x & y ſunt inter ſe primi numeri; tum ea reduci poteſt in hanc formulam $(x + y) \times (x^2 - xy + y^2) = v^2$, unde facile conſtat $x + y$ & $x^2 - xy + y^2$ vel eſſe primos inter ſe, vel pro communi menſurâ habere numerum 3; unde $x + y$ vel $= u^2$ vel $= 3u^2$; & ſic $x^2 - xy + y^2$ vel $= v^2$ vel $= 3v^2$.

8. In reſolvendis pluribus hujuſce generis problematibus præſtat quantitates aſſumere, quæ generaliter vel particulariter reſolvunt quoſdam caſus problematum; tum ex aſſumptis quantitatibus & reliquis problematum caſibus ſæpe erui poſſunt eorum reſolutiones.

Ex. Invenire tres numeros ejuſmodi, ut ſumma vel differentia ſingulorum duorum ſit quadratus. Pro tribus numeris vel aſſumantur $a^2x^2 + b^2y^2$, $2abxy$ & $a^2y^2 + b^2x^2$; vel $f^2g^2 + 1$, $2fg$ & $f^2 + g^2$; vel $\frac{1}{2}(f^4g^4 + g^4 + f^4 + 1)$, $\frac{1}{2}(f^4g^4 - g^4 - f^4 + 1) + 2f^2g^2$ & $\frac{1}{2}(f^4g^4 - g^4 - f^4 + 1) - 2f^2g^2$; &c. Omnes hæ aſſumptæ quantitates ſatisfaciunt quatuor caſibus dati problematis; tres quantitates $6a^{18}b^2 + 24a^{14}b^6 - 92a^{10}b^{10} + 24a^6b^{14} + 6a^2b^{18}$, $a^{20} + 21a^{16}b^4 - 6a^{12}b^8 - 6a^8b^{12} + 21a^4b^{16} + b^{20}$ & $10a^{18}b^2 - 24a^{14}b^6 + 60a^{10}b^{10} - 24a^6b^{14} + 10a^2b^{18}$; ubi a & b ſunt integri numeri & $6a^4b^4 + b^8$ minor eſt quam $3a^8$, erunt tres numeri, qui reſpondent problemati; ſed in his formulis haud continentur univerſi numeri, qui problemati reſpondent: generales valores ſingulorum numerorum quæſiti continent tres quantitates ad libitum aſſumendas.

Ex. 2. 1. Sint $x^2 - xy + y^2$ & $x + y$ quadrati: numeri $x = 2pq - p^2$ & $y = q^2 - p^2$ ſatisfaciunt priori caſui, nempe ut $x^2 - xy + y^2$ ſit

ſit quadratus; poſteriori caſui $x+y=q^2+2pq-2p^2=\overline{q+p}^2-3p^2=v^2$ ſatisfaciunt $p=2mn$ & $q=3m^2-2mn+n^2$.

2. Sint x^2-xy+y^2 & $x+y$ reſpective = triplo quadrato: numeri $x=2p^2-2pq-q^2$ & $y=p^2+2pq-2q^2$; vel $x=2p^2-2pq-q^2$ & $y=p^2-4pq+q^2$; vel $x=p^2+2pq-2q^2$ & $y=-p^2+4pq-q^2$ priori caſui ſatisfaciunt: fiat $x+y$ 1mo. $=3p^2-3q^2$; vel 2do. $=3p^2-6pq$; vel 3tio. $=6pq-3q^2=3v^2$; unde 1mo. $p=m^2+n^2$ & $q=2mn$; vel 2do. $p=2m^2$ & $q=m^2-n^2$; vel 3tio. $p=m^2+n^2$ & $q=2m^2$.

9. Ex notâ radicum extractione ſæpe deduci poſſunt novæ æquationes; quarum incognitarum quantitatum, ſi modo correſpondentes valores ſint rationales, tum correſpondentes valores incognitarum quantitatum datæ æquationis etiam erunt rationales quantitates.

Fingatur enim ſingula irrationalis quantitas in eâ contenta eſſe rationalis fractio vel integer numerus; & exinde deduci poſſunt novæ æquationes prædictæ.

Ex. Sit æquatio $ay^2+\overline{bx+c}\,y+dx^2+ex+f=0$; tum per notam extractionem erit $y=\sqrt{\frac{\overline{bx+c}^2-4a\times(dx^2+ex+f)}{4a^2}}-\frac{bx+c}{2a}$, unde $\overline{bx+c}^2-4a\times(dx^2+ex+f)=Ax^2+Bx+C$ erit quadratus $=v^2$; pro x ſcribatur $z-\frac{B}{2A}$ & reſultat $Az^2-\frac{B^2}{4A}+C=v^2$.

Ex. 2. Sit æquatio $y^3-\overline{a+bx}\,y+cx^2+dx+e=0$; tum erit $y=\sqrt[3]{cx^2+dx+e+\sqrt{\frac{\overline{cx^2+dx+e}^2}{4}+\frac{\overline{a+bx}^3}{27}}}$; fiat $+\frac{(cx^2+dx+e)^2}{4}+\frac{\overline{a+bx}^3}{27}=\overline{Ax^2+Bx+C}^2$ quadratus $=v^2$, & $cx^2+dx+e=\overline{Ax^2+Bx+C}=w^3$ cubo.

1. Invenire, utrum data methodus ſit generalis reſolutio dati problematis, necne.

Hoc plerumque conſtabit ex ſubſtitutionibus in ejus reſolutione uſitatis; ſi omnes ſubſtitutiones pro incognitâ vel incognitis quantitatibus in datâ reſolutione contentis ſint maxime generales, tum reſolutio eſt generalis.

2. 1. Quantitates x, y, z, v, &c. reſpective denotent omnes integros numeros; tum quantitas x denotat omnes integros numeros; fractio $\frac{x}{y}$ denotat omnes rationales quantitates; radix $\sqrt[m]{x}$ denotat omnes radices, quarum poteſtates ſunt m, integrorum numerorum; radix $\sqrt[m]{\frac{x}{y}}$ denotat radices (m) poteſtatis omnium rationalium quantitatum; &c.

2. 2. Quantitas (α) denotet quamcunque rationalem quantitatem $= \frac{x}{y}$, tum $\sqrt[m]{\alpha}$ denotat radices (m) poteſtatis omnium rationalium quantitatum; &c.

2. 3. Si in datâ reſolutione quantitas α haud aſcendat ad majores dimenſiones quam unam; tum ea reſolutio eſt generalis, continet enim omnes rationales quantitates.

2. 4. In numeratore ſit x quicunque integer numerus, qui haud aſcendit ad majores quam unam dimenſionem; & nullam habet coefficientem præter unitatem; tum reſolutio continet omnes integros numeros.

2. 5. Sint α & β, n & m quicunque integri numeri; tum poteſtas α^n ad datam diſtantiam habet plures valores quam β^m, ſi modo m major ſit quam n: ſit enim H data quantitas, ad quam proxime aſcendunt numeri α^n & β^m; tum numerus integrorum valorum poteſtatum α^n & β^m inter o & H contentorum quam proxime $:: H^{\frac{1}{n}} : H^{\frac{1}{m}} :: H^{\frac{m-n}{nm}} : 1$.

Sint $a\alpha^n$ & $b\beta^m$ proxime $= H$, ubi a & b ſunt integri numeri; tum erunt

erunt numeri integrorum valorum quantitatum $a\alpha^n$ & $b\beta^m$ inter o & H contentorum quam proxime :: $b^{\frac{1}{m}} H^{\frac{m-n}{nm}} : a^{\frac{1}{n}}$.

2. 6. Et ſic progredi liceat ad detegendas quam proxime rationes, quas habet numerus valorum in unâ datâ reſolutione ad numerum valorum in alterâ datâ reſolutione; cum ax^m, by^n, cz^r, &c.; $p\pi^b$, $q\varrho^i$, $r\sigma^k$, &c. reſpective in datis reſolutionibus contineantur; literis a, b, c, &c.; p, q, r, &c.; m, n, r, &c.; b, i, k, &c. integros numeros reſpective denotantibus.

3. 1. Datâ æquatione $(L = o)$ incognitas quantitates $(x, y, z,$ &c.$)$ involvente, ubi $x, y, z,$ &c. quoſcunque integros numeros vel rationales quantitates reſpective denotant; formulam generalem incognitarum quantitatum in eâ contentarum detegere, ſi modo generalem formulam recipiant: 1mo. inveniantur diviſores A, B, C, &c. datæ æquationis, ſi modo ea ullos habeat; fiant ſinguli diviſores nihilo reſpective æquales, & reducitur data æquatio in duas vel plures alias $A = o$, $B = o$, $C = o$, &c.; ſint $x = P$, $x = Q$, $x = R$, &c. generales reſolutiones æquationum $A = o$, $B = o$, $C = o$, &c.; tum erunt $x = P$, $x = Q$, $x = R$, &c. generales reſolutiones æquationis $L = o$.

3. 2. Si vero diviſores haud recipiat data æquatio, tum generalis reſolutio quantitatis (x) continet $(n - 1)$ quantitates $(y, z, v,$ &c.$)$; hæ $(n - 1)$ diverſæ quantitates ſæpe in pauciores facile reduci poſſunt; e. g. ſi quantitates, in quibus involvuntur (m) quantitates $(y, z, v,$ &c.$)$, ſint $A \times (ay + bz + cv +$ &c.$)^n + B \times (ay + bz + cv +$ &c.$)^{n-1} + C\,(ay + bz + cv +$ &c.$)^{n-2} +$ &c.; ſcribatur W pro $ay + bz + cv +$ &c. in datâ æquatione $L = o$, & reducetur ea in alteram $(n - m + 1)$ incognitas quantitates involventem: vel in genere ita ſcribantur rationales, &c. quantitates generaliter aſſumptæ $(\pi, \varrho, \sigma, \tau,$ &c.$)$ pro incognitis quantitatibus $(y, z, v,$ &c.$)$, ut ex aſſumptis quantitatibus datis facile acquiri poſſint incognitæ quantitates $(y, z, v,$ &c.$)$; &, ſi ex hâc methodo reductionis pauciores contineantur incognitæ quantitates in reſultante quam in datâ æquatione; tum in generali reſolutione quantitatis (x) numerus aſſumptarum quanti-

unius incognitæ quantitatis terminis vero alterius; & deinde e præcedentibus ſubſtitutionibus conſtabunt valores incognitarum quantitatum datæ æquationis quæſiti.

Ex. Sit $13a^2+1=x^2$; hinc proximus valor quantitatis (x) erit $3a$; ſupponatur $x=3a+b$; quo pro (x) in datâ æquatione ſubſtituto, reſultat $4a^2+1=6ab+b^2$: aſſumatur $a=b+c$, & exinde $4b^2+8bc+4c^2+1=6b^2+6bc+b^2$, hoc eſt $2bc+4c^2+1=3b^2$: aſſumatur etiam $b=c+d$, & exinde $3c^2+1=4cd+3d^2$: aſſumatur $c=d+e$, & reſultat $2de+3e^2+1=4d^2$: aſſumatur iterum $d=e+f$, quo ſubſtituto pro ſuo valore, reſultat $e^2+1=6ef+4f^2$: ſupponatur $e=6f+g$, & reſultat $6fg+g^2+1=4f^2$: capiatur $f=g+h$, & reſultat $3g^2+1=2gh+4h^2$: ſupponatur $g=h+k$, quo valore pro g in ultimâ æquatione ſubſtituto, reſultat $4hk+3k^2+1=3h^2$; ex quâ æquatione colligitur unum valorem quantitatis $k=1$, & quantitatis $h=2$; unde $g=h+k=3$, $f=g+h=5$, $e=6f+g=33$, $d=e+f=38$, $c=d+e=71$, $b=c+d=109$, $a=b+c=180$, $x=3a+b=649$.

Et ſic de correſpondentibus integris valoribus incognitarum quantitatum æquationis plures quam duas dimenſiones habentis, detegendis.

Et eadem methodus ad æquationem tres vel plures incognitas quantitates habentem applicari poteſt.

Hæc methodus paululum mutata fractionales correſpondentes valores incognitarum quantitatum inveniet.

Sed in quam plurimis caſibus hæc methodus haud multo magis præſtabit quam continua tentamina: e. g. ſit $xy=399$; nulla in hoc caſu quantitas pro approximatione aſſumi poteſt: & ſic in quam plurimis caſibus.

1. Hæ æquationes ſæpe nullos admittunt integros, nec forſan rationales etiamque nec irrationales dati generis, correſpondentes valores incognitarum quantitatum in iis contentarum.

E methodis prius traditis impoſſibiles radices detegendi conſtabunt nonnullæ æquationes hujuſmodi; etiamque limites inter quos conſiſtant.

ſiſtant poſſibiles datarum æquationum valores incognitarum quantitatum.

2. Si poſſibiles dentur radices, tum nonnunquam æquationes nullos habent integros nec forſan rationales valores, nec irrationales dati generis incognitarum quantitatum in iis contentarum ſibi ipſis correſpondentes.

Quod prædictæ incognitæ quantitates in nonnullis æquationibus nullos habent integros nec rationales, &c. correſpondentes valores, in nonnullis caſibus ab reſolutionibus datarum æquationum facile perſpici poteſt. e. g. Sit data æquatio $ax^m y^n + bx^{m+1} y^{n-1} + cx^{m+2} y^{n-2} + \&c. = 0$; in eâ pro x ſcribatur πy, reſultat $a\pi^m + b\pi^{m+1} + c\pi^{m+2} + \&c. = 0$; tum, ſi modo ex reſolutione conſtet hanc æquationem nullam habere rationalem radicem, nulli dantur rationales valores incognitarum quantitatum $(x \,\&\, y)$ correſpondentes. e. 2. Sit $(x + y + z + v + \&c.)^n = A$, tum, ſi radix n poteſtatis quantitatis (A) haud ſit rationalis quantitas, etiam haud dantur rationales correſpondentes valores incognitarum quantitatum x, y, z, v, &c.

2. Idem etiam conſtabit ex reſiduis a diviſione quantitatum in datis æquationibus contentarum per quaſcunque datas ortis; etiamque ex formulis diviſorum, quos recipiunt prædictæ quantitates in datis æquationibus contentæ; quod in ſubſequentibus tradetur.

3. Idem etiam conſtabit ex hoc, quod nulli inveniantur correſpondentes integri valores incognitarum quantitatum inter numeros detegendos, unde probari poteſt nullos omnino dari correſpondentes integros valores; dantur enim limites ſemper hujuſce generis, qui plerumque haud ſine maximâ difficultate deduci poſſunt. Et ſic de pluribus methodis.

PROB. LVIII.

Datis duabus vel pluribus (n) *æquationibus plures incognitas quantitates* (x, y, &c.) *habentibus, invenire integros correſpondentes valores incognitarum quantitatum* (x, y, &c.)

Inveniatur

Inveniatur proximus valor unius vel plurium incognitarum quantitatum terminis vero reliquarum & datarum quantitatum: qui valor auctus vel diminutus per aſſumptam quantitatem; vel, ſi modo proximi valores plurium incognitarum quantitatum aſſumantur, ſubſtituantur hi valores aucti vel diminuti per aſſumptas quantitates pro eorum reſpectivis valoribus in ſingulis æquationibus; & ſic deinceps; uſque donec inveniantur correſpondentes incognitarum quantitatum valores, vel quidam valor incognitæ quantitatis evadat (1) vel o, vel quantitas cognita, &c.

Ex. Sint duæ æquationes $x^2 = y^2 + 5z^2 - 12$ & $7x^2 = 10y^2 + 28z^2 - 69$: aſſumatur $x = y + 2z - v$, quâ quantitate pro ſuo valore in datis æquationibus ſubſtitutâ; reſultant $y^2 + 4yz + 4z^2 - 2yv - 4zv + v^2 = y^2 + 5z^2 - 12$, & exinde $v^2 - 4zv - 2yv = z^2 - 4yz - 12$, & $7v^2 - 28zv - 14yv + 28yz = 3y^2 - 69$: In hoc caſu ſupponatur e priori æquatione $v = +z + w$, & e poſteriori $v = +y - u$; unde, ſi in duabus præcedentibus æquationibus pro v ſcribatur $z + w$ & pro y ſcribatur $z + w + u$, reſultant duæ æquationes $2z^2 + 2zw + w^2 = 12 + 2zu - 2wu$, & $10z^2 + 20zw + 10w^2 - 8zu + 20wu = 69 - 3u^2$: aſſumatur quantitas $w = o$, & exinde $u = 1$ & $z = 3$; & $v = z + w = 3$ & $y = z + w + u = 4$ & $x = y + 2z - v = 7$.

Cor. Eadem methodus applicari poteſt ad (n) æquationes $(n+1)$ vel plures incognitas quantitates habentes: ſed maxime creſcit eundo labor: & ſæpe nulla dari poteſt methodus niſi per continua tentamina.

Hæc methodus etiam fractionales correſpondentes valores incognitarum quantitatum tandem inveniet.

PROB. LIX.

Datis uno, duobus vel pluribus integris valoribus e ſingulis incognitis quantitatibus in datâ æquatione duas vel plures incognitas quantitates habente; & datâ exinde methodo alterum integrum valorem e ſingulis incognitis

tis quantitatibus deducendi; infinitos integros valores e ſingulis incognitis quantitatibus deducere.

1. Repetatur eadem operatio continuo, quæ ex uno vel pluribus valoribus alios deducat, & perficitur problema.

Ex. 1. Sit æquatio $ax^2 + 1 = y^2$, inveniantur r & s correſpondentes integri valores quantitatum $(x$ & $y)$: alter integer valor quantitatis (x) inveniatur $2rs$; ejus vero correſpondens valor quantitatis (y) erit $2ar^2 + 1 = 2s^2 - 1$: integer valor incognitæ quantitatis (x) erit $2^m s^m - \overline{m-1} \cdot 2^{m-2} s^{m-2} + \overline{m-2} \cdot \frac{m-3}{2} 2^{m-4} s^{m-4} - \overline{m-3} \times \frac{m-4}{2} \times \frac{m-5}{3} 2^{m-6} s^{m-6} + \overline{m-4} \times \frac{m-5}{2} \times \frac{m-6}{3} \times \frac{m-7}{4} 2^{m-8} s^{m-8} -$ &c. in r ducta; & correſpondens valor incognitæ quantitatis (y) erit $2^m s^{m+1} - \overline{m+1}\, 2^{m-2} s^{m-1} + \overline{m+1} \times \frac{m-2}{2} 2^{m-4} s^{m-3} - \overline{m+1} \times \frac{m-3}{2} \times \frac{m-4}{3} 2^{m-6} s^{m-5} + \overline{m+1} \times \frac{m-4}{2} \times \frac{m-5}{3} \times \frac{m-6}{4} 2^{m-8} s^{m-7} - \overline{m+1} \times \frac{m-5}{2} \times \frac{m-6}{3} \times \frac{m-7}{4} \times \frac{m-8}{5} \times 2^{m-10} s^{m-9} +$ &c. Si in hiſce ſeriebus pro s^2 ſcribatur $ar^2 + 1$, reſultat ſeries in terminis quantitatis r. Hæ ſeries terminant, cum poteſtas quantitatis (s) evadat negativus numerus.

2. Iidem valores quantitatis (x) deſignari poſſunt per ſubſequentem methodum $r \times \left(2s - 1 + \frac{1}{1}\right) \times \left(2s - 1 + \frac{2s-1}{2s}\right) \times \left(2s - 1 + \frac{4s^2 - 2s - 1}{4s^2 - 1}\right) \times \left(2s - 1 + \frac{8s^3 - 4s^2 - 4s + 1}{8s^3 - 4s}\right) \times$ &c. i. e. ſit $r = 2$, & $2s = 6 = 5\frac{1}{1}$; valor quantitatis x erit $2 \times 5\frac{1}{1} \times 5\frac{5}{6} \times 5\frac{29}{35} \times 5\frac{169}{204} \times$ &c. ubi fractionis cujuſque numerator æquat denominatorem ſuum dempto denominatore proxime præcedente, denominator

i vero

vero numeratorem æquat termini proxime præcedentis in impropriam fractionem reducti: hoc problema primus invenit Vicecomes Brounker.

Hæ methodi minime inveniunt omnes poſſibiles valores: e quibuſdam datis valoribus multis modis inveſtigari poſſunt alii: ſed quoniam ex hypotheſi nulla dari poteſt regula, quæ omnes valores continet: ergo nulla regula inveniri poteſt, quæ aliam haud magis generalem admittit: & e nullâ generali ratiocinatione deduci poſſunt tales regulæ, ex eâ enim nulli oriuntur limites.

Ex. 2. Sit $ax^2+bx+c=v^2$, & ſint p & $q=\sqrt{ap^2+bp+c}$ correſpondentes valores incognitarum quantitatum x & y, plures earundem quantitatum valores correſpondentes detegere: aſſumantur $dp+eq+f=x$ & $gp+bq+l=v$; ſubſtituantur hæ quantitates pro x & v in æquatione $ax^2+bx+c=v^2$, reſultat $a(d^2p^2+2dfp+f^2+e^2q^2)+bdp+bf+c+(2adep+2afe+be)q=g^2p^2+2gpl+b^2q^2+l^2+(2gbp+2bl)q$; in hâc æquatione pro q^2 ſcribatur ap^2+bp+c, & exoritur $(ad^2+a^2e^2-g^2-ab^2)p^2+(2adf+bd-2gl+abe^2-bb^2)p+(af^2+bf-l^2-cb^2+ace^2+c)+(2adep+2afe+be-2gbp-2bl)q=0$; nunc fiant ſinguli hi termini $ad^2+a^2e^2-g^2-ab=0$, $2adf+bd+abe^2-2gl-bb^2=0$, $af^2+bf-l^2-cb^2+ace^2+c=0$, $2ade-2gb=0$, $2afe+be-2bl=0$; unde $\frac{ade}{b}=g$ & $l=\frac{2afe+be}{2b}$; in primâ æquatione pro g ſcribatur $\frac{ade}{b}$, reſultat $d^2b^2+ae^2b^2-ad^2e^2-b^4=-(b^2-d^2)\times(b^2-ae^2)=0$; ſed non poteſt $b^2-ae^2=0$, ni a ſit quadratum, ergo $b^2-d^2=0$, & conſequenter $b=\pm d$; ſcribantur hi valores d, ae & $\frac{2afe+be}{2d}$ pro b, g & l in ſecundâ æquatione, reſultat $2adf+bd+abe^2-2ae\times\frac{2afe+be}{2d}-bd^2=-\frac{1}{d}(d^2-ae^2)\times(bd-b-2af)$; unde $f=\frac{bd-b}{2a}$; ſcribantur

hæ quantitates pro ſuis valoribus in æquatione $af^2+bf-l^2-cb^2+ace^2+c=o$, & reſultat æquatio $\frac{1}{4a}(b^2-4ac)\times(d^2-ae^2-1)=o$: inveniantur correſpondentes integri valores quantitatum d & e in æquatione $d^2=ae^2+1$, qui ſint α & π reſpective; tum erunt $x=dp+eq+f=p\alpha+q\pi+\frac{b}{2a}\times(\alpha-1)$, & $y=ap\pi+q\alpha+\frac{b\pi}{2}$. Unde ſucceſſivi valores incognitæ quantitatis (x) erunt p, $p\alpha+q\pi+\frac{b\times(\alpha-1)}{2a}$, $2p\alpha^2+2\pi\alpha q-p+\frac{b\times(\alpha^2-1)}{a}$, &c.; ſint E, F, G reſpective tres ſucceſſivi valores hujus ſeriei, erit $2\alpha F-E+b\times\frac{\alpha-1}{a}=G$: ſucceſſivi valores incognitæ quantitatis (y) invenientur, q, $ap\pi+\alpha q+\frac{b\pi}{2}$, $2a\times\pi\alpha p+2\alpha^2 q-q+b\pi\alpha$, &c.: ſint tres ſucceſſivi termini hujus ſeriei reſpective P, Q, R; tum $R=2\alpha Q-P$.

Æquatio $v^2+(A+Bx)\times v+Cx^2+Dx+E=o$ facile in prædictam $y^2=ax^2+bx+c$, vel etiam in hanc $y^2=ax^2+d$ mutari poteſt; nihil enim aliud requiritur niſi ut deſtruatur ſecundus datæ æquationis terminus.

1. Coefficiens a nunquam debet eſſe quadratum; ſi enim hoc fiat, haud infinitos valores incognitarum quantitatum (x & y) admittet data æquatio.

2. In hujuſmodi reſolutionibus, cum numerus valorum e ſingulis incognitis quantitatibus independentium in datâ reſolutione ſit major, tum plerumque generalior erit reſolutio.

3. Sint r & s valores correſpondentes incognitarum quantitatum x & y in æquatione $ax^2+1=y^2$, tum $\frac{s}{r}$ erit quam proxime $=\sqrt{a}$; eo autem magis appropinquat ad radicem $\sqrt{a}$, quo majores ſunt valores r & s: ſed hoc magis pertinet ad infinitas ſeries.

O 2. 1. In

2. 1. In datâ algebraicâ æquatione e fractionibus & irrationalibus quantitatibus liberatâ, maximæ dimenſiones incognitarum quantitatum $(x, y, z,$ &c.) ſint (m); & in eâ pro $(x, y, z,$ &c.) ſcribantur $au^n + bu^{n-1} + cu^{n-2} +$ &c.; $a'u^{n'} + b'u^{n'-1} + c'u^{n'-2} +$ &c.; $a''u^{n''} + b''u^{n''-1} + c''u^{n''-2} +$ &c., &c.; reſultet æquatio $Au^s + Bu^{s-1} + Cu^{s-2} +$ &c.: fiant coefficientes $A = o$, $B = o$, $C = o$, &c.: ſingularum incognitarum quantitatum $a, b, c,$ &c.; $a', b', c',$ &c.; a'', b'', c'', &c., &c.; in reſultantibus æquationibus contentarum inveniatur correſpondens rationalis valor, ſi modo poſſibile ſit; tum conſequentur infiniti correſpondentes rationales & integrales valores incognitarum datæ æquationis quantitatum $(x, y, z,$ &c.): pro literâ (u) enim aſſumi poteſt omnis numerus vel rationalis quantitas.

In hoc caſu numerus incognitarum quantitatum $a, b, c,$ &c.; a', b', c', &c.; a'', b'', c'', &c., &c. major eſſe debet per numerum binarium vel tres vel quatuor, &c. quam numerus $(s + 1)$ æquationum $A = o$, $B = o$, $C = o$, &c.

Cum minores ſint indices n, n', n'', &c.; tum generalior erit valor quantitatum $(x, y, z,$ &c.) per hanc methodum inventus.

Annon reſolutio ex hoc fonte hauriri poteſt, e numero diverſarum incognitarum quantitatum in datâ æquatione contentarum & ejus dimenſionibus plerumque pendet.

2.2. Pro $x, y, z,$&c. reſpective aſſumi poſſunt quantitates hujuſce formulæ $\frac{au^n + bu^{n-1} + \&c.}{pu^m + qu^{m-1} + \&c.}$, $\frac{a'u^{n'} + b'u^{n'-1} + \&c.}{p'u^{m'} + q'u^{m'-1} + \&c.}$, &c.; quibus pro ſuis valoribus in datâ æquatione ſubſtitutis, & ſingulis reſultantis æquationis terminis ad communem denominatorem reductis, cujus numerator ſit $Au^s + Bu^{s-1} + Cu^{s-2} + \&c. = o$: fiant coefficientes $A = o$, $B = o$, $C = o$, &c; deinde inveniatur correſpondens valor ſingulæ incognitæ quantitatis $a, b,$ &c.; $p, q,$ &c.; a', b', &c.; p', q', &c.; & exinde detegentur infiniti correſpondentes valores incognitarum quantitatum $(x, y, z,$ &c.) in datâ æquatione contentarum.

2.3. Generalior erit reſolutio; ſi modo pro prædictis quantitatibus $(x, y, z,$ &c.) in datâ æquatione ſcribantur quantitates hujuſce formu-

læ $au^{n} + (b + cw + \&c.)\,u^{n-1} + (d + ew + \&c.)\,u^{n-2} + \&c.$, &c. vel generaliter quæcunque rationales functiones literarum u, w, U, &c.; ubi u, w, U,&c. quoſcunque numeros vel rationales quantitates ad libitum aſſumendas reſpective denotant; & reducantur ſinguli æquationis reſultantis termini ad communem denominatorem: deinde fiant ſinguli termini numeratoris, in quibus inveniuntur diverſæ dimenſiones incognitarum quantitatum (u, w, U, &c.) nihilo æquales, & inveniatur correſpondens valor e ſingulis incognitis quantitatibus (a, b, &c.; p, q, &c.; a', b', &c.; p', q', &c.) in reſultantibus æquationibus.

Eadem principia etiam applicari poſſunt ad plurimos valores incognitarum datæ æquationis quantitatum detegendos, quarum valores datam habent irrationalitatem; etiamque ad plures æquationes plures incognitas quantitates habentes; quæ vel ſunt integri numeri vel rationales vel irrationales dati generis quantitates.

PROB. LX.

Sit quicunque numerus per vulgarem notationem Indorum nunc in uſu ſecundum ſummam unitatum etiamque numerorum 10, 10^2, 10^3, *&c. vel quamcunque aliam conſimilem notationem progredientem; eam transformare in aliam ſecundum ſummam unitatum etiamque numerorum* α, α^2, α^3, α^4, *&c. progredientem.*

Dividatur numerus N per α, & ſit quotus N' & reſiduum R; i. e. α| N | N'; dividatur etiam N' per α, & ſit quotus N'' & reſiduum R',

$$\frac{\alpha N'}{R}$$

& ſic dividatur N'' per α, & ſit quotus N''' & reſiduum R''; & ſic continuo dividatur quotus per α uſque donec minor fiat quam α, qui ſit N'^{m}; tum erit notatio quæſita $N'^{m} \ldots R'''' R''' R'' R' R$.

Ex. Transformare numerum $37987 = N$ per vulgarem notationem in notationem ſecundum unitates, 6, 6^2, 6^3, &c. progredientem; operatio

operatio erit

```
        6          6          6          6
6 | 37987 | 6331 = N' | 1055 = N'' | 175 = N''' | 29 = N'''' | 4 = N'''''
    36      6          6          12          24
    19      33         45         55  5m. reſid. 5 = R''''
    18      30         42         54
    18      31         35  4m. reſid. 1 = R'''
    18      30         30
     7  2m re-
        ſiduum 1   3m. reſid. 5 = R''
     6
prim. reſ. 1 = R
```

unde notatio quæſita erit $N''''' R'''' R''' R'' R' R = 451511$.

Eadem omnino erit methodus additionis, ſubtractionis, multiplicationis, diviſionis, radicum extractionis, &c. numerorum vel eorum fractionum ſic deſignatorum, ac prædictarum operationum numerorum vel fractionum per vulgarem notationem deſignatorum.

De proprietatibus integrorum numerorum.

THEOR. XLVI.

1. Sit $N = a \times b \times c \times d \times$ &c.; dividatur M per a, & ſit quotiens M' & reſiduum p: dividatur M' per b & ſit quotiens M'' & reſiduum q, &c. i. e. ſit $a \left| \frac{M}{p} \right| \overset{b}{} \frac{M'}{q} \left| \overset{c}{} \frac{M''}{r} \right| \overset{d}{} \frac{M'''}{s} \Big|$ &c.; tum reſiduum e diviſione quantatis M per N erit $p + a \times q + ab \times r + abc \times s +$ &c.

2. Numeri $a, b, c, d, e,$ &c. dividantur per α, & ſint reſpectiva reſidua $\pi, \varrho, \sigma, \tau,$ &c. tum reſiduum numeri $ma + nb + rc + sd + te +$ &c. per α diviſi idem erit, ac reſiduum numeri $m\pi + n\varrho + r\sigma + s\tau +$ &c. per α diviſi.

3. Sit quantitas $a^m b^n c^r d^s$ &c. quicunque numerus; ubi $a, b, c, d,$ &c. reſpective primi ſint reipſa & inter ſe numeri; tum numerus diviſorum,

viſorum, quos admittit prædictus numerus, erit $(m+1)\times(n+1)\times(r+1)\times(s+1)\times$ &c.

Erunt enim $a, a^2, a^3, \ldots a^m$; $b, ab, a^2b, a^3b, a^4b, \ldots a^mb$; $b^2, ab^2, a^2b^2, a^3b^2, \ldots a^mb^2$; $b^3, ab^3, \ldots a^mb^3$; &c.; $b^n, ab^n, a^2b^n, \ldots a^mb^n$; ac, bc; a^2c, abc, b^2c; &c.; ac^2, bc^2, &c.; &c.; unde ſumma omnium diviſorum æqualis erit contento $(1+a+a^2+a^3+\ldots+a^m)\times(1+b+b^2+b^3+\ldots+b^n)\times(1+c+c^2+c^3+\ldots c^r)\times(1+d+d^2+d^3+\ldots+d^s)\times$ &c.; & conſequenter numerus diviſorum erit $(m+1)\times(n+1)\times(r+1)\times(s+1)\times$ &c.

Cor. Hinc datis primis diviſoribus datæ quantitatis & eorum poteſtatibus; facile deduci poteſt numerus diviſorum, quos admittit data quantitas.

Cor. 2. Hinc facile inveniri poſſunt infiniti numeri, qui datum numerum diviſorum admittunt; etiamque minimus numerus, qui datum numerum diviſorum admittit: aſſumantur enim a, b, c, d, &c. minimi numeri inter ſe primi & indices m, n, r, s, &c. ita ut $(m+1)\times(n+1)\times(r+1)\times(s+1)\times$ &c. $=$ dato numero, erit $a^m b^n c^r d^s$ &c. minimus numerus.

Cor. 3. Ubi numerus diviſorum ſit impar, ibi ille numerus erit quadratus; ſi numerus diviſorum ſit impar, tum $m+1$, $n+1$, $r+1$, $s+1$, &c. erunt reſpective impares numeri; & conſequenter m, n, r, s, &c. pares numeri, & exinde quantitas $a^m b^n c^r d^s$ &c. quadratus.

4. Sit quantitas 2^n-1 primus numerus; quantitas $2^{n-1}\times(2^n-1)$ erit perfectus numerus; ſumma enim omnium ejus diviſorum per theorem. præcedens erit (ſi $1+2+4+8\ldots 2^{n-1}=2^n-1=A$ & $1+2^n-1=2^n=B$) $A\times B=2^n\times(2^n-1)$: ex hoc facto ſubtrahatur numerus ipſe $2^{n-1}\times(2^n-1)$, & reſiduum erit $2^{n-1}\times(2^n-1)$: ergo ſumma omnium diviſorum præter numerum ipſum erit æqualis numero ipſi.

5. Sint N & M amicabiles numeri, i. e. $N=SM-M$ & $M=SN-N$, ubi SM & SN denotant ſummas diviſorum quantitatum M & N reſpective; ſint 2^nx & 2^nyz amicabiles numeri, ubi x, y, z ſunt

ſunt primi numeri: ſumma autem diviſorum prioris numeri erit $2^{n+1} - 1 + (2^n - 1) \times x = 2^n y z$; poſterioris vero $2^{n+1} - 1 + (2^{n+1} - 1) \times (y + z) + (2^n - 1) \times y z = 2^n x$: e priori æquatione $x = \frac{2^n y z - 2^{n+1} + 1}{2^n - 1}$, & exinde e poſteriori $z = \frac{(2^n - 1) \times y + 2^{n+1} - 1}{y - 2^n + 1}$ $= 2^n - 1 + \frac{2^{2n}}{y - 2^n + 1}$; unde y talis primus numerus aſſumi debet, ut $y - 2^n + 1$ ſit diviſor numeri 2^{2n}, & z & x ſint etiam primi numeri.

Hinc inveniuntur amicabiles numeri 284 & 220; etiamque 18416 & 17296; &c. Eodem fere modo inveniri poſſunt infiniti alii amicabiles numeri; aſſumendo pro 2^n & 2^n quoſcunque alios numeros, & pro x, y, z numeros conſtantes e pluribus primis numeris.

Sint a & b dati numeri; & z, y, x, &c. α, β, γ, &c. primi numeri quæſiti, & $a x y z$ &c. $= N$ & $b \alpha \beta \gamma$ &c. $= M$ amicabiles numeri: ſint m & n ſummæ diviſorum numerorum a & b reſpective, tum $m \times (x + 1) \times (y + 1) \times (z + 1) \times$ &c. $- N = b \alpha \beta \gamma \delta$ &c. & $n \times (\alpha + 1) \times (\beta + 1) \times (\gamma + 1) \times$ &c. $- M = a x y z$ &c.

Facile conſtat, per hunc modum, viz. ex aſſumptis quantitatibus, quarum diviſores innoteſcunt; & æquationibus quæſitas relationes exprimentibus, infinitæ deduci poſſunt conſimiles propoſitiones; quales ſunt methodi inveniendi numeros diverſos, quorum aggregata diviſorum ſunt inter ſe æqualia (e. g. ſi includatur inter diviſores numerus ipſe, tum 6 & 11; 10 & 17; 14, 15 & 23, &c. erunt numeri hujuſce generis); vel hæc aggregata quamcunque inter ſe habeant relationem, &c. vel inveniri poſſunt numeri, quorum ſummæ diviſorum ad numeros ipſos datam habeant rationem.

6. Sit numerus $a^m b^n c^r d^s$ &c. $\times N = P$, cujus ſumma diviſorum erit $\frac{a^{m+1} - 1}{a - 1} \times \frac{b^{n+1} - 1}{b - 1} \times \frac{c^{r+1} - 1}{c - 1} \times \frac{d^{s+1} - 1}{d - 1} \times$ &c. $= H$: ducatur numerus (P) in numerum $Q = a^\alpha b^\beta c^\gamma d^\delta$ &c.; ubi α, β, γ, δ, &c. ſunt

ſunt integri numeri; tum erit ſumma diviſorum numeri $(P \times Q) = \frac{a^{m+\alpha+1}-1}{a-1} \times \frac{b^{n+\beta+1}-1}{b-1} \times \frac{c^{r+\gamma+1}-1}{c-1} \times \frac{d^{s+\delta+1}-1}{d-1} \times$ &c. $= K$; dividatur ſumma K per ſummam H, reſultat $\frac{a^{m+\alpha+1}-1}{a^{m+1}-1} \times \frac{b^{n+\beta+1}-1}{b^{n+1}-1} \times \frac{c^{r+\gamma+1}-1}{c^{r+1}-1} \times \frac{d^{s+\delta+1}-1}{d^{s+1}-1} \times$ &c. $= a^{\alpha} b^{\beta} c^{\gamma} d^{\delta}$ &c. &c.: ſi m, n, r, &c. ſint multo majores quam α, β, γ, &c.; tum quotiens erit prope $a^{\alpha} b^{\beta} c^{\gamma} d^{\delta}$ &c.

7. Sit $A = 1.2.3.4.5.6.7.8 \ldots (l-2).(l-1)$; ducatur hæc quantitas in l, & ſumma diviſorum quantitatis reſultantis non habet ad ſummam diviſorum quantitatis A majorem rationem quam $l+1 : 1$, non habet minorem rationem quam $l : 1$. Ducatur contentum (A) in l, & numerum (b) minorem quam l; tum ſumma diviſorum numeri $l \times A$ major erit quam ſumma diviſorum numeri $b \times A$: &c.

7.2. Sit $N = 1.2.3.4.5.6 \ldots l$, tum numerus N habet ad aggregatum e ſingulis ejus diviſoribus minorem rationem, quam habet quicunque numerus minor quam N ad aggregatum e ſingulis ejus diviſoribus.

8. Sit $a^x b^z = A$, & $(x+1) \times (z+1) =$ maxim. ubi a & b denotant invariabiles quantitates: tum erit $a^{x+1} = b^{z+1}$.

Inveniatur enim fluxio quantitatis $a^x b^z = A$, quæ erit $a^x b^z \times (\dot{x} \times \log. a - \dot{z} \times \log. b) = 0$; unde $\dot{z} = \frac{\log. a}{\log. b} \times \dot{x}$: at fluxio rectanguli $(\overline{x+1} \times \overline{z+1})$ erit nihilo æqualis, quoniam rectangulum ipſum eſt maximum, i. e. $x\dot{z} - z\dot{x} - \dot{x} + \dot{z} = 0$: in hâc æquatione pro $\dot{z}$ ſcribatur ejus valor prius traditus, reſultat $(x+1) \times \log. a = (z+1) \times \log. b$, & conſequenter $a^{x+1} = b^{z+1}$. q. e. d.

8. 2. Si vero plures dentur factores, i. e. $a^x b^z c^y d^v \times$ &c. $= A$; & requiratur, ut $\overline{x+1}.\overline{z+1}.\overline{y+1}.\overline{v+1}$. &c. evadat maxim. erunt $x+1 : z+1 :: \log. b : \log. a$; $x+1 : y+1 :: \log. c : \log. a$; $x+1 : v+1 :: \log. d : \log. a$, &c.; & conſequenter quantitates $x+1$, $z+1$,

$y+1$,

$y+1$, $v+1$, &c. erunt in ratione reciprocâ quantitatum log. a, log. b, log. c, &c.; & exinde $a^{x+1} = b^{z+1} = c^{y+1} = d^{v+1} =$ &c.

9. 1. Iiſdem poſitis, ex iiſdem principiis probari poteſt contentum $\frac{a^{x+1}-1}{a-1} \times \frac{b^{z+1}-1}{b-1} \times \frac{c^{y+1}-1}{c-1} \times \frac{d^{v+1}-1}{d-1} \times$ &c. eſſe maxim. cum $a^{x+1} = b^{z+1} = c^{y+1} = d^{v+1} =$ &c.

2. Sint a, b, c, d, &c. primi; & x, y, z, v, &c. integri numeri; tum erit $(x+1) \times (z+1) \times (y+1) \times (v+1) \times$ &c. numerus diviſorum, & $\frac{a^{x+1}-1}{a-1} \times \frac{b^{z+1}-1}{b-1} \times \frac{c^{y+1}-1}{c-1} \times$ &c. ſumma diviſorum quantitatis A: in hoc caſu forſan haud aſſumi poſſunt æquales quantitates a^{x+1}, b^{z+1}, c^{y+1}, d^{v+1}, &c. ſed eo major erit prædictus numerus & ſumma, quo propiores ad æqualitatem inter ſe accedant.

3. Si vero numeri a, b, c, d, &c. haud dentur; tum numerus & ſumma diviſorum erit major, cum a, b, c, d, &c. ſint prope inter ſe æquales, & major ſit numerus quantitatum a, b, c, d, &c.

10. Erit $Sn = S(n-1) + S(n-2) - S(n-5) - S(n-7) + S(n-12) + S(n-15) - S(n-22) - S(n-26) + S(n-35) + S(n-40) -$ &c.; ubi $S(n)$, &c. denotat ſummam diviſorum numeri n, &c. hìc ſigna $+$ & $-$ geminata terminos hujus progreſſionis alternatim afficiunt.

Omnes numeri in hâc formulâ $\frac{3z^2 \pm z}{2}$, ubi z eſt integer numerus, continentur.

Series terminat, cum terminus $S(x)$ evadat negativus vel o; ſi evadat $S(o)$, pro $S(o)$ ultimo ſeriei termino ſcribatur n. *Demonſtratio.*

Sit $z = xS1 + x^2S2 + x^3S3 + x^4S4 + x^5S5 +$ &c. unde $z = 1(x + x^2 + x^3 + x^4 + x^5 +$ &c.$) + 2(x^2 + x^4 + x^6 + x^8 +$ &c.$) + 3(x^3 + x^6 + x^9 + x^{12} +$ &c.$) + 4(x^4 + x^8 + x^{12} +$ &c.$) + 5(x^5 + x^{10} +$ &c.$) +$ &c. $= \frac{1 \times x}{1-x} + \frac{2 \times x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \frac{4x^4}{1-x^4} +$ &c., exinde $\frac{z\dot{x}}{x} = \frac{\dot{x}}{1-x} + \frac{2x\dot{x}}{1-x^2} + \frac{3x^2\dot{x}}{1-x^3} +$ &c., cujus fluens erit $\int \frac{z\dot{x}}{x} =$

X x log.

$\log.(1-x)+\log.(1-x^2)+\log.(1-x^3)+\&c. = \log.(1-x)\cdot(1-x^2)(1-x^3)(1-x^4)\,\&c. = \log.(1-x-x^2+x^5+x^7-x^{12}-\&c.)$, unde $z=\frac{x+2x^2-5x^5-7x^7+12x^{12}+15x^{15}-\&c.}{1-x-x^2+x^5+x^7-x^{12}-x^{15}+\&c.}$ $= xS1+x^2S2+x^3S3+x^4S4+\&c.$; ex quâ æquatione ſequitur $S1=1$, $S2=S1+2$, $S3=S2+S1$, $S4=S3+S2$, $S5=S4+S3-5$, &c.; & in genere $Sn=S(n-1)+S(n-2)-S(n-5)-S(n-7)+\&c.$

11. Sint a, b, c, d, e, f, &c. primi numeri; tum numerus integrorum numerorum, qui non dividi poſſunt per primos a, b, c, d, &c. erit ad numerum integrorum, qui dividi poſſunt per prædictos primos :: $1-\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}+\&c.\right)+\left(\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc}+\frac{1}{ad}+\frac{1}{bd}+\frac{1}{cd}+\frac{1}{ae}+\&c.\right)-\left(\frac{1}{abc}+\frac{1}{abd}+\frac{1}{acd}+\frac{1}{bcd}+\frac{1}{abe}+\&c.\right)+\left(\frac{1}{abcd}+\frac{1}{abce}+\frac{1}{acde}+\frac{1}{bcde}+\&c.\right)-\left(\frac{1}{abcde}+\frac{1}{abcdf}+\frac{1}{acdef}+\frac{1}{bcdef}+\&c.\right)+\left(\frac{1}{abcdef}+\&c.\right)-\&c. = P : 1-P.$

12. Sint a, b, c, d, e, &c. reſpective primi numeri 2, 3, 5, 7, 11, 13, &c. pro a ſcribatur 2, & ſit $p=\frac{1}{2}$; pro a & b ſcribantur 2 & 3 reſpective, & erit $p'=\left(1-\frac{1}{2}\right)\times\frac{1}{3}=\frac{1}{6}$; pro a, b & c ſcribantur 2, 3 & 5, & evadet $p''=\left(1-\left(\frac{1}{2}+\frac{1}{3}\right)+\frac{1}{2.3}\right)\times\frac{1}{5}=\frac{1}{15}$; pro a, b, c & d ſcribantur reſpective 2, 3, 5 & 7, & evadet $p'''=\left(1-\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{5}\right)+\left(\frac{1}{2.3}+\frac{1}{2.5}+\frac{1}{3.5}\right)-\frac{1}{2.3.5}\right)\times\frac{1}{7}=\frac{4}{105}$; & ſimiliter $p''''=\left(1-\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}\right)+\left(\frac{1}{2.3}+\frac{1}{2.5}+\frac{1}{3.5}+\frac{1}{2.7}\right.\right.$

i + 1

$+\frac{1}{3.7}+\frac{1}{5.7}\Big)-\Big(\frac{1}{2.3.5}+\frac{1}{2.3.7}+\frac{1}{2.5.7}+\frac{1}{3.5.7}\Big)+\frac{1}{2.3.5.7}\Big)\times\frac{1}{11}$ &c.; tum erit $1=p+p'+p''+p'''+$ &c. in infinitum.

13. Sit numerus $N=p^{\alpha}\times q^{\beta}\times r^{\gamma}\times s^{\delta}\times$ &c.; ubi p, q, r, s, &c. primos numeros reſpective denotant; tum erunt $(p^{\alpha}-p^{\alpha-1})\times(q^{\beta}-q^{\beta-1})\times(r^{\gamma}-r^{\gamma-1})\times(s^{\delta}-s^{\delta-1})\times$ &c. numeri minores quam N, & ad eum N primi.

14. Sit datus numerus $a+b$ & aſſumatur $10c+b=(10-\pi)\times m$ vel $100c+b=(100-\pi)\times m$, &c. ubi a, b, c, π, & m integri ſunt numeri; tum $a-10c+10m$, &c. dividi poteſt per π, ſi modo datus numerus $a+b$ per π dividi poſſit.

Pro b ſcribatur $\overline{10-\pi}\,m-10c$ & evadet $a+b=a-10c+10m-\pi m$, unde conſtat propoſ.: ſi pro π ſcribatur 7, fit methodus domini Kiraaſt.

15. Sit $\frac{f+e}{x}$ integer numerus; & m diviſor quantitatis (f), & $m-x=n$; tum $\frac{\frac{nf}{m}+e}{x}=\frac{\frac{\overline{m-x}f}{m}+e}{x}=\frac{f+e}{x}-\frac{f}{m}$ erit integer numerus.

Ex. 1. Sit numerus $fedcba$ multiplex numeri (x); ubi f, e, d, c, b, a denotant figuras in ſuis reſpectivis locis; dicatur $10-x=n$, ducatur $n\times f$ & addatur e; ſummæ $\begin{matrix}nf\\e\end{matrix}$ adjiciatur in directum $dcba$; & numerus $\begin{matrix}nf\\e\end{matrix}\,dcba$ erit multiplex numeri (x), quod docuit Fontinelle; & ſic $nfe\,dcba$, $nfed\,cba$, &c. erunt multiplices numeri (x): 2. ſit vero $100-x=n$ & numeri $n\times f\;edcba$, $nfe\;dcba$, $nfed\;cba$, &c. erunt multiplices numeri (x), ſi modo numerus $fedcba$ ſit multiplex numeri (x). Sit numerus $fedcb=41573$ multiplex numeri $(x=7)$

 &

& 10 — $x = 3$; ergo $\underset{edcb}{nf} = \underset{1573}{3 \times 4} = 13573$, $\underset{dcb}{nfe} = \underset{573}{3 \times 41} = 12873$, &c. Sit $100 - x = n = 93$, & $\underset{dcb}{nf} = \underset{1573}{93 \times 4} = 38773$, $\underset{dcb}{nfe} = \underset{573}{93 \times 41} = 38703$, &c. multiplices erunt numeri $x = 7$. Facile conſtat exemplum e ſcribendo pro n ejus valorem $10 - x$, vel $100 - x$, &c.

16. Sit quilibet numerus $fedcba$ dividendus per α: aſſumo $10 - \alpha$ vel $100 - \alpha = n$, &c. e. g. ſit $100 - \alpha$; tum operatio erit

$100 - n \,|\, fedcba \,|\, fedc$
$\quad\underline{fedc - n \times fedc}$
$\quad + fedc \times n + ba$ primum reſiduum, &c. ſic continuo repetitâ operatione ad veram quotientem continuâ approximatione accedere licet.

Ex. Sit 5430856 numerus per numerum 83 dividendus. In hoc caſu $100 - 83 = 17 = n$, tum operatio erit

prima quotiens	54308 . 56
$17 \times \frac{54308}{100} + 56 =$	9232 . 92
$17 \times \frac{9232}{100} + 92 =$	1570 . 36
$17 \times \frac{1570}{100} + 36 =$	267 . 26
$17 \times \frac{267}{100} + 26 =$	45 . 65
$17 \times \frac{45}{100} + 65 =$	8 . | 30
$17 \times \frac{8}{100} + 30 =$	1 . | 66
$\frac{17 \cdot 1}{100} + 66 =$	. 8 | 3
	&c.
Quotiens	65432

Hæc fuit methodus Joannis Andræ Caſtelvetri.

THEOR.

THEOR. XLVII.

1. Sint x & z quicunque integri numeri; & ſint a & b numeri, quorum nullus invenitur communis diviſor; tum $ax+bz$ conficere poteſt ullum numerum, qui ſuperat numerum $a\times b-a-b$.

2. Sit n integer numerus; tum $nax+nbz$ poteſt conficere quemlibet numerum per n diviſibilem, qui ſuperat $n\times a\times b-na-nb$.

3. Omnis integer numerus vel eſt triangulus; vel e duobus aut tribus triangulis compoſitus; i. e. conſtat ex uno, duobus, vel tribus ſubſequentibus numeris 1, 3, 6, 10, &c.

4. Omnis integer numerus eſt pentagonus; vel e duobus, tribus, quatuor vel quinque pentagonis compoſitus; i. e. conſtat ex uno, duobus, tribus, quatuor vel quinque ſubſequentibus numeris 1, 4, 10, 20, 35, &c. &c. conſtat etiam ex uno, duobus, tribus, quatuor, 5, 6, 7, vel octo hexagonis numeris 1, 5, 15, 35, 70, 126, 210, &c.

5. Omnis integer numerus eſt quadratus; vel e duobus, tribus vel quatuor quadratis compoſitus.

6. Quantitates a^2-b^2, vel a^2-b^2+2, ubi a & b vel ſunt integri numeri vel o; poſſunt conficere quemlibet numerum.

7. Quantitas $pa^2+qb^2+rc^2+sd^2$; ubi p, q, r & s ſunt numeri inter ſe primi; & a, b, c & d ſunt quilibet integri numeri vel o: vel magis generaliter $pa^2+ba+qb^2+kb+rc^2+lc+sd^2+md+n+\alpha ab+\beta ac+\gamma ad+\delta bc+\varepsilon bd+\zeta cd$, ubi $b, k, l, m, n, \alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ ſunt etiam integri numeri, poteſt conficere quemlibet numerum, qui ſuperat datum numerum inveſtigandum.

8. Quantitas $(a^2\pm ab+b^2)+(p^2\pm pq+q^2)$; ubi a & b, p & q ſunt integri numeri, poteſt æquari cuicunque numero.

9. Omnis integer numerus vel eſt cubus; vel e duobus, tribus, 4, 5, 6, 7, 8, vel novem cubis compoſitus: eſt etiam quadrato-quadratus; vel e duobus, tribus, &c. uſque ad novemdecim compoſitus, & ſic deinceps: conſimilia etiam affirmari poſſunt (exceptis excipiendis) de eodem numero quantitatum earundem dimenſionum.

10. Omnis

10. Omnis numerus $4n+2$ diviſibilis per duo, non autem per quatuor, componitur e duobus vel tribus quadratis.

11. In genere omnis integer numerus (adhibitis limitibus e quantitatibus datis exortis) compoſitus eſt e quodam finito numero quantitatum hujuſce generis $ax^m + bx^n + cx^r +$ &c. vel e quibuſdam numeris quantitatum $ax^m + bx^n + cx^r +$ &c. & $ex^\alpha + fx^\beta + gx^\gamma +$ &c. &c. ubi m, n, r, &c.; α, β, γ, &c.; a, b, c, &c.; e, f, g, &c. integros denotant numeros.

Cavendum autem eſt, ne prædictos limites pro nihilo habeamus: forſan prædictæ quantitates haud incipiunt ab unitate; & inveſtigandus eſt limes a quo generaliter incipiunt: & forſan ſunt quantitates hujuſce generis $3x^4 + 6x^3 + 24$, ubi omnes quantitates diviſibiles erunt per 3; & conſequenter nullum poſſunt conficere numerum, præter eos, qui dividi poſſint per numerum 3, &c.

11. 2. Conſimilibus principiis inveniri poſſunt omnes numeri, quos conficere poſſunt quæcunque quantitates dati generis; e. g. quos conficere poteſt quantitas $\overline{ax+e}^2 + \overline{bx+f}^2 + \overline{cx+g}^2 + \overline{dx+e}^2$.

In omnibus hiſce caſibus ſæpe diverſis modis componi poſſunt numeri; e. g. dantur numeri, qui componi poſſunt pluribus modis ex uno vel duobus vel tribus vel quatuor quadratis.

Facile adjici poſſunt conſimiles propoſitiones de fractionalibus & irrationalibus quantitatibus.

THEOR. XLVIII.

Sit $N = a^2 + mb^2$, & $M = p^2 + mq^2$, ubi a, b, p, q & m integros denotant numeros; tum erit $N \times M = \overline{ap + mbq}^2 + m \times \overline{aq - bp}^2 = \overline{ap - mbq}^2 + m \times \overline{aq + bp}^2$.

THEOR. XLIX.

Sit $N = a^2 + rb^2$, & N^{2m+1} & N^{2m+2} componi poſſunt $(m+1)$ diverſis modis e quantitatibus hujuſce generis $p^2 + rq^2$.

Hæ

Hæ autem quantitates deſignari poſſunt per hanc methodum, viz. $N^{2m} = 4ra^2b^2 \times \overline{a^2+rb^2}^{2m-2} + \overline{a^2-rb^2}^{2} \times \overline{a^2+rb^2}^{2m-2}$ = generaliter

$$\frac{\overline{\overline{a+\sqrt{-r}b}^{2l} + \overline{a-\sqrt{-r}b}^{2l}}^{2}}{4} \times \overline{a^2+rb^2}^{2m-2l} + \frac{\overline{\overline{a+\sqrt{-r}b}^{2l} - \overline{a-\sqrt{-r}b}^{2l}}^{2}}{4} \times \overline{\sqrt{-1}}^{2} \times \overline{a^2+rb^2}^{2m-2l},$$

ubi l eſt quicunque integer numerus æqualis vel minor quam m: & $N^{2m+1} = a^2 \times \overline{a^2+rb^2}^{2m} + rb^2 \times \overline{a^2+rb^2}^{2m}$

$$= \text{generaliter } \frac{\overline{\overline{a+\sqrt{-r}b}^{2l+1} + \overline{a-\sqrt{-r}b}^{2l+1}}^{2}}{4} \times \overline{a^2+rb}^{2m-2l} + \frac{\overline{\overline{a+\sqrt{-r}b}^{2l+1} - \overline{a-\sqrt{-r}b}^{2l+1}}^{2}}{4} \times \overline{\sqrt{-1}}^{2} \times \overline{a^2+rb^2}^{2m-2l}.$$

Hoc theorema demonſtrari poteſt ex hâc propoſitione facile deducendâ; nempe erit $\overline{a+\sqrt{-r}b}^{l} = a^l + l\sqrt{-r}\,a^{l-1}b - l \times \frac{l-1}{2} r a^{l-2}b^2 - l \cdot \frac{l-1}{2} \cdot \frac{l-2}{3} r\sqrt{-r}\,a^{l-3}b^3 + l \cdot \frac{l-1}{2} \cdot \frac{l-2}{3} \cdot \frac{l-3}{4} r^2 a^{l-4}b^4 + l \cdot \frac{l-1}{2} \cdot \frac{l-2}{3} \cdot \frac{l-3}{4} \cdot \frac{l-4}{5} r^2\sqrt{-r}\,a^{l-5}b^5 -$ &c.: tum $\frac{\overline{a+\sqrt{-r}b}^{l} + \overline{a-\sqrt{-r}b}^{l}}{2} = a^l - l \cdot \frac{l-1}{2} r a^{l-2}b^2 + l \cdot \frac{l-1}{2} \cdot \frac{l-2}{3} \cdot \frac{l-3}{4} r^2 a^{l-4}b^4 -$ &c. $= P$, & $\frac{\overline{a+\sqrt{-r}b}^{l} - \overline{a-\sqrt{-r}b}^{l}}{2\sqrt{-1}} = \sqrt{r} \times (l a^{l-1}b - l \cdot \frac{l-1}{2} \cdot \frac{l-2}{3} r a^{l-3}b^3 + l \cdot \frac{l-1}{2} \cdot \frac{l-2}{3} \cdot \frac{l-3}{4} \cdot \frac{l-4}{5} r^2 a^{l-5}b^5 -$ &c.$) = \sqrt{r} \times Q$; unde $P^2 + rQ^2 = a^{2l} + lra^{2l-2}b^2 + l \cdot \frac{l-1}{2} r^2 a^{2l-4}b^4 +$ &c. $= \overline{a^2+rb^2}^{l}$.

Cor.

Cor. 1. Hinc conſtat, ſi numerus N componatur (n) diverſis modis e prædictis quantitatibus; ejus quadratum etiam componi (n) diverſis modis, ejus cubum & quadrato-quadratum $(2n)$ diverſis modis ex hiſce quantitatibus, & ejus $2m+1$ vel $2m+2$ poteſtatem $(\overline{m+1} \times n)$ diverſis modis e quantitatibus formulæ p^2+rq^2 componi: in quibuſdam autem caſibus duo vel plures valores fiunt inter ſe æquales.

Cor. 2. Sint N, M, R, S, &c. numeri, e quantitatibus prædictæ formulæ a^2+mb^2 (modis n, r, s, t, &c. diverſis) compoſiti; tum numerus $N^\alpha \times M^\beta \times R^\gamma \times S^\delta \times$ &c. (modis $\frac{\alpha+1}{2} \times (\beta+1) \times (\gamma+1) \times (\delta+1) \times$ &c. $\times n \times r \times s \times t \times$ &c. diverſis,) e quantitatibus prædictis p^2+mq^2 componitur. Si vero ullus factor $\alpha+1$ ſit impar, tum pro iſto factore ſcribatur $\alpha+2$; & ſic de reliquis.

Conſtat ex his duobus theorematibus conjunctim conſideratis.

Hic excipiendi ſunt ii caſus, in quibus duo vel plures valores fiunt inter ſe æquales.

Cor. 3. Hinc inveniri poſſunt numeri, qui e quantitatibus hujuſmodi $(p^2+m\cdot q^2)$, quoties (n) quis velit, componuntur. Sit enim $2n=(\alpha+1)\times(\beta+1)\times(\gamma+1)\times$&c. ſi α, β, γ, δ, &c. ſint impares numeri: ſin unus α ſit par, tum pro $\alpha+1$ ſcribatur $\alpha+2$, &c. & aſſumantur numeri N, M, R, S, &c. ſolummodo e duobus quadratis compoſiti; tum $N^\alpha M^\beta R^\gamma S^\delta$ &c. erit numerus quæſitus. Et ſic generalior reddi poteſt ſolutio.

Hinc etiam deduci poteſt, quoties datus numerus e duobus quadratis componitur: ſit $N=a\times b\times c\times d\times$ &c. & cognoſcatur quoties a, b, c, d, &c. conſtant e duobus quadratis, ergo inveniri ex hoc theoremate poteſt, quoties numerus N conſtat e duobus quadratis.

Cor. 4. Sit numerus $a^2+mb^2=A=p\times q\times r\times s\times t\times$ &c. ubi p, q, r, s, t, &c. ſint primi numeri, & $p=\alpha^2+m\beta^2$; tum erit $q\times r$ $s\times$&c. $=\gamma^2+m\delta^2$; & ſi A ſit primus numerus, tum non erit $A=ma^2+b^2=mc^2+d^2$.

Enim

Enim $(a^2 - c^2)\,A = a^2(mc^2 + d^2) - c^2(ma^2 + b^2) = a^2d^2 - b^2c^2 = (ad + bc) \times (ad - bc)$; ſi A ſit primus, tum is continetur vel in factore $ad + bc$ vel in factore $ad - bc$; at $2A = ma^2 + b^2 + mc^2 + d^2$, & $2A - 2ad - 2bc = \overline{m-1}\,a^2 + \overline{m-1}\,c^2 + a^2 + c^2 + b^2 + d^2 - 2ad - 2bc = (m-1) \times (a^2 + c^2) + \overline{a-d}^2 + \overline{b-c}^2$; ſed hæc quantitas ob m haud minorem quam 1 erit affirmativa, ergo A major erit quam $ad + bc$, & conſequenter in eo non continetur.

2. Sit $mp^2 + q^2 = P$ primus numerus diviſor numeri $ma^2 + b^2 = N$; quotiens $\frac{ma^2 + b^2}{mp^2 + q^2}$ eandem habet formulam. Enim $p^2(ma^2 + b^2) - a^2(mp^2 + q^2) = p^2b^2 - a^2q^2 = (pb + aq) \times (pb - aq)$ per P diviſibilis erit; unde $pb + aq$ vel $pb - aq$ eſt diviſibilis per primum numerum P, & exinde $aq \pm bp = rP = mrp^2 + rq^2$; hinc $a = \frac{p(mrp \mp b)}{q} + rq$; unde $\frac{p(mrp \mp b)}{q}$, & conſequenter $\frac{mrp \mp b}{q}$ eſt integer numerus; ponatur $mrp \mp b = \mp sq$, & exinde $b = sq \pm mrp$, & $a = \mp sp + rq$, & $N = ma^2 + b^2 = (mp^2 + q^2) \times (mr^2 + s^2)$, & conſequenter quotiens $\frac{ma^2 + b^2}{mp^2 + q^2} = mr^2 + s^2$.

Erit $m \times (a^2 + mb^2) = m^2b^2 + ma^2 = c^2 + md^2$.

THEOR. L.

Sit $(a^2 + b^2 + c^2 + d^2)\,(p^2 + q^2 + r^2 + s^2) = x^2 + y^2 + z^2 + v^2$ ubi a, b, c, d & p, q, r, s ſunt integri numeri; tum poſſunt eſſe integri valores quantitatum x, y, z & v reſpective; viz. $x = ap + bq + cr + ds$, $y = aq - bp \pm cs \mp dr$, $z = ar \mp bs - cp \pm dq$, $v = as \pm br \mp cq - dp$.

Conſimilia etiam applicari poſſunt ad plures hujuſmodi quantitates.

THEOR. LI.

Erit $(a^2+ab+b^2)\times(c^2+cd+d^2)=\overline{ac+ad+bd}^2+(ac+ad+bd)\times(bc-ad)+\overline{bc-ad}^2=\overline{ac+bd+bc}^2+(ac+bd+bc)\times(ad-bc)+\overline{ad-bc}^2=\overline{ac+ad+bc}^2+(ac+ad+bc)\times(bd-ac)+\overline{bd-ac}^2=\overline{ad+bd+bc}^2+(ad+bd+bc)\times(ac-bd)+\overline{ac-bd}^2=P.$

Cor. 1. Hinc, ſi modo ducatur quantitas formulæ $p^2\pm pq+q^2$ (ubi p & q vel $=0$ vel integri ſunt numeri) in diverſam quantitatem ejuſdem formulæ; tum reſultat quantitas ejuſdem formulæ.

Cor. 2. Rectangulum $(a^2+ab+b^2)\times(c^2+cd+d^2)$ componitur duobus diverſis modis e quantitatibus hujuſce formulæ p^2+pq+q^2; duobus etiam diverſis modis e quantitatibus hujuſce formulæ p^2-pq+q^2 (ubi p & q vel $=0$ vel ſunt integri numeri); nam differentiæ $bc-ad$ & $ad-bc$, $bd-ac$ & $ac-bd$ ſunt inter ſe reſpective æquales & negativæ.

Cor. 3. Hinc conſtat, ſi numeri N & M componantur (n & m) diverſis modis e quantitatibus formulæ p^2+pq+q^2, numerum $N\times M$ componi $2n\times m$ diverſis modis e quantitatibus prædictæ formulæ.

Omnes numeri, qui continentur in formulâ p^2+pq+q^2, etiam continebuntur in formulâ p^2-pq+q^2; & vice verſâ omnes numeri, qui continentur in formulâ p^2-pq+q^2, etiam in formulâ p^2+pq+q^2 continebuntur; ſed numeri in formulâ p^2-pq+q^2 bis, cum numeri in formulâ p^2+pq+q^2 ſemel occurrant. Conſtat e theorem. ſubſequente.

Cor. 4. Omnes diviſores numerorum hujuſce formulæ $p^2\pm pq+q^2$; ubi p & q ſunt numeri inter ſe primi; erunt etiam numeri ejuſdem formulæ.

Probari poteſt hoc coroll. per methodum ei in theor. 49. cor. 4. traditæ conſimilem.

Cor.

Cor. 5. Summa duorum cuborum $a^3 + b^3$ habet diviſores, qui componuntur e diviſoribus numeri $a + b$, & numeris in formulâ $p^2 \pm pq + q^2$ contentis. In his caſibus ſint a & b inter ſe primi.

Cor. 6. Eadem applicari poſſunt ad quantitates hujuſce formulæ $a^2 + mab + nb^2$; facile enim reduci poſſunt in quantitates præcedentis formulæ; & ſic de pluribus hujuſcemodi quantitatibus.

THEOR. LII.

1. Sint n, a, α & m integri numeri; tum reſiduum ex diviſione quantitatis $\overline{na + \alpha}^m = n^m a^m + m \times n^{m-1} a^{m-1} \alpha + m \cdot \frac{m-1}{2} n^{m-2} a^{m-2} \alpha^2 + \ldots + mna\alpha^{m-1} + \alpha^m$ per n, idem erit ac reſiduum quantitatis α^m per n. Sit m impar numerus & α negativa quantitas; tum, ſi poteſtas α^m per n diviſa relinquit l, $\overline{na - \alpha}^m$ per n diviſa, relinquit $n - l$: & ſic $\overline{\alpha - na}^m$ per n diviſa, idem habebit reſiduum ac α^m per n diviſa.

2. Sit n primus & a integer numerus; ſi poteſtas a^n per n diviſa habeat reſiduum l, tum $\overline{a + 1}^n = a^n + na^{n-1} + n \cdot \frac{n-1}{2} a^{n-2} + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} a^{n-3} \ldots + n \cdot \frac{n-1}{2} a^2 + na + 1$ per n diviſa, habet idem reſiduum ac numerus $a^n + 1$ per n diviſus, & conſequenter ejus reſiduum erit $l + 1$.

Cor. Hinc reſiduum e diviſione numeri a^n per n idem erit ac reſiduum numeri a per n; & conſequenter $\frac{a^n - a}{n}$ erit integer numerus; unde, ſi a haud dividi poteſt per n (quoniam $\frac{a^n - a}{n} = a \times \frac{a^{n-1} - 1}{n}$) $\frac{a^{n-1} - 1}{n}$ erit integer numerus: aliter.

Sit n primus & a integer numerus, tum $a^{n-1} - 1$ ſemper

dividi poteſt per n, ni a per n dividi poſſit. Si enim $a^{n-1}-1$ dividi poſſit per n, tum a^n-a; ſi vero a^n-a, tum $\overline{a+1}^n-a-1$ dividi poteſt per n: enim $\overline{a+1}^n=a^n+na^{n-1}+n\times\frac{n-1}{2}a^{n-2}\ldots na+1$; cujus ſeriei ſinguli termini (modo n ſit primus numerus) præter primum & ultimum per n dividi poſſunt, ergo $\overline{a+1}^n-a^n-1$ diviſionem per n admittet; ſed a^n-a diviſionem per n admittet, ergo $\overline{a+1}^n-a-1$; & exinde, ſi vera ſit propoſitio, cum a ſit quidam numerus, vera etiam erit, quicunque numerus ſit a.

Cor. 1. Sit n primus numerus, & $\frac{a^{n-1}-b^{n-1}}{n}=\frac{a^{n-1}-1}{n}-\frac{b^{n-1}-1}{n}$ erit integer numerus, cum a & b per n dividi nequeant.

Cor. 2. Sint m, n, p, q, &c. numeri primi & inæquales, ſitque A minimus communis dividuus eorum unitate diminutorum, puta ipſorum $m-1, n-1, p-1, q-1$, &c. tum a^A diviſa per $mnpq$ &c. 1 relinquit: niſi a dividi poſſit per aliquem horum numerorum m, n, p, q, &c. Literis enim a, b, c, d, &c: α integros numeros denotantibus, erit per theor. $a^{m-1}=mb+1$, unde reſiduum ex diviſione quantitatis

$$a^{(m-1)\times\alpha}=\overline{mb+1}^{\alpha}=m^{\alpha}b^{\alpha}+\alpha m^{\alpha-1}b^{\alpha-1}+\alpha\cdot\frac{\alpha-1}{2}m^{\alpha-2}b^{\alpha-2}+\text{\&c.}$$

$$\ldots+\alpha\cdot\frac{\alpha-1}{2}m^2b^2+\alpha mb+1$$ per m erit 1; ſit $\alpha=n-1$, & reſiduum ex diviſione quantitatis $a^{(m-1)\times(n-1)}=\overline{mb+1}^{n-1}=ne+1$ per n erit 1, ni $a^{m-1}=mb+1$ dividi poteſt per n; unde $a^{(m-1)\times(n-1)}-1$ dividi poteſt per m & n; & quoniam m & n ſunt inter ſe primi, dividi etiam poteſt per $m\times n$; & conſequenter $a^{(m-1)\times(n-1)}$ diviſa per $m\times n$ relinquit 1. Et ſimiliter eadem methodus ratiocinandi applicari poteſt ad plures factores $p-1$, $q-1$, &c.

Cor. 3. Sit n primus numerus; & omnis potentia $\frac{a^{n-1}}{n}\times\frac{a^{n-1}}{n}\times$

a^{n-1}

$\frac{a^{n-1}}{n}$ &c. $= \frac{a^{m.\overline{n-1}}}{n^m}$, cujus exponens eſt $m.\overline{n-1}$ diviſa per n^m, vel o vel 1 relinquit.

Cor. 4. Sit $2n+1$ primus & p integer numerus; tum $\frac{2^n+1}{2n+1}$ erit integer numerus, ſi $n = 4p+1$ vel $4p+2$; ſin aliter, $\frac{2^n-1}{2n+1}$ erit integer numerus: erit $\frac{3^n+1}{2n+1}$, ſi $n = 6p+2$ vel $6p+3$; ſin aliter $\frac{3^n-1}{2n+1}$ erit integer numerus. Nam $a^{2n}-1$ ſemper diviſibilis erit per $2n+1$, ni a diviſibilis ſit per $2n+1$; ſed $a^{2n}-1 = (a^n+1) \times (a^n-1)$ erit diviſibilis per $2n+1$: utrum factor a^n+1 vel a^n-1 dividi poteſt per $2n+1$, facile erui poteſt.

Cor. 5. Sit p maximus communis diviſor quantitatum m & $2n$, tum a^p-b^p diviſibilis erit per $2n+1$, ſi a^m-b^m diviſibilis ſit per $2n+1$ primum numerum. Utræque enim quantitates a^m-b^m & $a^{2n}-b^{2n}$ recipiunt maximum communem diviſorem a^p-b^p; ergo, ſi $2n+1$ dividat utraſque quantitates a^m-b^m & $a^{2n}-b^{2n}$, primus numerus $2n+1$ dividet earum maximum communem diviſorem a^p-b^p.

Facile conſtat ex operatione in prob. 47. traditâ, i.e. ex inveniendo communes diviſores quantitatum a^m-b^m & $a^{2n}-b^{2n}$.

Cor. 6. Sint reſidua e diviſione numeri A per numeros n, m, r, s, t, &c. numeri reſpectivi α, β, γ, δ, &c., i.e. $nx+\alpha = A$, $my+\beta = A$, $rv+\gamma = A$, $sw+\delta = A$, &c.; ubi x, y, v, w, &c. denotant integros numeros; ex æquatione $nx+a = my+\beta$ per prob. 52. inveniantur integri valores X & Y quantitatum x & y in terminis quantitatis π: deinde æquentur duo numeri $nX+\alpha = rv+\gamma$; & ex prædicto problemate inveniantur integri valores X' & V quantitatum X & v in terminis quantitatis ϱ; deinde æquentur duo numeri $nX'+\alpha = sw+\delta$; & ex æquatione reſultante inveniantur X'' & W integri valores quantitatum X' & w in terminis quantitatis σ; & ſic deinceps:

ceps: hinc innoteſcent integri valores quantitatum x, y, z, v, &c. in terminis quantitatis σ, &c.

Cor. 7. Summa duorum quadratorum $a^2 + b^2$ hujuſce formulæ per nullum primum numerum hujus formæ $4n - 1$ dividi poteſt, niſi utriuſque radix a & b ſit diviſibilis per $4n - 1$.

Differentia enim $a^{4n-2} - b^{4n-2}$ per $4n - 1$ dividi poteſt, unde $a^{4n-2} - b^{4n-2} + 2b^{4n-2} = a^{4n-2} + b^{4n-2}$ per $4n - 1$ non dividi poteſt; at $a^{4n-2} + b^{4n-2}$ per $a^2 + b^2$ dividi poteſt, ergo $a^2 + b^2$ per $4n - 1$ non dividi poteſt.

2. Et ex iiſdem principiis probari poteſt ſummam $a^{2^m} + b^{2^m}$ non alios admittere diviſores, præter eos qui continentur in hâc formâ $2^{m+1} n + 1$.

Cor. 8. $\overline{c^n + ap}^{\frac{p-1}{n}} = c^{p-1} + \frac{p-1}{n} c^{\frac{p-1}{n} \times n-1} ap +$ &c. per p primum numerum diviſâ relinquit o vel 1; in hoc & ſubſequentibus cor. α, a, c & $\frac{p-1}{n}$ integri ſint numeri.

Cor. 9. Sit $ab^n = c^n \pm ap$ & p primus numerus, tum $a^{\frac{p-1}{n}}$ per p diviſa relinquet o vel unitatem: nam $\overline{c^n \pm \alpha p}^{\frac{p-1}{n}} = a^{\frac{p-1}{n}} b^{p-1}$, & $\overline{c^n \pm \alpha p}^{\frac{p-1}{n}}$ & b^{p-1} per p diviſæ relinquunt unitatem, unde $a^{\frac{p-1}{n}} - 1$ diviſibilis eſt per p: ſit $p = m \times n + 1$, tum $a^m - 1$ diviſibilis erit per p, cum $ab^n - c^n$ diviſibilis ſit per p.

Cor. 10. Si formula $a^m - 1$ diviſibilis ſit per primum numerum $p = mn + 1$; tum ita aſſumi poſſunt numeri x & y, ut $ax^n - y^n$ ſit per primum numerum p diviſibilis; cum enim x^{mn} & y^{mn} per p diviſæ relinquant unitatem, formula $a^m x^{mn} - y^{mn}$ per p erit diviſibilis, dummodo neque x neque y per p ſit diviſibilis; cum enim $a^m x^{mn} - y^{mn} = (ax^n - y^n)(a^{m-1} x^{mn-n} + a^{m-2} x^{mn-2n} y^n + a^{m-3} x^{mn-3n} y^{2n} + \ldots y^{mn-n})$; ſi prior factor nunquam, tum poſterior ſemper diviſibilis erit per p: in poſteriore factore $a^{m-1} x^{mn-n} + a^{m-2} x^{mn-2n} y^n + \ldots y^{mn-n}$ pro y ſcribantur reſpective 1, 2, 3, 4, &c., $p - n(mn - n + 1)$; quantitatum reſultantium in-

inveniantur differentiæ, ita ut exterminentur quantitates a & x, & ultima reſultans differentia erit $1.2.3.4.5 \ldots (mn - n)$; ſed quoniam omnes quantitates reſultantes ſunt diviſibiles per p, ergo omnes earum differentiæ, & conſequenter ultima prædicta $1.2.3.4 \ldots (mn - n)$; ſed hoc contentum non diviſibile eſt per $mn+1$; ergo poſterior factor non eſt ſemper diviſibilis per p, & conſequenter datur caſus, in quo $ax^n - y^n$ diviſibilis erit per p.

THEOR. LIII.

1. Sint reſidua e diviſione numerorum A, B, C, &c. per numerum a reſpective α, β, γ, &c., i. e. ſint $A = ba + \alpha$, $B = ka + \beta$, $C = la + \gamma$, &c; ubi b, k, l, &c. ſunt integri numeri; tum reſiduum e diviſione numeri $A \times B \times C \times$ &c. $= (ba + \alpha) \times (ka + \beta) \times (la + \gamma) \times$ &c. per a idem erit ac reſiduum e diviſione numeri $\alpha \times \beta \times \gamma \times$ &c. per a.

2. Et vice verſâ ſint reſidua e diviſione numerorum A & $A \times B$ per a reſpective α & $\alpha \times \beta$, ubi α ſit primus ad numerum a; tum reſiduum e diviſione numeri B per a erit β. Sint enim reſidua e diviſione numerorum A & B per a reſpective α & γ; tum erit $\alpha \times \gamma$ reſiduum e diviſione producti $A \times B$ per a; ſed per hypotheſin hoc reſiduum erit $\alpha \times \beta$; ergo $\alpha \times \gamma = m \times a + \alpha\beta$, unde $\gamma = \frac{ma}{\alpha} + \beta$; ſed, quoniam α & a ſunt inter ſe primi numeri, & $\frac{ma}{\alpha}$ eſt integer numerus, erit α diviſor quantitatis m, ergo reſiduum $\gamma = \beta$.

Cor. 1. Sint reſidua ex diviſione numerorum A, B, C, &c. per n reſpective 1, tum reſiduum ex diviſione contenti $A \times B \times C \times$ &c. per n erit 1.

Cor. 2. Sit q primus numerus, & a^q per primum numerum p diviſus relinquat unitatem; tum a^q erit minima poteſtas quantitatis a, quæ per p diviſa relinquit unitatem; ni numerus a per p diviſus relinquit unitatem. Si enim a^q haud ſit minima poteſtas, ſit a^r minima poteſtas

poteſtas quantitatis a, quæ per n diviſa, relinquet 1; & ſint reſpectiva reſidua e diviſione numerorum $a, a^2, a^3, \ldots a^r$ per n reſpective $\alpha, \beta, \gamma, \delta$, &c.... 1; deinde reſidua e diviſione numerorum $a^{r+1} = a^r \times a$, a^{r+2}, a^{r+3}, &c.... a^{2r} erunt reſpective $\alpha, \beta, \gamma, \delta$, &c. 1; unde reſidua e diviſione numerorum $a^r, a^{2r}, a^{3r}, a^{4r}$, &c. per n erunt 1; ſed in hâc ſerie numerorum $r, 2r, 3r, 4r$, &c. haud datur primus numerus niſi r, & exinde r & q erunt inter ſe æquales.

3. Sint $\alpha, \beta, \gamma, \delta$, &c. reſidua e diviſione numerorum A, B, C, D, &c. per n; tum erunt α^m; $\alpha\beta, \alpha\gamma, \alpha\delta, \beta\gamma, \beta\delta, \gamma\delta$, &c.; $\alpha\beta\gamma, \alpha\beta\delta$, &c. & in genere $\alpha^b \beta^k \gamma^l$ &c. reſidua e diviſione numerorum formulæ $A^b B^k C^l$ &c. per n: ubi A, B, C, &c. ſint dati; & b, k, l, &c. quicunque integri numeri: vel b, k, l, &c. ſint dati; & A, B, C, &c. quicunque integri numeri; ſint π, ρ, σ, &c. non reſidua e diviſione quantitatum prædictæ formulæ $A^b B^k C^l$ &c. per primum numerum n; tum erunt $\alpha \times \pi, \beta \times \pi, \gamma \times \pi$, &c.; $\alpha \times \rho, \beta \times \rho, \gamma \times \rho$, &c.; $\alpha \times \sigma, \beta \times \sigma, \gamma \times \sigma$, &c.; &c.; etiam non reſidua.

Cor. 1. Hinc numerus non reſiduorum e diviſione numerorum 1^m, $2^m, 3^m, 4^m$, &c. in infinitum, vel numerorum $1, a, a^2, a^3, a^4$, &c. in infinitum, vel generaliter omnium numerorum formulæ $A^b B^k C^l$ &c. per primum numerum p vel erit (0) vel n vel $2n$ vel $3n$ vel $4n$, &c., ubi n eſt numerus reſiduorum ex eâdem diviſione exortorum.

Cor. 2. Sit p primus numerus & a minima poteſtas quantitatis a, quæ per p diviſa relinquit unitatem; & ſit q minor quam $p-1$ & haud major erit quam $\frac{p-1}{2}$; ſit minor quam $\frac{p-1}{2}$ & haud major erit quam $\frac{p-1}{3}$; ſit minor quam $\frac{p-1}{3}$ & haud major erit quam $\frac{p-1}{4}$; & ſic deinceps.

Cor. 3. Sint $\alpha, \beta, \gamma, \delta$, &c. reſidua e diviſione quadratorum 1, 4, 9, 16, 25, &c. per primum numerum p, & (π) non reſiduum; tum $\alpha\pi$, $\beta\pi, \gamma\pi, \delta\pi$, &c. erunt omnia non reſidua; & productum e quibuſ-

que duobus non reſiduis, viz. $\alpha\pi \times \beta\pi = \alpha\beta \times \pi^2$ erit reſiduum, ubi $\alpha\beta$ & π^2 ſunt reſidua.

4. Sint reſidua e diviſione numerorum $1^m, 2^m, 3^m, 4^m, 5^m$, &c. per n reſpective $\alpha, \beta, \gamma, \delta$, &c. ubi m eſt impar numerus; tum reſidua erunt α & $n-\alpha$; β & $n-\beta$; γ & $n-\gamma$; &c. i. e. α & $-\alpha$, β & $-\beta$, γ & $-\gamma$, &c. inter ſe complementi. Reſidua enim ex diviſione numeri $(kn-\alpha)^m$ per n, erunt α^m & $-\alpha^m$, &c.

Sint $\alpha, \beta, \gamma, \delta$, &c. reſidua e diviſione numerorum $1, a, a^2, a^3$, &c. in infinitum per primum p; & ſi occurrat unus complementus α, tum ſemper occurrent complementi: ſi enim duo complementi α & $-\alpha$ occurrant; tum occurret etiam $\frac{-\alpha}{\alpha} = -1$, & conſequenter complementi $-1 \times \alpha = -\alpha$, $-1 \times \beta = -\beta$, $-\gamma$, $-\delta$, &c.

Sit m par & p primus numerus, & ſi unus occurrat complementus α & $-\alpha$ e diviſione numerorum $1^m, 2^m, 3^m, 4^m, 5^m$, &c. per primum numerum p, tum ſemper occurrent complementi; idem etiam aſſeri poteſt, ſi unus complementus occurrat e diviſione $A^h B^k C^l$ &c. per primum numerum p; ubi A, B, C, &c. ſint dati; & h, k, l, &c. quicunque integri numeri; vel h, k, l, &c. dati, & A, B, C, &c. quicunque integri numeri.

Hæc conſtant e principiis in priori caſu traditis.

THEOR. LIV.

1. Sit r maximus communis diviſor numerorum m & $n-1$, ubi n eſt primus numerus; tum erit $\frac{n-1}{r}+1$ numerus reſiduorum e diviſione numerorum $1^m, 2^m, 3^m, 4^m, 5^m, 6^m$, &c. in infinitum per n, quorum duo ſemper erunt 0 & 1.

Numerus reſiduorum, quæ ſunt eadem, e diviſione numerorum $1^r, 2^r, 3^r, 4^r, \ldots \overline{n-1}^r$ per primum numerum (n) non poteſt eſſe major quam r.

2. Sit $a^l b^k c^i$ &c. $\times B = a^\pi b^\rho c^\sigma$ &c. $\times \frac{a^\pi b^\rho c^\sigma \text{ \&c.} - 1}{2} \times \frac{a^\pi b^\rho c^\sigma \text{ \&c.} - 2}{3}$ $\ldots \frac{a^\pi b^\rho c^\sigma \text{ \&c.} - b + 1}{b}$; ubi a, b, c, &c. primi, & B primus ad numeros a, b, c, &c.; etiamque π, ρ, σ, &c. dati ſint numeri, & b integer minor quam $a^\pi b^\rho c^\sigma$ &c. $= L$: ſint l', k', i', &c. valores quantitatum l, k, i, &c. (quibus reſpective correſpondeant valores b', b'', b''', &c. quantitatis b) iſtiuſmodi, ut $l' + \pi' \times (a^\pi - b') = p$, $k' + \rho' \times (a^\rho - b'') = q$, $i' + \sigma' \times (a^\sigma - b''') = r$, &c. ubi π', ρ', σ', &c. ſunt dati numeri, evadant minimi: tum haud plura poſſunt eſſe diverſa reſidua e diviſione numerorum 1^m, 2^m, 3^m, 4^m, 5^m, &c. ubi $m = a^\pi b^\rho c^\sigma$ &c. per numerum $a^p b^q c^r$ &c., quam $a^{\pi'} b^{\rho'} c^{\sigma'}$ &c. -1: idem etiam aſſeri poteſt de diviſione prædictorum numerorum per $a^{p'} b^{q'} c^{r'}$ &c. ſi modo p', q', r', &c. ſint reſpective non majores quam p, q, r, &c.

3. Sit numerus $A = abcd$ &c., ubi a, b, c, d, &c. ſunt inter ſe primi, & ſit numerus reſiduorum e diviſione per quantitates a, b, c, d, &c. reſpective m, n, r, s, &c. tum haud plura poſſunt eſſe reſidua e diviſione per A quam $mnrs$, &c.; frequenter autem quædam ex his fiant bis, ter, &c. inter ſe æqualia.

Sint enim α & β reſpective reſidua e diviſione per quantitates a & b, & x & y integri numeri quæſiti; aſſumatur $ax + \alpha = by + \beta = H$, ſint X & Y correſpondentes valores quantitatum (x & y) acquiſiti; tum per prob. 52. erunt $b\pi + X$ & $a\pi + Y$, ubi π eſt quicunque integer numerus, generales valores quantitatum x & y, & conſequenter unum ſolummodo datur reſiduum e diviſione quantitatum H per rectangulum $a \times b$; viz. $aX + \alpha = bY + \beta$, unde e ſingulis (m, n, &c.) reſiduis inter ſe comparatis facile conſtat propoſitio.

Hinc non poſſunt eſſe plura reſidua e diviſione quadratorum per numerum A quam $\frac{a+1}{2} \times \frac{b+1}{2} \times \frac{c+1}{2} \times$ &c.; forſan vero haud tam multa inveniuntur: hic pro uno reſiduo aſſumitur (o).

Cor.

Cor. 1. Sint α, β, γ, δ, &c. reſidua e diviſione numerorum 1^m, 2^m, 3^m, 4^m, 5^m, &c. in infinitum per quemlibet numerum n; & ſint a, b, c, d, e, &c. non reſidua; tum $n \times r + a$ vel $n \times r + b$, &c. ubi r eſt integer numerus, non poſſunt conficere numerum b^m; e. g. reſidua e diviſione quadratorum 1, 4, 9, 16, & per 4 erunt o & 1; ergo non reſidua erunt 2 & 3; unde $4r + 2$ vel $4r + 3$ & conſequenter $4r - 1$ non poſſunt conficere quadratum. e. 2. reſidua e diſivione cuborum 1, 8, 27, 64, &c. per 7 erunt 1 & 6, & conſequenter ejus non reſidua erunt 2, 3, 4 & 5; unde $7r + 2$ vel $7r + 3$ vel $7r + 4$ vel $7r + 5$ non poſſunt conficere cubum.

Cor. 2. Sint a', b', c', d', &c. numeri, qui non ſint ſumma quorumcunque duorum reſiduorum α, β, γ, δ, &c.; & a'', b'', c'', d'', &c. numeri, qui non ſint ſumma quorumcunque trium reſiduorum, & ſic deinceps; tum neque $n \times r + a'$ neque $n \times r + b'$, &c. poſſunt eſſe ſumma duorum numerorum (m) poteſtatis, viz. $b^m + k^m$: & ſimiliter neque $n \times r + a''$ neque $nr + b''$, &c. poſſunt eſſe ſumma trium numerorum $b^m + k^m + l^m$; &c. e. g. reſidua e diviſione cuborum per 7 erunt o, 1 & 6; ſummæ e quibuſque duobus reſiduis poſſunt eſſe o, 1, 2, 6 & $5 = 6 + 6 - 7$; ergo 3 & 4 ſunt numeri, qui non poſſunt eſſe duorum prædictorum reſiduorum ſumma, & conſequenter $7r + 3$ & $7r + 4$ non poteſt eſſe ſumma duorum affirmativorum cuborum; e. 2. reſidua e diviſione numerorum 1^5, 2^5, 3^5, 4^5, 5^5, &c. per 11, erunt o, 1 & 10; ſummæ e quibuſque duobus erunt o, 1, 10, 2 & 9, & conſequenter 3, 4, 5, 6, 7 & 8 haud continentur in prædictis ſummis, unde neque $11r + 3$ neque $11r + 4$, &c. poteſt eſſe æqualis ſummæ numerorum $b^5 + k^5$. e. 3. ſummæ e quibuſque tribus prædictis reſiduis o, 1 & 10 erunt o, 1, 2, 3, 8, 9 & 10; unde 4, 5, 6 & 7 in his ſummis haud continentur; ergo neque $n \times r + 4$ neque $nr + 5$ neque $nr + 6$ neque $nr + 7$ poteſt eſſe æqualis ſummæ trium numerorum $b^5 + k^5 + l^5$. e. 4. ſumma vel differentia e quibuſque duobus prædictis reſiduis o, 1 & 10 erunt o, 1, 10, 2 & 9; vel e prædictis re-

reſiduis 0, 1 & 6 erunt 0, 1, 2, 6 & 5; ergo nullus numerus formularum $11r + 3$, $11r + 4$, $11r + 5$, $11r + 6$, &c. poteſt eſſe æqualis ſummæ vel differentiæ $b^5 \pm k^5$; & ſimiliter nullus numerus formularum $7r + 3$ & $7r + 4$ poteſt eſſe æqualis numeris $b^3 \pm k^3$.

Cor. 3. In genere, ſint α, β, γ, δ, &c. reſidua e diviſione numerorum 1^m, 2^m, 3^m, 4^m, & per n; & α', β', γ', δ', &c. reſidua e diviſione $1^{m'}$, $2^{m'}$, $3^{m'}$, $4^{m'}$, &c., &c., per n; ſi numerus H haud contineatur in quocunque numero hujuſce formulæ $A\alpha \pm B\beta \pm C\gamma \pm$ &c.; tum $nr + H$ non poteſt eſſe numerus formulæ $Ab^m \pm Bk^m \pm Cl^m \pm$ &c; & ſi numerus H' haud contineatur in formulâ $A\alpha \pm B\beta \pm C\gamma \pm$ &c. $\pm A'\alpha' \pm B'\beta' \pm C'\gamma' \pm$ &c.; tum $nr + H'$ non poteſt eſſe numerus formulæ $Ab^m \pm Bk^m \pm Cl^m \pm$ &c. $\pm A'b'^{m'} \pm B'k'^{m'} \pm C'l'^{m'} \pm$ &c.; ubi b, k, l, &c.; b', k', l', &c., m, m', &c., A, B, C, &c., A', B', C', &c. integri ſunt numeri.

Cor. 4. Quadrati 1, 4, 9, 16, &c. per primum numerum formulæ $4n - 1$ diviſi, non habent complementos α & $-\alpha$; ergo ſumma duorum quadratorum $(a^2 + b^2)$ non dividi poteſt per $4n - 1$, ni a^2 & b^2 dividi poſſunt per $4n - 1$: hinc ſumma duorum quadratorum inter ſe primorum non dividi poſſunt per numerum compoſitum formulæ $4n - 1$; omnis enim compoſitus numerus formulæ $4n - 1$ habet primos diviſores ejuſdem formulæ.

4. Sint a, b, c, d, e, &c. non reſidua e diviſione numerorum 1, a, a^2, a^3, a^4, a^5, &c. in infinitum per numerum n; tum $nr + a$, $nr + b$, $nr + c$, $nr + d$, &c., ubi r eſt integer numerus, non poteſt eſſe ulla poteſtas quantitatis a.

Sint α, β, γ, δ, &c. non reſidua e diviſione numerorum a, a^2, a^3, &c. per n; & α', β', γ', δ', &c. non reſidua e diviſione numerorum b, b^2, b^3, b^4, b^5, &c. per n; & ſi numerus H' haud contineatur in formulâ $A\alpha \pm B\beta \pm C\gamma \pm$ &c. $\pm A'\alpha' \pm B'\beta' \pm C'\gamma' \pm$ &c.; tum $n \times r + H'$ haud poteſt eſſe numerus formulæ $Aa^b \pm Ba^k \pm Ca^l \pm$ &c. $\pm A'b^{b'} \pm B'b^{k'} \pm C'b^{l'} \pm$ &c., &c.; ubi omnes prædictæ literæ integros denotant affirmativos numeros.

Cor.

Cor. Sit m impar numerus, tum erunt duo numeri (a^m & b^m) ad n primi formulæ x^m, quorum ſumma $a^m + b^m$ dividi poteſt per quemcunque primum numerum n; ſit m par numerus, & erunt vel duo vel tres numeri formulæ x^m, quorum ſumma $a^m + b^m$, vel $a^m + b^m + c^m$ dividi poteſt per prædictum numerum.

5. Ex his principiis ſæpe probari poteſt nullum aſſignari poſſe integrum valorem ſingularum incognitarum quantitatum in datâ æquatione contentarum; i. e. dividatur utraque pars æquationis per eundem quemlibet numerum, & ſi ex principiis prius traditis conſtet duo reſidua exinde reſultantia nunquam evadere poſſe inter ſe æqualia, tum haud dantur integri valores prædictarum incognitarum quantitatum. e. g. ſit $13x^4 + 2 = 7x^6 + 5x^{12}$; dividatur hujuſce æquationis utraque pars per 13; prior pars $13x^4 + 2$ per 13 diviſa relinquit 2; poſterior vero pars $7x^6 + 5x^{12}$ per 13 diviſa non habere poteſt reſiduum 2; poteſtas enim x^6 per 13 diviſa habet reſidua 0, 1, 12, & conſequenter $7x^6$ per 13 diviſa habet reſidua 0, $7 \times 1 = 7$, 6; & ſimiliter $5x^{12}$ per 13 diviſa habet reſidua 0, 5: ſed ex additione numeri 0 vel 7 vel 6 ad numerum 0 vel 5 non exoriri poteſt numerus (2); ergo duo reſidua ex utrâque æquationis parte haud poſſunt eſſe inter ſe æqualia.

Haud refert, annon diviſor ſit coefficiens datæ æquationis.
Pro diviſoribus præſtat aſſumere numeros, qui præbent pauciora reſidua e diviſione datæ æquationis per eos, quam quique alii minores numeri: hi autem numeri e theorem. prius traditis facile conſtant.

Sint $\alpha, \beta, \gamma, \delta$, &c. quicunque numeri in datâ quantitate A, quæ per numerum p ſemper dividi poteſt, necne; in eâ pro $\alpha, \beta, \gamma, \delta$, &c. ſcribantur quæcunque functiones incognitarum quantitatum ($\pi, \varrho, \sigma, \tau$, &c.) ita aſſumptæ, ut (ſi π, ϱ, σ, &c. ſint integri numeri) $\alpha, \beta, \gamma, \delta$, &c. ſint integri numeri; tum reſultans quantitas eodem modo ſemper per p diviſibilis erit, necne.

THEOR.

PROB. LXI.

Invenire formulas diviſorum quarundam datarum quantitatum.

1. 1. Reducantur datæ quantitates in alias, quarum formulæ diviſorum innoteſcunt, & exinde deduci poſſunt formulæ diviſorum datarum quantitatum.

2. Aſſumantur quantitates datarum formularum & in ſeſe ducantur, exinde reſultant quantitates, quæ recipiunt diviſores dati generis: e. g. aſſumantur quantitates datæ formulæ $4n+1$, viz. $4x+1$, $4y+1$, $4z+1$, &c., ubi literæ x, y, z, &c. integros denotant numeros; earum contentum erit numerus eandem formulam habens; contentum $(4x+1)\times(4y\pm1)\times(4z\pm1)\times$ &c. eandem habet formulam, ſi numerus factorum formulæ $(4n-1)$ in eo contentorum ſit par; ſin aliter formulam $(4n-1)$ habet. Conſtat ex multiplicando factores in ſeſe.

3. Si omnes diviſores quarundam quantitatum ſub datâ formulâ contineantur; & inveniantur aliæ formulæ, in quibus conſiſtant omnes quantitates datæ formulæ; tum omnes diviſores prædictarum quantitatum in his formulis conſiſtent.

Ex. 1. Sint quæcunque quantitates a^2+nb^2, ubi a & b ſunt quicunque inter ſe primi numeri, & n eſt datus numerus: tum plerumque ejus diviſores in eâdem formulâ p^2+nq^2 conſiſtent: hæc formula mutari poteſt in alteram $nx+\alpha$, ubi x ſit integer & α numerus invariabilis; ſi dividatur quantitas p^2+nq^2 per n, nam eadem erunt reſidua ac ea ex diviſione quadratorum (p^2) per n; ſint α, β, γ, δ, &c. reſpective reſidua ex diviſione omnium quadratorum per n; ergo primi diviſores numerorum formulæ a^2+nb^2 continebuntur in formulis $nx+\alpha$, $nx+\beta$, $nx+\gamma$, &c.

Sit n primus numerus, tum numerus formularum prædictarum erit $\frac{n+1}{2}$; numerus enim reſiduorum α, β, γ, &c. erit $\frac{n-1}{2}$.

Ex. 2.

Ex. 2. Numerus formularum $(4nx+\alpha)$ primorum diviſorum prædictarum quantitatum (a^2+nb^2) erit $n-1$: totidem enim diverſa reſidua habent quantitatum primi diviſores hujuſce formulæ per $4n$ diviſi.

Si vero n ſit hujuſce formulæ $(4x-1)$ quantitas; tum hæ $(n-1)$ formulæ $(4nx+\alpha)$ ſemper reduci poſſunt in $\frac{n-1}{2}$ numerum formularum hujuſce generis $(2nx+\alpha)$. Sit $n=2$ & erit $4nx+\alpha = 8x+1$ & $8x+3$ reſpective. Diviſores 2 & n etiam habet data æquatio.

Sit $n=p\times q\times r\times s\times$ &c. ubi p, q, r, s, &c. primos numeros reſpective denotent, tum numerus primorum diviſorum formularum $(4nx+\alpha)$ erit $=(p-1)\times(q-1)\times(r-1)\times$ &c. ſit vero $n = 2pqr$ &c. tum numerus prædictus erit $2\times(p-1)\times(q-1)\times(r-1)\times$ &c. vel diviſores erunt vel 2, vel p, q, r, &c.

Sit $4nx+\alpha$ formula diviſoris quantitatis a^2+nb^2, tum $4nx-\alpha$ minime poteſt eſſe formula diviſoris prædictæ quantitatis.

Omnis formula $(4pqx+\alpha)$, quæ continet diviſores quantitatis (a^2+pqb^2), etiam erit formula, quæ continet diviſores quantitatis (pa^2+qb^2): enim $p\times(a^2+pqb^2)=pa^2+p^2qb^2=pa^2+qc^2$.

Sit $4nx+\alpha$ formula, quæ continet diviſores hujuſce formulæ a^2+nb^2, tum omnis primus numerus $A=4nx+\alpha$ erit etiam hujuſce formulæ a^2+nb^2.

2. Sit $a^2-nb^2=A$, ubi $n=p\times q\times r\times s\times$ &c. vel $= 2pqrs$ &c. etiamque p, q, r, s, &c. primos denotent numeros; tum numerus primorum diviſorum $(4nx+\alpha)$ erit ac in præcedente caſu vel $(p-1)\times(q-1)\times(r-1)\times$ &c. vel $2\times(p-1)\times(q-1)\times(r-1)\times$ &c.

Si hoc in caſu $4nx+\alpha$ ſit formula diviſorum, tum $4nx-\alpha$ erit etiam formula ejus diviſorum.

Sit $n=4x+1$, tum formulæ primorum diviſorum exprimi poſſunt per formulam $2nx\pm\alpha$: erunt etiam primi diviſores $2, p, q, r$, &c.

Facile

Facile etiam inveſtigari poſſunt omnes diverſi valores quantitatis (α), quorum unus ſemper erit 1.

Si numerus P habeat formulam $8n+1$ vel $8n+3$, & in formulam $2a^2+b^2$ reſolvi nequeat; tum non erit primus numerus.

Cor. 1. Si nullo modo (n) ſit aggregatum quadrati (a^2) & trigonalis $\frac{b^2+b}{2}$; tum $8n+1$, quæ non $= 8a^2+4b^2+4b+1 = 2\times 4a^2 + (2b+1)^2 = 2p^2+q^2$, non eſt primus.

Cor. 2. Si (n) nullo modo in hâc formâ $a^2+a+\frac{1}{2}(b^2+b)$ contineatur; tum $8n+3$ nullo modo in hâc formulâ $8a^2+8a+2+4b^2+4b+1 = 2(2a+1)^2+(2b+1)^2 = 2p^2+q^2$ contineri poteſt, & conſequenter non eſt primus.

3. Sit n numerus primus, etiamque ad numerum (m) primus; tum non datur formula $na+\alpha$, quæ haud continet numerum formulæ p^2+mq^2: dividatur enim quantitas p^2 per n, reſultent reſidua $\alpha, \beta, \gamma, \delta$, &c.; dividatur etiam $m\times q^2$ per eundem numerum n, & reſultant eadem reſidua $(\alpha, \beta, \gamma, \delta$, &c.) ducta in reſiduum (π) ex diviſione numeri m per n diviſi; ſi π ſit unum ex reſiduis $(\alpha, \beta, \gamma$, &c.) tum reſultabunt reſidua $(\alpha, \beta, \gamma, \delta$, &c.), ſin aliter omnia non reſidua: his duobus conjunctis, omnia reſidua minora quam n reſultabunt.

4. Formulæ diviſorum quantitatum a^2+ab+b^2 ubi a & b ſunt inter ſe primi, erunt p^2+pq+q^2; hinc formulæ prædictæ non poſſunt habere formulas $3x+2$, &c.

Hinc facile inveniri poſſunt multi numeri, qui nunquam poſſunt eſſe quadrati: inveniatur quantitatis formula; quæ nunquam poteſt eſſe diviſor quantitatis $a^2 \pm nb^2$; e formulâ quantitatis ſubtrahatur nb^2; & reſultat formula, quæ nunquam poteſt eſſe quadratus: e. g. quantitas $4nx-1$ nunquam poteſt eſſe diviſor quantitatis a^2+nb^2, unde $(4nx-1)\times u$ (ubi u ſit integer numerus) haud poteſt æquari a^2+nb^2, & exinde (ſi modo ſupponatur $b=1$) $4nxu-u-n$ haud poteſt eſſe quadratus (a^2); & ſic de aliis.

THEOR.

THEOR. LV.

1. Sit P quæcunque integralis functio literarum x, y, z, &c., in eâ pro x ſcribatur $= x + ny + mz +$ &c. $+ a$; ubi literæ x, y, z, &c.; n, m, &c. a integros affirmativos vel negativos denotant numeros, reſultet functio integralis Q; tum omnes valores functionum P & Q e ſubſtitutione omnium integrorum numerorum pro literis x, y, z, &c. reſultantes iidem evadent.

Ex. 1. Sit functio $x^2 - xy + y^2$; in eâ pro y ſcribatur $x + y$, reſultabit $x^2 - x \times \overline{x+y} + \overline{x+y}^2 = x^2 + xy + y^2$, unde omnes valores functionis $x^2 + xy + y^2$ e ſubſtitutione omnium numerorum pro x & y iidem erunt ac omnes valores functionis $x^2 - xy + y^2$ e ſubſtitutione omnium integrorum numerorum pro prædictis quantitatibus (x & y).

Et facile infinitæ conſimiles propoſitiones deduci poſſunt.

Ex. 2. Sit $Ax^2 - Bxy + Cy^2$; pro x ſcribatur $z - a$, reſultat $Az^2 - \overline{2Aa + B}\,zy + (Aa^2 + Ba + C)\,y^2$, ergo omnes numeri in his duobus formulis contenti ſunt iidem.

2. Sit $Ax = ap^n + bp^{n-1} + cp^{n-2} + dp^{n-3} +$ &c. $+ P$, ubi literæ A, a, b, c, d, &c. P integros datos vel affirmativos vel negativos, & x & p quoſcunque integros numeros reſpective denotant: tum, ſi modo dentur integri valores quantitatum x & p, qui ſint α & π, dabitur etiam valor quantitatis p, qui non major erit quam $\frac{1}{2}A$.

Scribatur enim $\pi \pm \mu A$, ubi μ eſt integer numerus vel affirmativus vel negativus, pro p in datâ æquatione, & reſultat $Ax = a \times \overline{\pi \pm \mu A}^n + b \times \overline{\pi \pm \mu A}^{n-1} + c \times \overline{\pi \pm \mu A}^{n-2} +$ &c. $+ P$: ſi vero $a\pi^n + b\pi^{n-1} + c\pi^{n-2} +$ &c. $+ P$ dividi poſſit per A, tum etiam prædicta quantitas $a \times \overline{\pi \pm \mu A}^n + b \times \overline{\pi \pm \mu A}^{n-1} + c \times \overline{\pi \pm \mu A}^{n-2} +$ &c. $+ P$ dividi poteſt per A, quicunque ſit valor quantitatis μ; ita vero aſſumi poteſt

 valor

valor quantitatis μ, ut $\pi \pm \mu A$ haud major ſit quam $\frac{1}{2}A$; ergo dabitur valor $\pi \pm \mu A$ quantitatis x, qui haud major erit quam $\frac{1}{2}A$.

LEMMA.

1. Sit $a^2 + b^2$ quadratus, & a & b inter ſe primi; erunt $a = p^2 - q^2$, $b = 2pq$, & exinde $a^2 + b^2 = \overline{p^2 + q^2}^2$, & p & q inter ſe primi; eorumque alter erit par, alter impar, aliter enim a & b non poſſunt eſſe primi inter ſe numeri.

2. Sit $a^2 - b^2$ quadratus, & a & b inter ſe primi; erunt $a = p^2 + q^2$ & $b = p^2 - q^2$ vel $b = 2pq$, ubi p & q ſunt inter ſe primi, eorumque alter par & alter impar.

THEOR. LVI.

1. Si per ſubſtitutionem rationalem incognitarum quantitatum reſultantis æquationis pro incognitis quantitatibus datæ æquationis reduci poſſit data æquatio in alteram ei conſimilem; tum ex uno correſpondente valore ſingularum incognitarum quantitatum reſultantis æquationis, quæ datæ conſimilis eſt, erui poſſunt plures valores e prædictâ ſubſtitutione.

2. Si valores incognitarum quantitatum in reſultante æquatione ſint minores quam valores correſpondentium incognitarum quantitatum in datâ æquatione, & continuo per conſimilem ſubſtitutionem deprimi poſſit reſultans æquatio in alteram, cujus incognitæ quantitates adhuc ſint minores; &c.; tum vel e parvis valoribus datæ æquationis deduci poſſunt plures alii; vel nullos omnino integros correſpondentes valores incognitarum quantitatum admittit data æquatio.

3. Vel magis generaliter, ſi modo per reciprocam rationalem ſubſtitutionem, i. e. ſubſtitutionem rationalem incognitarum reſultantis æquationis quantitatum pro incognitis quantitatibus datæ æquationis,

nis, quæ præbet etiam rationalem ſubſtitutionem incognitarum quantitatum datæ æquationis pro incognitis quantitatibus reſultantis æquationis; reducatur data æquatio in alteram; tum poſſunt eſſe correſpondentes rationales valores incognitarum datæ æquationis quantitatum, necne; prout poſſunt eſſe rationales valores incognitarum deductæ æquationis quantitatum, necne.

Ex. 1. Sit $x^4 + ay^4 = z^2$, ubi a ſit integer numerus; fingatur $z = m^2 + an^2$, unde $z^2 = (m^2 - an^2)^2 + a(2mn)^2$; & exinde $y^2 = 2mn$, & $x^2 = m^2 - an^2$; ſit $x = p^2 - aq^2$, unde $x^2 = (p^2 + aq^2)^2 - a(2pq)^2$, exinde $m = p^2 + aq^2$ & $n = 2pq$; ſubſtituantur hi valores in æquatione $y^2 = 2mn$, reſultat $y^2 = 4pq(p^2 + aq^2)$; deinde fiat $p = s^2$, $q = t^2$, $u^2 = p^2 + aq^2$, exorietur $y = 2stu$, & exinde $s^4 + at^4 = u^2$ datæ æquationi ſimilis, & $x = s^4 - at^4$, $y = 2stu$, $z = (s^4 + at^4)^2 + a(2s^2t^2)^2$.

Ex. 2. Summa duorum biquadratorum $(\alpha^4 + \beta^4)$ non poteſt eſſe quadratus. Primo, ſi α & β habeant communem diviſorem, i. e. $\alpha = ca$ & $\beta = cb$, & conſequenter $\alpha^4 + \beta^4 = c^4(a^4 + b^4)$; $a^4 + b^4$ erit quadratus, ſi $\alpha^4 + \beta^4$ ſit quadratus, ubi a & b ſunt numeri inter ſe primi.

Sit $a^4 + b^4$ quadratus, tum erit $a^2 = p^2 - q^2$ & $b^2 = 2pq$, ubi p & q ſunt numeri inter ſe primi, eorumque alter (q) par, alter (p) impar, aliter $p^2 - q^2$ non poteſt eſſe quadratus; at, quoniam $2pq$ eſt quadratus, erunt utrique p & $2q$ quadrati: ut vero $p^2 - q^2$ fiat quadratus, erunt $p = m^2 + n^2$ & $q = 2mn$, ubi m & n ſunt inter ſe primi numeri, eorum alter par & alter impar: ſed quoniam $2q$ eſt quadratus, erit $4mn$ ſeu mn quadratus, unde m & n erunt quadrati; ſint ergo $m = x^2$ & $n = y^2$, & erit $p = m^2 + n^2 = x^4 + y^4$ quadratus; hinc, ſi $a^4 + b^4$ ſit quadratus, $x^4 + y^4$ erit quadratus, ubi x & y ſunt minores numeri quam a & b: & ſimiliter duo minores biquadrati, quorum ſumma eſt quadratus, inveniri poſſunt; & ſic deinceps in infinitum, ergo nulli aſſignari poſſunt biquadrati hujuſce generis.

Hinc tres numeri p, $2q$ & $p^2 - q^2$ & exinde $2pq(p^2 - q^2)$; etiamque $pq(p^2 + q^2)$, non poſſunt eſſe quadrati.

2. Ex eadem ratiocinatione ſequitur $a^4 - b^4$ & exinde $ab(a^2 - b^2)$, $2ab(a^2 + b^2)$, & $2a^4 + 2b^4 = \overline{a^2+b^2}^2 + \overline{a^2-b^2}^2$ non poſſe quadratum eſſe, ni $a = b$.

Cor. 1. Summa $\overline{a^2+b^2}^2 + \overline{a^2-b^2}^2 = 2a^4 + 2b^4$ non poteſt eſſe quadratus, quoniam $\overline{a^2+b^2}^2 - \overline{a^2-b^2}^2$ eſt quadratus; aliter $(\overline{a^2+b^2}^2 + \overline{a^2-b^2}^2) \times (\overline{a^2+b^2}^2 - \overline{a^2-b^2}^2) = \overline{a^2+b^2}^4 - \overline{a^2-b^2}^4 = p^4 - q^4$ eſſet quadratus.

Cor. 2. Differentia $2a^4 - 2b^4$ nequit eſſe quadratus. Sint enim a & b primi numeri & $2a^4 - 2b^4$ quadratus, erunt a & b impares numeri: at quoniam $2(a-b)(a+b)(a^2+b^2)$ eſt quadratus, erit $\left(\frac{a-b}{2}\right)\left(\frac{a+b}{2}\right)\left(\frac{a^2+b^2}{2}\right)$ quadratus; qui factores, cum ſint inter ſe primi, erunt quadrati: ſit $\frac{a-b}{2} = p^2$ & $\frac{a+b}{2} = q^2$; unde $a = p^2 + q^2$ & $b = q^2 - p^2$, exinde $\frac{a^2+b^2}{2} = p^4 + q^4$; ſed $p^4 + q^4$ nequit eſſe quadratus, ergo $2a^4 + 2b^4$ non eſt quadratus.

Cor. 3. Sint m, a & b inter ſe primi numeri; & $m(a^2 - mb^2) \times (a^2 + mb^2)$ non poteſt eſſe quadratus; ſi enim ſit quadratus, tum m, $a^2 - mb^2$ & $a^2 + mb^2$ erunt reſpeſpective quadrati; ponatur $m = p^2$, ergo $a^2 - p^2b^2$ & $a^2 + p^2b^2$, & conſequenter $a^4 - p^4b^4$ erit quadratus, quod eſt abſurdum. Et ſimiliter conſtat $2m(a^2 - mb^2) \times (a^2 + mb^2)$ non poſſe quadratum eſſe.

Si m non ſit primus reſpectu habito ad a, ponantur $m = rs$ & $a = rc$, ubi r & s ſint inter ſe primi; i. e. ſupponatur numerus (m) nullum habere diviſorem quadratum, aliter reduci poteſt; unde $r^5sc^4 - r^3s^3b^4 = ma^4 - m^3b^4$ & $2r^5sc^4 - 2r^3s^3b^4$ eſſent quadrati: ſi autem hi numeri ſint quadrati, tum eorum primi diviſores r, s, &c. erunt quadrati, quod facile e præcedentibus patet eſſe abſurdum.

Cor. 4. Formulæ $ma^4 + m^3b^4$ & $2ma^4 + 2m^3b^4$ non poſſunt eſſe quadrati. Si enim $mp^2 - mq^2$ fuerit quadratus, $mp^2 + mq^2$ non poteſt

eſſe

eſſe quadratus, aliter evadet $m^2(p^4-q^4)$ vel $2m^2(p^4-q^4)$ quadratus. Ponatur $mp^2-mq^2=\frac{(p-q)^2\times a^2}{b^2}$, unde $mp+mq=\frac{a^2p-a^2q}{b^2}$, & exinde $q=\frac{p(a^2-mb^2)}{a^2+mb^2}$: ſit $p=a^2+mb^2$, unde $q=a^2-mb^2$, & $p^2+q^2=2a^4+2m^2b^4$; ergo $m(p^2+q^2)=2ma^4+2m^3b^4$, deinde $2mp^2+2mq^2=4ma^4+4m^3b^4$ non poſſunt eſſe quadrati.

Cor. 5. Nullus trigonalis numerus $x.\frac{x+1}{2}$ præter unitatem poteſt eſſe biquadratus numerus; enim x & $\frac{x+1}{2}$, vel $\frac{x}{2}$ & $x+1$ ſunt inter ſe primi, ergo erunt biquadrati, viz. 1mo. ſint $x=p^4$ & $\frac{x+1}{2}=q^4$, tum erit $p^4+1=2q^4$, unde $4q^4=2p^4+2$, quod prius probatur impoſſibile eſſe; & ſic de altero caſu.

Ex. 3. Nullus numerus formulæ a^4+2b^4 poteſt eſſe quadratus. Fingatur $(a^2+\frac{m}{n}b^2)^2=a^4+2b^4$, ubi a & b, m & n ſunt numeri inter ſe primi; & exinde $\frac{b^2}{a^2}=\frac{2mn}{2n^2-m^2}$, quæ fractio vel irreducibilis eſt, vel ex diviſione per a evadet irreducibilis: ponantur $2mn$ & $2n^2-m^2$ eſſe numeri inter ſe primi, & evadet $b^2=2mn$ & $a^2=2n^2-m^2$; in hoc caſu m eſt impar numerus, & n vel par vel impar; in priori caſu non poteſt $2n^2=a^2+m^2$, in poſteriori caſu non poteſt $b^2=2mn$. Habeant numeri $2mn$ & $2n^2-m^2$ communem diviſorem 2, & conſequenter m erit par numerus; ponatur $m=2k$, tum erit n numerus impar; ergo $\frac{b^2}{a^2}=\frac{2kn}{n^2-2k^2}$, ubi $2kn$ & n^2-2k^2 erunt numeri inter ſe primi: hinc ob b & a inter ſe primos erit $b^2=2kn$ & $a^2=n^2-2k^2$; ſed $2kn$ non eſt quadratus, ni k ſit par numerus, & exinde n & $2k$ debebunt eſſe quadrati: fiat $n=c^2$ & $2k=4d^2$, ubi c eſt impar;

[illegible] $= \frac{2p}{q}$ [illegible], &

[illegible] & p & q ſunt [illegible] $= r^4 + 2r$, ergo neceſſe [illegible], erit $= x^4 +$ [illegible] quadratus, ubi [illegible] minores adhuc in- [illegible] numerus [illegible] $q = 2$ [illegible], unde $\frac{c^2}{c^2}$

$=$ [illegible] primos erit $= qr$ & $=$ [illegible] erunt utrique quadrati; i. e. $q = x^2$ & [illegible] & unde [illegible] $+ 2y^4$ ſit quadratus, erit [illegible] quadratus. &c.

Cor. Si forma [illegible] non poteſt eſſe quadratus, tum forma [illegible] nullo pacto quadratus effici poterit.

Sit [illegible] $+ \frac{m}{n}$ [illegible], erit $kn^2b^2 = 2mna^2 + m^2b^2$ & $\frac{b^2}{a^2} =$ [illegible] ſed quoniam $a^4 + kb^4$ nequit eſſe quadratus, tum etiam [illegible] ſeu [illegible] $- m^2)$ quadratus eſſe non poterit: fiat $m =$ [illegible] prodibit $2\alpha\beta(k\beta^2y^4 - \alpha^2x^4)$ ſeu $2k\alpha\beta^3y^4 - 2\alpha^3\beta x^4$, quæ formula non poteſt eſſe quadratus.

Ex hinc facile deduci poteſt formulas $\alpha^3\beta x^4 + \alpha\beta^3y^4$, $\alpha^3\beta x^4 -$ [illegible], $2\alpha^3\beta x^4 - 2\alpha\beta^3y^4$, $2\alpha^3\beta x^4 + 2\alpha\beta^3y^4$, $2\alpha^3\beta x^4$ [illegible] non poſſe eſſe quadratos.

[illegible] 4. Nullus cubus unitate auctus præter 8 poteſt eſſe quadratus, [illegible] non poteſt $\frac{a^3}{b^3} + 1 = v^2$ & conſequ[illegible] $+ b^6$ [illegible] $= u^2$.

 Erit

Erit 1mo. $b \times (a+b) \times (a^2 - ab + b^2) = a^3 b + b^4$ quadratus, cum $b \times (a+b) = a^2 - ab + b^2$, & conſequenter $a = 2b$ caſus excipiendus: ponatur $a + b = c$, reſultat $cb(c^2 - 3cb + 3b^2)$, qui non poteſt eſſe quadratus, ni $c = 3b$, ubi b & c ſunt inter ſe primi numeri: 1mo. ſit c multiplum numeri 3, viz. $c = 3d$; in hoc caſu factores c & $c^2 - 3bc + 3b^2$ communem habent diviſorem 3; & $cb(c^2 - 3cb + 3b^2) = 9db(b^2 - 3bd + 3d^2)$; ergo, ſi quantitas $cb(c^2 - 3cb + 3b^2)$ ſit quadratus, quantitas $db(b^2 - 3bd + 3d^2)$ ejuſdem formulæ, quæ conſtat e minoribus numeris, erit etiam quadratus; & ſic deinceps: 2do. ſit c non diviſibilis per 3, tum erunt b, c & $c^2 - 3cb + 3b^2$ quadrati; fiat $c^2 - 3bc + 3b^2 = (\frac{m}{n} b - c)^2$, unde $\frac{b}{c} = \frac{3n^2 - 2mn}{3n^2 - m^2}$ vel $= \frac{2mn - 3n^2}{m^2 - 3n^2}$, cujus fractionis termini erunt inter ſe primi, niſi m per 3 diviſibilis ſit: ſit m per 3 non diviſibilis, erit $c = 3n^2 - m^2$ vel $= m^2 - 3n^2$, & $b = 3n^2 - 2mn$ vel $= 2mn - 3n^2$; at $3n^2 - m^2$ non poteſt eſſe quadratus, ergo ponatur $c = m^2 - 3n^2 = (m - \frac{p}{q} n)^2$, unde $\frac{m}{n} = \frac{3q^2 + p^2}{2pq}$. & $\frac{b}{n^2} = \frac{2m}{n} - 3 = \frac{3q^2 - 3pq + p^2}{pq}$; igitur $pq \times (3q^2 - 3pq + p^2)$ erit quadratus, quæ quantitas eandem habet formulam ac præcedens, & conſtat ex minoribus numeris; & ſic deinceps: 2do. ſit $m = 3k$, erit $\frac{b}{c} = \frac{n^2 - 2kn}{n^2 - 3k^2}$, unde $c = n^2 - 3k^2$ vel $= 3k^2 - n^2$ & $b = n^2 - 2kn$ vel $= 2kn - n^2$; ſed $3k^2 - n^2$ nequit eſſe quadratus, ponatur igitur $c = n^2 - 3k^2 = (n - \frac{p}{q} k)^2$, exinde $\frac{n}{k} = \frac{3q^2 + p^2}{2pq}$, & $\frac{b}{n^2} = 1 - \frac{2k}{n} = \frac{p^2 + 3q^2 - 4pq}{3q^2 + p^2}$; & conſequenter $(p^2 + 3q^2)(p - q)(p - 3q)$ erit quadratus; ponatur $p - q = t$ & $p - 3q = u$, unde formula $(p^2 + 3q^2)(p - q)(p - 3q) = tu(3t^2 - 3tu + u^2)$, quæ eandem formulam habet ac

quan-

quantitas $cb(c^2-3cb+3b^2)$, & ex minoribus numeris conſtat, erit quadratus, &c.

THEOR. LVII.

Omnes poteſtates a^{4m}, ubi a & m ſint integri numeri, habent ſuas figuras deſinentes in 0, 1, 5, 6: & omnes quadrati numeri; i. e. a^2 habent ſuas figuras deſinentes in 0, 1, 5, 6, 4, 9: omnes quadrati numeri habent pro ſuâ penultimâ figurâ 0, 2, 4, 6, 8; ni ultima figura radicis ſit 6 vel 4, in quo caſu ſemper habent imparem numerum: ſi ultima figura ſit 5, tum penultima omnis paris poteſtatis erit 2, imparis vero vel 2 vel 7.

Sint 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 reſpective ultima figura radicis, tum erunt 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 reſpective correſpondens ultima figura imparium poteſtatum 1, 5, 9, 13, 17, &c.; & 0, 1, 8, 7, 4, 5, 6, 3, 2, 9 reſpective correſpondens ultima figura cæterarum imparium poteſtatum 3, 7, 11, 15, 19, &c.; & 0, 1, 4, 9, 6, 5, 6, 9, 4, 1 erunt reſpective correſpondens ultima figura parium poteſtatum 2, 6, 10, 14, &c.; & 0, 1, 6, 1, 6, 5, 6, 1, 6, 1 correſpondens ultima figura parium poteſtatum 4, 8, 12, 16, &c.

Cor. Quadrati quatuor haud poſſunt eſſe in arithmeticâ progreſſione.

Ex theoremate per additionem facile deduci poteſt coroll.

PROB. LXII.

Datâ æquatione duas vel plures (x, y, *&c.*) *incognitas quantitates habente, quarum valores ſint rationales; invenire æquationem, cujus incognita quantitas* (w) *quamcunque habeat algebraicam relationem ad radices incognitarum quantitatum datæ æquationis.*

1. Si relatio ſit algebraica inter quantitatem aſſumptam (w) & correſpondentes valores incognitarum quantitatum, tum prius perficitur pro-

problema e reductione plurium æquationum in unam, ita ut incognitæ quantitates exterminentur.

2. Si vero algebraica relatio haud ſolummodo conſiſtat inter correſpondentes rationales radices incognitarum quantitatum, ſed involvat vel diverſas rationales radices ejuſdem incognitæ quantitatis vel rationales haud correſpondentes radices e diverſis incognitis quantitatibus: inveniatur relatio inter quaſcunque diverſas radices incognitarum quantitatum, quæ ſemper inveniri poteſt, cum detur generalis methodus detegendi unam radicem; methodus enim, quæ generaliter inveniat unam radicem, etiam plures (ſi modo plures ſint) radices deduceret.

Ita reducantur æquationes reſultantes in unam ut exterminentur incognitæ quantitates, & fit problema.

3. Si vero haud inveniri poſſit relatio inter diverſas prædictas radices, tum ſolummodo e continuis approximationibus, correctionibus & conjecturis progredi liceat.

Et ſic de ſummis e ſingulis integris valoribus functionum prædictorum generum, ita ut hæ ſummæ fiant quantitates dati generis; & de transformandis pluribus hujuſce generis æquationibus.

PROB. LXIII.

Datis quibuſcunque quantitatibus vel numeris, ex iis formare dati generis quantitates.

Aſſumantur quædam ex his quantitatibus, & ex iis inveniantur quantitates reſultantes: ſi vero ex his quantitatibus haud formentur quantitates dati generis; continuo corrigantur vel approximentur, uſque donec ad veros valores perventum eſt.

Ex. Sint quantitates datæ omnes cubi numeri, & quantitas dati generis quadratus numerus; requiratur invenire cubum numerum, qui additus omnibus ſuis partibus aliquotis conficiat quadratum.

Aſſumantur multi cubi numeri minimi, & inveniantur aggregata

e ſingulis horum diviſoribus, & quando cubi (primi, ſecundi, tertii, aliive) variorum primorum non habent ſua diviſorum aggregata ita per primorum paria deſignanda (qualia habet aggregatum diviſorum cubi $343 = 7^3$: aggregatum enim eſt $1 + 7 + 49 + 343 = 400 = 20 \times 20 = 2 \times 2 \times 2 \times 2 \times 5 \times 5 = 2^2 \times 2^2 \times 5^2$) poteſt tamen compoſitus e duobus, tribus, pluribuſve talibus continuo multiplicatis (qui itidem cubus erit) habere diviſorum ſuorum aggregatum (quod eſt compoſitum ex aggregatis illis continue multiplicatis) ſic deſignandum: nimirum, ſi cubi ſic componendi ita eligantur, ut primi, qui in deſignandis aggregatorum aliquibus ſunt ſolitarii, comites ſibi acquirant ex ſimilibus, qui in aliis aggregatorum occurrunt itidem ſolitarii; i. e. continuo corrigantur.

Sic pro cubo numeri 47, diviſorum aggregatum $1 + 47 + 47^2 + 47^3 = 106080$ (numeris primis deſignatum) eſt $2^2 . 2^2 . 2 . 3 . 5 . 13 . 17$: quod (præter paria) habet ſolitarios 2, 3, 5, 13, 17; cum quo ſi componatur cubus numeri 5, qui habet aggregatum diviſorum $2^2 . 3 . 13$; hic occurrunt ſolitarii $3 . 13$; qui (comites facti prius ſolitariis 3, 13) relinquunt jam ſolitarios 2, 5, 17: ſi tum prius ſumpti componantur porro cum cubo numeri 13, qui diviſorum aggregatum habet $2^2 \times 5 \times 7 \times 17$; comites habentur prius ſolitariis 5, 17; ſed ſolitarii ſuperſunt 2, 7: tum aſſumatur cubus numeri 41, cujus habetur diviſorum aggregatum $2^2 \times 3 \times 7 \times 29^2$, & exinde præter paria habentur ſolitarii 2, 3: & ſic pro cubo numeri 11, diviſorum aggregatum eſt $2^2 \times 2 \times 3 \times 61$, unde numeris 2, 3 habentur comites, ſed manet 61 ſolitarius; & pro cubo numeri 27 habetur diviſorum aggregatum $2^2 \times 11^2 \times 61$; unde numerus 61 habebit comitem, & radix cubi quæſiti, ex hâc continuâ methodo corrigendi præcedentes aſſumptos numeros, reſultabit $47 \times 5 \times 13 \times 41 \times 11 \times 27$: & ſic inveniri poſſunt infinitæ aliæ hujuſce problematis ſolutiones.

Et ſic de omnibus hujuſce generis problematibus reſolvendis.

Hic viſum eſt ſubiungere duas vel tres ſubſequentes proprietates primorum & integrorum numerorum.

1. Omnis

1. Omnis par numerus conſtat e duobus primis numeris, & omnis impar numerus vel eſt primus numerus, vel conſtat e tribus primis numeris, &c.

2. Erit $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\frac{1}{13}+\frac{1}{17}+$ &c. in infinitum major quam quæcunque finita quantitas.

Cor. 1. Contentum $1\times2\times3\times5\times7\times11\times13\times$ &c. in infinitum ad contentum $(1+1)\times(2+1)\times(3+1)\times(5+1)\times(7+1)\times(11+1)\times(13+1)\times$ &c. in infinitum: habet minorem quam quamcunque finitam rationem.

Cor. 2. Hinc ſemper inveniri poteſt numerus, cujus diviſorum ſumma habet ad numerum ipſum, majorem quam quamcunque datam rationem.

3. Haud dantur tres primi numeri in arithmeticâ progreſſione, quorum communis differentia haud diviſibilis ſit per numerum 6; ni 3 ſit primus terminus arithmeticæ ſeriei, in quo caſu poſſunt eſſe tres & haud plures termini ejuſdem arithmeticæ ſeriei primi numeri, & quorum communis differentia haud diviſibilis ſit per 6: hic excipiantur duæ arithmeticæ ſeries 1, 2, 3 & 1, 3, 5, 7.

Haud dantur quinque primi numeri in arithmetica progreſſione, quorum communis differentia haud diviſibilis ſit per 30; ni 5 ſit primus terminus arithmeticæ ſeriei, in quo caſu poſſunt eſſe quinque termini, & haud plures, qui ſunt primi numeri ejuſdem arithmeticæ ſeriei: & ſic in genere haud dantur 3, 5, 7, 11, 13, 17, &c. primi numeri in arithmeticâ progreſſione & quorum communes differentiæ haud reſpective diviſibiles ſunt per numeros $1\times2\times3$, $1\times2\times3\times5$, $1\times2\times3\times5\times7$, $1\times2\times3\times5\times7\times11$, $1\times2\times3\times5\times7\times11\times13$, $1\times2\times3\times5\times7\times11\times13\times17$, &c. ni 3, 5, 7, 11, 13, 17, &c. ſint reſpective primi termini arithmeticæ ſeriei, in quibus caſibus poſſunt eſſe ſolummodo 3, 5, 7, 11, 13, 17, &c. termini arithmeticarum ſerierum reſpective, quorum differentiæ prædictos diviſores haud admittunt.

Hæc proprietas primorum numerorum facile demonſtrari, & plures ex eodem fonte hauriri poſſunt. Facile enim conſtat e præcedentibus per continuam additionem ejuſdem numeri ad primum numerum ſemper produci numerum per tres, quinque, &c. diviſibilem, ni numerus adjectus per tres, quinque, &c. ſit diviſibilis: numeri aliis modis ita ſæpe conſtituti ſunt, ut ex eorum conſtitutione conſtabit eos diviſores recipere & conſequenter haud primos eſſe.

4. Haud dantur quicunque primi numeri in arithmeticâ progreſſione, quorum aggregatum eſt primus numerus.

5. Sit n numerus primus, & $\frac{1 \times 2 \times 3 \times 4 \ldots (n-2) \times (n-1) + 1}{n}$ erit integer numerus, e. g. $\frac{1 \times 2 + 1}{3} = 1$, $\frac{1 \cdot 2 \cdot 3 \cdot 4 + 1}{4} = 5$, $\frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 + 1}{7} = 103$, &c. Hanc maxime elegantem primorum numerorum proprietatem invenit vir clariſſimus, rerumque mathematicarum peritiſſimus Joannes Wilſon Armiger.

Sed datâ aliquâ quantitate, cujus n ſit diviſor, facile deduci poſſunt infinitæ aliæ, quarum eadem quantitas (n) erit diviſor: e. g. $\frac{1 \cdot 2 \cdot 3 \ldots (n-2) - 1}{n}$, $\frac{1 \cdot 2^2 \cdot 3 \ldots (n-3) + 1}{n}$, $\frac{1 \cdot 2^2 \cdot 3^2 \ldots (n-4) - 1}{n}$ $\ldots \frac{2^2\, 3^2\, 4^2\, 5^2 \ldots \overline{\frac{n-1}{4}}^2 \pm 1}{n}$ (ubi erit $+1$, quando $\frac{n-1}{2}$ ſit par numerus, ſin aliter -1) integri erunt numeri.

Demonſtrationes vero hujuſmodi propoſitionum eo magis difficiles erunt, quod nulla fingi poteſt notatio, quæ primum numerum exprimit.

6. Sit $x^r = x \cdot (x-1) \cdot (x-2) \ldots (x-r+1) + Px \cdot (x-1) \cdot (x-2) \ldots (x-r+2) + Qx \cdot (x-1) \cdot (x-2) \ldots (x-r+3) + Rx \cdot (x-1) \cdot (x-2) \ldots (x-r+4) + \ldots + Hx \cdot (x-1) + Ix$; tum

tum erit $1^r + 2^r + 3^r + 4^r + \ldots + x^r = \frac{1}{r+1}(x+1).x.(x-1)$ $.(x-2)\ldots(x-r+1) + \frac{A}{r}(x+1).x.(x-1).(x-2)\ldots$ $(x-r+2) + \frac{B}{r-1}(x+1).x.(x-1).(x-2)\ldots(x-r+3)$ $+ \frac{C}{r-2}(x+1).x.(x-1).(x-2)\ldots(x-r+4) + \ldots + \frac{H}{3}$ $(x+1).x.(x-1) + \frac{I}{2}(x+1).x = S.$

Sint $A = 1+2+3+4+\ldots+(r-2)+(r-1)$, $A' = 1+2+3+\ldots+(r-3)+(r-2)$, $A'' = 1+2+3+\ldots+(r-4)+(r-3)$, $A''' = 1+2+3+\ldots(r-5)+(r-4)$, &c.; ſummæ rectangulorum e quibuſque duobus numeris $1, 2, 3, 4, \ldots r-2, r-1$; $1, 2, 3, 4, \ldots r-3, r-2$; $1, 2, 3, 4, \ldots r-4, r-3$; &c.; dicantur reſpective B, B', B'', &c.; contentorum e quibuſque tribus, quatuor, &c. prædictis numeris dicantur reſpective C, C', C'', &c.; D, D', D'', &c.; E, E', E'', &c.; &c.; tum erunt $P = A$, $Q = PA' - B$, $R = QA'' - PB' + C$, $S = RA''' - QB'' + PC' - D$; & ſic deinceps.

Cor. 1. Hinc $1 \times 2 \times 3 \times 4 \times \ldots \times (r-1) \times r \times (r+1) \times S$ ſemper erit diviſibilis per $(x+1).x$.

Cor. 2. Sit $x+1$ primus numerus, qui major eſt quam $r+1$; tum prædictum aggregatum S diviſibile erit per $x+1$. Sit x primus numerus major quam $r+1$, & aggregatum S diviſibile erit per x.

Cor. 3. 1. Sit $x+1$ primus numerus, & $a+b+c+d+$ &c. minor quam x; ſint etiam $\alpha, \beta, \gamma, \delta$, &c. prædicti numeri $1, 2, 3, 4, 5, \ldots x$; tum aggregatum e ſingulis valoribus contentorum $\alpha^a \beta^b \gamma^c \delta^d$ &c. $+ \alpha^b \beta^a \gamma^c \delta^d$ &c. $+ \alpha^a \beta^b \gamma^d \delta^c$ &c. $+ \alpha^b \beta^a \gamma^d \delta^c$ &c. $+$ &c. diviſibile erit per $x+1$.

2do. Sit x primus numerus & $a+b+c+d+$ &c. minor quam $x-1$, tum prædictum aggregatum diviſibile erit per x.

Patet

Patet e prob. tertio primi libri; ex eo enim conſtat ſingulum terminum in aggregato per methodum in prædicto problemate traditam, diviſibilem eſſe per $x+1$ & ejus poteſtates in priori caſu; & per x & ejus poteſtates in poſteriori caſu.

7. Sit $a+b+c+d+\&c.=x$, per cor. 1. prob. 3. inveniatur contentum e ſingulis (x) radicibus $=\alpha\times\beta\times\gamma\times\delta\times\&c.=1.2.3.4\ldots(x-1)\times x$; per theor. ſingulus terminus in prædicto contento ſic invento diviſibilis erit per $x+1$ & ejus poteſtates; ſi $x+1$ ſit primus numerus; ni terminus qui ſit ſumma ex ejus (x) poteſtatibus, i. e. $1^x+2^x+3^x+4^x\ldots+\overline{x-1}^x+x^x=T$; ſed per theor. 52. ſingulus terminus 1^x, 2^x, 3^x, 4^x, &c. per $x+1$ diviſus relinquit unitatem, ergo prædicta ſumma per $x+1$ diviſa relinquit x, & conſequenter contentum $1.2.3.4\ldots(x-1).x+1$ per $x+1$ diviſum relinquit o; ſi $x+1$ ſit primus numerus.

Cor. Hinc contentum $(z-1).(z-2).(z-3)\ldots(z-x)=z^x-Az^{x-1}+Bz^{x-2}-Cz^{x-3}\ldots+Pz^2-Qz+1.2.3.4\ldots(x-1).x$, ubi z eſt integer numerus per $x+1$ diviſum vel relinquit o vel x: ſi z per $x+1$ haud ſit diviſibilis, tum facile conſtat unum e factoribus $z-1$, $z-2$, $x-3$, .. $(z-x)$ per $x+1$ eſſe diviſibilem, & exinde reſiduum prædictum $=o$; ſi vero z per $x+1$ ſit diviſibilis, tum reſiduum ex diviſione prædicti contenti per $x+1$ idem erit ac reſiduum ex diviſione contenti $1.2.3.4\ldots(x-1).x$ per $x+1$.

Aliter e prædictis conſtat numeros A, B, C, D, ... P & Q diviſibiles eſſe per $x+1$; etiamque poteſtatem z^x per $x+1$ diviſam relinquere 1, ſi z haud dividi poſſit per $x+1$; & ultimo contentum $1.2.3.4\ldots(x-1).x$ per $x+1$ diviſum relinquere z; unde, ſi z^x haud dividi poſſit per $x+1$, ſumma $z^x+1.2.3.4\ldots(x-1).x$ diviſa per $x+1$, relinquit $1+x$, i. e. o, & exinde conſtat cor.

8. Sit $1^r+2^r+3^r+4^r\ldots+x^r=S$, ubi r ſit impar numerus, tum erit $1.2.3.4\ldots r\times(r+1)\times S$ ſemper diviſibilis per $(x+1)^2\times x^2$.

2. Sit

9. Sit r par numerus, & erit $1 . 2 . 3 . 4 \ldots \times r \times (r+1) \times S$ ſemper diviſibilis per $x . (x+1) . (2x+1)$.

Cor. Iiſdem literis eaſdem quantitates denotantibus, ſit $a+b+c+d+$ &c. minor quam x & ſumma $a+b+c+d+$ &c. impar numerus, etiamque $x+1$ primus numerus; tum aggregatum e ſingulis quantitatibus hujuſce formulæ $\alpha^a \beta^b \gamma^c \delta^d$ &c. $+ \alpha^b \beta^a \gamma^c \delta^d$ &c. $+ \alpha^a \beta^b \gamma^d \delta^c$ &c. + &c. ſemper diviſibile erit per $(x+1)^2$.

2. Sit ſumma $a+b+c+d+$ &c. impar numerus & minor quam $x-1$, & x ſit primus numerus; tum erit prædictum aggregatum ſemper diviſibile per x^2.

3. Sit $a+b+c+d+$ &c. minor quam $2x$, & $a, b, c, d,$ &c. reſpective pares numeri, & $2x+1$ primus numerus; tum aggregatum prædictum dividi poteſt per $2x+1$.

10. Sit S minimus communis denominator fractionum $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \ldots \frac{1}{m}, \frac{1}{m+1}$; & $P = 1^m + 2^m + 3^m + 4^m + 5^m + \ldots + x^m$ & $Q = 1 + 2^2 + 3^2 + 4^2 + 5^2 + \ldots + x^2$; tum (ſi m ſit par numerus) $S \times P$ ſemper dividi poteſt per Q.

2. Sit m impar numerus, & $1+2+3+4+5+\ldots+x=p$, & $1^3+2^3+3^3+4^3+5^3+\ldots+x^3=p'$; tum $S \times D$ dividi poteſt per p & p'.

In quibuſdam aliis etiam caſibus, ut facile conſtat, P dividi poteſt per Q in priori caſu; & per p, etiamque per p' in poſteriori caſu.

11. Sint $\alpha, \beta, \gamma, \delta,$ &c. radices æquationis $z^n - 1 = 0$, & $a+b+c+d+$ &c. haud major quam n, tum erit aggregatum e ſingulis valoribus contentorum $\alpha^a \beta^b \gamma^c \delta^d$ &c. $+ \alpha^b \beta^a \gamma^c \delta^d$ &c. $+ \alpha^a \beta^c \gamma^b \delta^d$ &c. + &c. $= P$ diviſibile per nullum primum numerum majorem quam n; etiamque ſi numerus literarum $\alpha, \beta, \gamma, \delta,$ &c. in ſingulis contentis $\alpha^a \beta^b \gamma^c \delta^d$ &c. ſit m, tum diviſibile erit per nullum primum numerum majorem quam $m-1$, niſi per n vel primos numeros ejus (n) diviſores, &c.

THEOR.

THEOR. LVIII.

1. $\overline{(a+b)\times b+a^2}$; 2. $\overline{(a+b+c)\times c+(a+b)\times b+a^2}$ vel $\overline{(a+b+c)\times c+\overline{(a+b)\times b+a^2}\times c+\overline{(a+b)\times b+a^2}\times b+a^3}$; 3. $(a+b+c+d)\times d+(a+b+c)\times c+(a+b)\times b+a^2$ vel $\overline{(a+b+c+d)\times d+\overline{(a+b+c)\times c+\overline{(a+b)\times b+a^2}}\times d+\overline{(a+b+c)\times c+\overline{(a+b)\times b+a^2}\times c+\overline{(a+b)\times b+a^2}\times b+a^3}=$ P vel $P\times d+\overline{\overline{(a+b+c)\times c+\overline{(a+b)\times b+a^2}\times c+\overline{(a+b)\times b+a^2}\times b+a^3}\times c+\overline{\overline{(a+b)\times b+a^2}\times b+a^3}\times b+a^4}$; 4. $(a+b+c+d+e)\times e+(a+b+c+d)\times d+(a+b+c)\times c+(a+b)\times b+a^2=P$, vel $P\times e+\overline{(a+b+c+d)\times d+\overline{(a+b+c)\times c+\overline{(a+b)\times b+a^2}}\times d+\overline{(a+b+c)\times c+\overline{(a+b)\times b+a^2}\times c+\overline{(a+b)\times b+a^2}\times b+a^3}=\mathfrak{Q}$, vel $\mathfrak{Q}\times e+\overline{\overline{(a+b+c+d)\times d+\overline{(a+b+c)\times c+(a+b)\times b+a^2}\times d+\overline{(a+b+c)\times c+(a+b)\times b+a^2}\times c+\overline{(a+b)\times b+a^2}\times b+a^3}\times d+\overline{(a+b+c)\times c+\overline{(a+b)\times b+a^2}\times c+\overline{(a+b)\times b+a^2}\times b+a^3}\times c+\overline{\overline{(a+b)\times b+a^2}\times b+a^3}\times b+a^4}$; & ſic in infinitum eadem erit functio quantitatum a, b, c, d, e, f, &c. & conſequenter ſemper eadem manebit, utcunque mutetur ordo literarum a, b, c, d, e, f, &c. Quantitas $\overline{\overline{\overline{(a+b)\times b+a^2}\times b+a^3}\times b+a^4}$ &c. eadem erit functio quantitatum a & b, &c.

2. Sit F functio, in quâ literæ $(x, y, z$, &c.) ſimiliter involvuntur: ſint $\phi(e, f, g$, &c.), $\phi(e, f, h$, &c.), $\phi(f, g, h$, &c.), &c. functiones, in quibus e, f, g, &c. e, f, h, &c.; e, g, h, &c.; &c.; reſpective ſimiliter involvuntur: denotent etiam $\pi(e)$, $\pi(f)$, $\pi(g)$, $\pi(h)$, &c. eaſdem functiones literarum e, f, g, h, &c.; in functione F pro literis $(x, y, z, v$, &c.) ſcribantur reſpective $\pi(e)+\phi(f, g, h$, &c.), $\pi(f)$

$+ \varphi(e, g, h, \&c.)$, $\pi(g) + \varphi(e, f, h, \&c.)$, $\pi(h) + \varphi(e, f, g, \&c.)$, &c. reſultabit quantitas in quâ ſimiliter involventur literæ e, f, g, h, &c.

THEOR. LIX.

Sint a, b, c, d, e, &c. h, k, l, m, n quæcunque quantitates; tum erit

$$a \times b \times (a+b) + (a+b) \times c \times (a+b+c) + (a+b+c) \times d \times (a+b+c+d) + (a+b+c+d) \times e \times (a+b+c+d+e) + \&c. \ldots + (a+b+c+d+\&c.+h) \times k \times (a+b+c+d+\&c.+h+k) + (a+b+c+d+\&c.+h+k) \times l \times (a+b+c+d+\&c.+h+k+l) + (a+b+c+d+\&c.+h+k+l) \times m \times (a+b+c+d+\&c.+h+k+l+m) + (a+b+c+d+\&c.+h+k+l+m) \times n \times (a+b+c+d+\&c.+h+k+l+m+n) = n \times m \times (n+m) + (n+m) \times l \times (n+m+l) + (n+m+l) \times k \times (n+m+l+k) + (n+m+l+k) \times h \times (n+m+l+k+h) + \&c. \ldots + (n+m+l+k+h+\&c.+e) \times d \times (n+m+l+k+h+\&c.+e+d) + (n+m+l+k+h+\&c.+e+d) \times c \times (n+m+l+k+h+\&c.+e+d+c) + (n+m+l+k+h+\&c.+e+d+c) \times b \times (n+m+l+k+h+\&c.+e+d+c+b) + (n+m+l+k+\&c.+e+d+c+b) \times a \times (n+m+l+k+h+\&c.+e+d+c+b+a).$$

Demonſtratur in theor. 18. noſt. curvarum proprietatum.

PROB. LXIV.

1. *Sit æquatio, quæ ſit functio quantitatum* (x, y, z, *&c.*), *in quâ continentur* (n) *coefficientes ad libitum aſſumendæ; dentur etiam* (n) *correſpondentes valores e ſingulis incognitis quantitatibus* (x, y, z, *&c.*) *invenire coefficientes prædictas.*

Pro incognitis quantitatibus $(x, y, z$, &c.) ſcribantur earum (n) correſpondentes valores, reſultant (n) æquationes totidem (n) incognitas

cognitas quantitates, i. e. (n) coefficientes ad libitum aſſumendas, habentes; e quibus detegi poſſunt coefficientes ipſæ.

2. Sint m æquationes ($n+m$) incognitas quantitates ($x, y, z,$ &c.) habentes, & in ſingulis hiſce æquationibus contineantur (r) incognitæ coefficientes ad libitum aſſumendæ; tum ex datis (r) correſpondentibus valoribus ſingularum incognitarum quantitatum ($x, y, z,$ &c.) per methodum in præcedente caſu traditam erui poſſunt coefficientes ipſæ.

PROB. LXV.

Invenire quantitatem, quæ evadat reſpective S, S', S'', S''', &c.; *cum* $x=\alpha$, $y=\beta$, $z=\gamma$, &c.; $x=\alpha'$, $y=\beta'$, $z=\gamma'$, &c.; $x=\alpha''$, $y=\beta''$, $z=\gamma''$, &c.

Aſſumantur quæcumque functiones ($H, K, L,$ &c.; $A, B, C, D,$ &c.) quantitatum ($x, y, z,$ &c.); in functionibus $H, K, L,$ &c. pro incognitis quantitatibus ($x, y, z,$ &c.) ſcribantur reſpective $\alpha, \beta, \gamma,$ &c.; $\alpha', \beta', \gamma',$ &c.; $\alpha'', \beta'', \gamma''$, &c.; $\alpha''', \beta''', \gamma'''$, &c.; reſultent quantitates $h, k, l,$ &c. reſpective: in functione (A) pro incognitis quantitatibus ($x, y, z,$ &c.) ſcribantur prædictæ quantitates $\alpha, \beta, \gamma,$ &c.; $\alpha', \beta', \gamma',$ &c.; $\alpha'', \beta'', \gamma''$, &c.; reſultent quantitates $a, a', a'', a''',$ &c.: & ſimiliter in functionibus $B, C, D,$ &c. ſcribantur pro incognitis quantitatibus ($x, y, z,$ &c.) prædictæ quantitates; reſultent quantitates $b, b', b'',$ &c.; $c, c', c'',$ &c.; d, d', d'', &c. Quantitas quæſita poteſt eſſe $\frac{H\times(A-a')\times(A-a'')\times(A-a''')\times \&c.}{h\times(a-a')\times(a-a'')\times(a-a''')\times \&c.}$

$$\times S + \frac{K\times(B-b)\times(B-b'')\times(B-b''')\times \&c.}{k\times(b'-b)\times(b'-b'')\times(b'-b''')\times \&c.}\times S' + \frac{L\times(C-c)\times(C-c')\times(C-c''')\times \&c.}{l\times(c''-c)\times(c''-c')\times(c''-c''')\times \&c.}\times S'' + \&c.$$

2. Invenire quantitatem, quæ evadat S^a, S^b, S^c, S^d, &c., cum $x=\alpha_1$,

i, vel

vel $x = \beta$, vel $x = \gamma$, vel $x = \delta$, &c. reſpective: quantitas quæſita poteſt eſſe $\frac{(x - \beta) \times (x - \gamma) \times \&c.}{(\alpha - \beta) \times (\alpha - \gamma) \times \&c.} \times S^a + \frac{(x - \alpha) \times (x - \gamma) \times \&c.}{(\beta - \alpha) \times (\beta - \gamma) \times \&c.} \times S^b + \frac{(x - \alpha) \times (x - \beta) \times \&c.}{(\gamma - \alpha) \times (\gamma - \beta) \times \&c.} \times S^c + \&c.$

SCHOLIUM.

Hactenus de algebraicis æquationibus; eadem etiam applicari poſſunt ad ullas alias æquationes, quæ e ſubſtitutione transformari poſſunt in algebraicas æquationes; e. g. ſit $x^{nv} + Ax^{(n-1)v} + Bx^{(n-2)\times v} +$ &c. $= o$, pro x^v ſcribatur z, & transformatur data exponentialis æquatio in algebraicam $x^n + Az^{n-1} + Bz^{n-2} +$ &c. $= o$, cujus radices ſint $\alpha, \beta, \gamma, \delta$, &c. & erit $x^v = \alpha$, $x^v = \beta$, &c. unde (n) erunt diverſi valores quantitatis (x^v).

1. Sit $e^v = a$, ubi e ſit data affirmativa quantitas & major quam 1; tum, ſi a ſit affirmativa, erit v affirmativa vel negativa quantitas, prout a major ſit quam 1, necne: ſi a ſit negativa quantitas; tum v erit fractio, cujus denominator (w) ſit par numerus, numerator (v) vero impar; etiamque erit affirmativa vel negativa, prout a major ſit quam 1, necne. Sit e affirmativa & minor quam 1; tum erit v negativa vel affirmativa quantitas prout a major ſit quam $+1$ vel -1, necne. Sit e negativa quantitas & major quam 1; tum erit v affirmativa vel negativa quantitas, prout a major ſit quam $+1$ vel -1, necne. Sit e minor quam 1; & v erit affirmativa vel negativa quantitas, prout a minor ſit quam $+1$ vel -1, necne.

2. Sit $x^x = a$; cum omnis quantitas evadat maxima vel minima, tum ejus fluxio nihilo erit æqualis; fluxio quantitatis x^x erit $x^x\dot{x} + x^x \log. x\dot{x}$, nihilo fiat æqualis, reſultat log. $x = -1$; deinde inve-

niatur quantitas, cujus hyperb. log. $= -1$, quæ erit ,63 &c. $= \varrho$; ſit $\varrho^{\prime} = \pi$; 1mo. ſit a affirmativa quantitas & major quam $\frac{1}{\pi}$, tum dabitur unus affirmativus valor quantitatis x; 2do. ſi a inter π & $\frac{1}{\pi}$ ponatur, tum erit unus valor quantitatis x affirmativus, alter vero negativus; 3^tio^. ſi a inter o & π ponatur, tum ſolummodo erit unus negativus valor quantitatis x. Sit a negativa quantitas & major quam $\frac{1}{\pi}$, tum dabitur affirmativus valor quantitatis x, qui erit fractio, cujus denominator ſit par & numerator impar numerus; 2do. ſi a inter π & $\frac{1}{\pi}$ ponatur, tum unum affirmativum & unum negativum valorem habet quantitas x; 3^tio^. ſi a minor ſit quam π, tum unum ſolummodo habet negativum valorem.

Sint vero plures exponentiales quantitates in datâ æquatione contentæ, etiamque exponentiales ſuperiorum ordinum; & inveniatur numerus diverſorum valorum, quos habet ſingula exponentialis diverſi generis quantitas: & componantur numeri reſultantes, & exinde ſæpe inveniri poteſt numerus valorum incognitæ quantitatis in datâ æquatione: & ſic de transformatione æquationum hujuſce generis; de inveniendis impoſſibilibus, affirmativis & negativis, integris & rationalibus earum radicibus. e. g. Sit $x^x \pm$ &c. $= a$, ubi a ſit integer numerus; &, ſi modo x ſit integer numerus, erit diviſor numeri a; &c. Omnes numeri minime conſtare poſſunt e finitâ multitudine numerorum x^x vel e^x, ubi e & x integros denotant numeros: ſucceſſivi enim termini in nimis magnâ ratione creſcunt & decreſcunt: ſic e. g. $a^{x+1} - 1$ ſemper ad minimum conſtat e pluribus quam $(a-1) \times x$ terminis hujus generis a^x. Sed de exponentialibus ſcribere non eſt animus; & hiſce Meditationibus finis jam imponendus.

ponendus eſt: hoc contentus, quod transformationes; methodos detegendi impoſſibiles, affirmativas & negativas radices; conſtitutiones, reductiones & reſolutiones; æquationum exterminationes irrationalium quantitatum; & proprietates integrorum numerorum & æquationum, &c.; modis ni fallor magis generalibus quam ab aliis factum eſt, tradiderim.

C O R R I G E N D A.

PAG. 2. lin. 11. pro $(m \times m - n + 1)$, lege $m \times (m - n + 1)$; p. 7. l. ult. dele &c.; p. 12. l. 27. pro $S^{a+b} + S^{a+c}$, lege $S^{a+b} \times S^{a+c}$; l. 28. pro $\times S^{c+d}$, lege $+ S^{c+d}$; p. 15. l. 15. pro $\frac{m+8}{3}$ lege $\frac{m+8}{3}D$; p. 16. l. 15. dele in eo; p. 18. l. 1. dele 11; l. 22. pro fient, lege fiet earum ſumma; p. 21. l. 13. pro Rx, lege Rx^2; l. 23. pro V lege U; p. 22. l. 20. pro fingatur, lege fingantur; p. 30. l. 12. pro ſimplicem, lege ſimplicem communem; p. 31. l. ult. pro A^2 ſemel, lege B^2; p 32. l. 16. pro poſſint, lege poſſunt; l. 30. pro quæ, lege qui; pag. 33. l. 9. & 21. & p. 33. l. 3. & 10. pro contentum, lege aggregatum contentorum; p. 33. l. 13. pro tertii, lege quarti; p. 34. l. 14. pro ſint, lege ſunt; p. 38. pro Theor. lege Theor. 2; p. 39. l. 23. pro $n-1$, lege $n-2$; p. 40. l. 22. pro x^{n-1} ſemel lege x^{n-2}; l. penult. pro $=$, lege $= B$; p. 41. l. ult. pro a ſemel lege α; p. 43. pro Theor. 2. lege Theor. 3; p. 46. l. 19. pro x^{n-3}, lege x^{n-2}; p. 46. pro Theor. lege Theor 4; p. 51. l. 11. pro ponentur, lege ponetur; p. 52. l. 2. pro quæ, lege qui; p. 53. pro Theor. lege Theor. 5; p. 54. pro Theor. lege Theor. 6; p. 57. l. 14. & 21. pro ex, lege de; p. 60. l. 1. pro & β, lege & γ; p. 63. l 7. pro literarum, lege numerus literarum; l. 10. pro 2Γ, lege $2TL$; l. 18, 19, 20. pro $\mp 2X$, lege $\pm 2X$; p. 65. l. 2. pro quadratis, lege quantitatibus; l. 11. pro $n-m-1$, lege $(n-m) \times (n-m-1)$; p. 66. l. ult. pro $\mp$, lege $\pm$; p. 67. l. 1. pro ib $+$, lege ib $-$, & pro $\pm$, lege $\mp$; l. 13. pro 2, lege & m evadat $m-1$; l. 16. pro 4, lege 4 & m evadat $m-2$; p. 76. l. 11. pro deducere, lege deducere regulas; p. 78. l. 14. pro 4, lege 4^2; p. 88. l. antepen. dele ab; p. 89. l. 15. pro $= o$, lege $= v$; p. 92, 93, & 96. pro Theor. lege Theor. 10, 11, & 12; p. 100. l. 18. pro in quibus, lege in quâ; l. 21. pro x, v, lege x, y, v; p. 102. l. 7. pro (x), lege (y); p. 105. l. 21. dele $= o$; p. 109. l. 9. pro earum, lege ſuas; p. 112. l. 1. pro ab unâ, lege per unam; l. ult. pro quoniam, lege , quoniam; p. 118. l. 12. pro x^{m-2}, lege x^{n-2}; p. 119. l. 15. pro r, lege a; p. 121. l. 21. pro $n-m$, lege $n-m+1$; p. 122. l. 10. pro quantitas, lege quantitates; p. 127. l. 24. pro primi, lege primi, etiamque diverſi; p. 129. l. 5. pro necne, lege vel integralis; p. 131. l. 11. pro conſtat, lege conſtant; p. 133. l. 4. pro quâ, lege eâque; p. 136. l. 14. pro quæ, lege qui; p. 140. l. 6. pro $2p$, lege p; p. 143. l. penult. pro ſuam, lege ſuum; p. 144. l. ult. pro r, s, lege q, r, s; p. 145. l. 18. & ult. pro $+ \frac{1}{3}q$, lege $- \frac{1}{3}q$; p. 146. l. 1. & 2. pro $+ \frac{1}{3}q$, lege $- \frac{1}{3}q$; l. 12. pro π & ϱ, lege π & ϱ'; p. 154. l. 2. pro eorum, lege iis; p. 161. l. 27. pro æquatio, lege quantitas, & pro & n erunt, lege & ex æquatis correſpondentibus terminis n reſultant; p. 164. l. 10. & penult. pro $\sqrt{(b^2)}$, lege $\sqrt{(b)}$; p. 176. l. 24. pro x^n, lege $x^n - px^{n-1}$; p. 178. l. 16. pro $\sqrt[n]{\ }$, lege $\sqrt[k]{\ }$, & ſemel, pro &c., lege &c., &c.; p. 181. l. 19. pro & erit, lege ubi; p. 183. l. 2. poſt; lege, ubi $1, \pi, \varrho, \sigma, \tau$ ſunt radices æquationis $x^5 - 1 = 0$; p. 189. l. 5. pro æquationes, lege æquationes n incognitas quantitates $(x, y, z, \&c.)$ habentes; p. 197. & 198. pro $V^{n-2}S$, lege $V^{n-1}S$, & pro $+ V^{n-3}R$, lege $- V^{n-1}R$, & pro $+ V^{n-3}TS$, lege $- V^{n-2}TS$; p. 202. l. 19. pro 4, lege 5; l. 23. pro radices, lege radices quantitatis (y); p. 205. l. 12. pro $- \frac{1}{2}c +$ ſemel, lege $- \frac{1}{2}c -$; p. 206. l. 5. pro $-$, lege $- n$, l. 21. pro quantitas, lege quantitatis; p. 207. l. 6. pro $= o$, lege $= o$, ubi n & m majores ſunt quam r & s reſpective; l. 20. pro $+$ &c., lege $+$ &c. $= o$, exortâ; l. 21. pro $m + n$, lege $m + n - 1$; l. antepen. pro poteſt, lege poſſunt; p. 209. l. 5. dele $= o$; p. 211. pro diviſibilum, lege diviſibilium; p. 217. l. 9. pro tot, lege totidem; p. 218. l. 22. dele æquationes; l. 19. pro recipiet, lege recipiat; p. 228. l. 8. pro præcedenta, lege præcedentia; p. 231. l. 15. pro $z^{p'm+d}$, lege $z^{p\,m+e}$; l. 11. pro progredientium, lege progredientibus; l. 14. pro prædictæ ſeries, lege prædictarum ſerierum; p. 235. l. 11. pro inveniant, lege invenient; p. 238. l. 22. pro reſpondentium, lege reſpondentium valorum; p. 240. l. antepen. pro annnon, lege annon; p. 241. l. 25. pro $\beta' = o$, lege $\beta'' = o$; p. 248. l. 1. & 2. pro $i\left(\frac{i+1}{2}\right) \ldots \left(\frac{i+b-1}{b}\right) - 1$, lege $(i+1)\left(\frac{i+2}{2}\right) \ldots \left(\frac{i+b-1}{b}\right) - 1$; p. 250. l. 9. pro x, y, lege y, x; p. 251. l. 26. pro &, lege ut; l. penult. ſemel dele vel; p. 252. l. penult. pro infinitæ, lege infinitas; p. 256. l. 21. pro duæ æquationes habent, lege termini duarum æquationum, in quibus inveniuntur maximæ dimenſiones, habent; l. 25. pro prædictæ æquationes, lege prædicti termini; p. 259. l. 14. pro involvent, lege involvet; p. 263. l. 17. pro x, lege v; l. 25. pro habet, lege habeat; p. 264. l. 1.

pro

CORRIGENDA.

pro x, ſcribatur v, & l. 2. pro x bis, ſcribatur v; l. 15. pro quantitatbus, lege quantitatibus; l. 16. pro $\varphi : a$, lege $\varphi : \alpha$; p. 274. l. 24. pro qui, lege quæ; p. 275. l. 12. pro valores, lege diviſores; p. 278. l. 17. pro vel, lege vel reſultant; l. 23. pro erit, lege unde; p. 287. l. 20. pro s^{s-2}, lege a^{s-2}; p. 289. l. 15. pro $4x$, lege $2x$; p. 291. l. 2. pro s, lege $s - \frac{1}{2}p\varrho$; l. 9. pro l, lege β; l. 16. pro P^6, P^4, & P^2, lege P^3, P^2, & P; p. 292. pro $3n+2$, lege $3n+3$; p. 294. l. 5. pro $=B$, lege $=B$, &c.; p. 296. l. ult. pro b, lege $\surd(b)$; p. 300. l. 21. pro 3, lege 4; p. 315. l. 12. pro 1, lege quod dividit [illegible] p. 316. l. 2. pro ab, lege $-ab$; p. 317. l. 5. pro y, lege $y +$ &c.; p. 324. l. 19. pro maximæ, lege maximæ vel minimæ; p. 325. l. 3. pro $-x^3$, lege $-z^3$; p. 327. l. 15. pro xx, lege xz; p. 329. l. 20. pro v^2, lege w^2; p. 331. l. 9. pro incognitas, lege incognitas (n); p. 332. l. 23. & 24. pro n, $n-1$, & $n-2$, lege n', $n'-1$, & $n'-2$; p. 337. l. 10. 13. & 14. pro v^2 & v, lege y^2 & y; l. 20. pro ab, lege ab^2; p. 340. l. 8. pro correſpondens, lege correſpondens rationalis; l. 19. pro progredientem, lege progrediens; l. ult. pro notationem, lege notationem deſignatum; p. 346. l. 17. pro erit, lege ſit; p. 353. pro N, lege A; p. 358. l. 13. pro $\times n-1$, lege $\times n-n$; p. 363. l. 7. pro diſivione, lege diviſione; p. 364. l. 26. & 27. pro non reſidua, lege reſidua; p. 365. l. 5. poſt numerum, lege ni $n-1$ & $m-1$ habeant communem diviſorem majorem quam 2; p. 366. l. 23. dele nam; p. 367. l. antepen. pro $4x+1$, lege $4x-1$; p. 368. l. 17. dele diviſi; l. 27. poſt nb^2, lege ubi $n = 4v + 1$; p. 373. l. 11. pro a^2, lege a^4; l. 14. pro a, lege 2; p. 374. l. 10. pro a^2, lege a^4; p. 377. l. 17. pro generum, lege generum inveniendis; p. 384. l. 3. pro e, g, h, lege f, g, h.

ADDENDA.

A D D E N D A.

PAG. 4. Cor. 1. Sit data æquatio $x^n - Px^{n-m} + \mathcal{Q}x^{n-2m} - Rx^{n-3m} \ldots \mp H^{n-\alpha\times m} \pm Kx^{n-\alpha m-1} \mp \&c. = A$; tum ſumma omnis poteſtatis (a); ubi a per m haud diviſibilis & minor eſt quam $\alpha \times m$, e ſingulis radicibus nihilo erit æqualis.

Cor. 2. Sit æquatio $x^n - px^{n-1} + qx^{n-2} - rx^{n-3} + \ldots \pm bx^{n-m} \mp kx^{n-m-1} \pm \&c. = 0$; ſit b primus terminus, qui continet functionem quantitatis z; tum eadem erit ſumma omnis poteſtatis e ſingulis radicibus, quæ minor eſt quam m, utcunque varietur quantitas z.

Cor. 3. Sit æquatio, cujus reſolutio cognoſcitur, exinde ſæpe deduci poteſt ſumma poteſtatis (a) e ſingulis radicibus.

Ex. Sit æquatio $x^n - nax^{n-2} + n.\frac{n-3}{2}a^2x^{n-4} - \&c. = -b$, cujus reſolutio eſt $x = \sqrt[n]{-\frac{1}{2}b + \sqrt{\frac{1}{4}b^2 - a^n}}\,(P) + \sqrt[n]{-\frac{1}{2}b - \sqrt{\frac{1}{4}b^2 - a^n}}\,(\mathcal{Q})$; ſint $\alpha, \beta, \gamma, \delta$, &c. reſpectivæ radices æquationis $x^n - 1 = 0$; & radices æquationis $x^n - nax^{n-2} + n.\frac{n-3}{2}a^2x^{n-4} - \&c. = -b$ erunt reſpective $\alpha P + \frac{1}{\alpha} \times \mathcal{Q}$, $\beta P + \frac{1}{\beta} \times \mathcal{Q}$, $\gamma P + \frac{1}{\gamma}\mathcal{Q}$, $\delta P + \frac{1}{\delta}\mathcal{Q}$, &c.: unde ſumma $(2r)$ poteſtatis e ſingulis radicibus (ubi $2r$ minor eſt quam n) $=$

$$\alpha \times \left(\overline{-\tfrac{1}{2}b + \sqrt{\tfrac{1}{4}b^2 - a^n}}\,^{\frac{1}{n}}\right)^{2r} + \frac{1}{\alpha} \times \left(\overline{-\tfrac{1}{2}b - \sqrt{\tfrac{1}{4}b^2 - a^n}}\,^{\frac{1}{n}}\right)^{2r} =$$

$(\alpha P +$

A D D E N D A.

$$(\alpha P+\frac{1}{\alpha}Q)^{2r}=\alpha^{2r}P^{2r}+2r\times\alpha^{2r-2}P^{2r-1}Q\ldots 2r\times\frac{2r-1}{2}\times\frac{2r-2}{3}\ldots\frac{r+1}{r}\times\alpha^{r}\times\frac{1}{\alpha^{r}}(1)\times P^{r}Q^{r}(a^{r})+\&c.$$

$$(\beta P+\frac{1}{\beta}Q)^{2r}=\beta^{2r}P^{2r}+2r\beta^{2r-2}P^{2r-1}Q\ldots 2r\times\frac{2r-1}{2}\times\frac{2r-2}{3}\ldots\frac{r+1}{r}\times a^{r}+\&c.$$

$$(\gamma P+\frac{1}{\gamma}Q)^{2r}=\gamma^{2r}P^{2r}+2r\gamma^{2r-2}P^{2r-1}Q\ldots 2r\times\frac{2r-1}{2}\times\frac{2r-2}{3}\ldots\frac{r+1}{r}\times a^{r}+\&c.$$

$$(\delta P+\frac{1}{\delta}Q)^{2r}=\delta^{2r}P^{2r}+2r\delta^{2r-2}P^{2r-1}Q\ldots 2r\times\frac{2r-1}{2}\times\frac{2r-2}{3}\ldots\frac{r+1}{r}\times a^{r}+\&c.$$

$$\&c. \quad = \quad \&c.$$

at, quoniam $\alpha^{2r}+\beta^{2r}+\gamma^{2r}+\delta^{2r}+\&c.=0$, $\alpha^{2r-2}+\beta^{2r-2}+\gamma^{2r-2}+\delta^{2r-2}+\&c.=0$, & ſic deinceps; erit (π) ſumma $(2r)$ poteſtatis e ſingulis radicibus datæ æquationis $=2r\times\frac{2r-1}{2}\times\frac{2r-2}{3}\ldots\frac{r+1}{r}\times n a^{r}$.

Erit per problema $\pi=(2\times n^{r}-2r\times n^{r-2}\times n.\frac{n-3}{2}+2r\times n^{r-3}\times n.\frac{n-4}{2}\times\frac{n-5}{3}-2r\times n^{r-4}\times n.\frac{n-5}{2}\times\frac{n-6}{3}\times\frac{n-7}{4}+\&c.$
$+2r\times\frac{r-3}{2}n^{r-4}\times(n.\frac{n-3}{2})^{2}-\&c.)\,a^{r}=2r\times\frac{2r-1}{2}\times\frac{2r-2}{3}\ldots\frac{r+1}{r}\times n a^{r}$.

Cor. 4. Sit æquatio $x^{n}-nax^{n-1}+n.\frac{n-1}{2}a^{2}x^{n-2}-n.\frac{n-1}{2}.\frac{n-2}{3}a^{3}x^{n-3}+\&c.=b$, cujus radices erunt $x=a+\sqrt[n]{b}$; unde ejus reſpective radices erunt $a+\alpha\sqrt[n]{b}$, $a+\beta\sqrt[n]{b}$, $a+\gamma\sqrt[n]{b}$, $a+\delta\sqrt[n]{b}$, &c.: ſumma r poteſtatis e ſingulis hiſce radicibus erit, ſi modo r minor ſit quam n,

A D D E N D A.

$(a+\alpha\sqrt[n]{b})^r = a^r + r a^{r-1}\alpha\sqrt[n]{b} + \&c.$

$(a+\beta\sqrt[n]{b})^r = a^r + r a^{r-1}\beta\sqrt[n]{b} + \&c.$

$(a+\gamma\sqrt[n]{b})^r = a^r + r a^{r-1}\gamma\sqrt[n]{b} + \&c.$

&c. $=$ &c.

at quoniam $\alpha+\beta+\gamma+\&c. = o$, $\alpha^2+\beta^2+\gamma^2+\&c. = o$; erit ſumma quæſita $= n \times a^r$.

Per prob. erit ſumma quæſita $= \Big(n^r - r \times n^{r-2} \times n . \frac{n-1}{2} + r \times n^{r-4} \times n . \frac{n-1}{2} \times \frac{n-2}{3} - r \times n^{r-6} \times n . \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} + \&c. + r \times \frac{r-3}{2} \times n^{r-4} \times \left(n . \frac{n-1}{2}\right)^2 - \&c.\Big)$

$a^r = n \times a^r$.

Hæ enim ſunt eædem ſummæ per duos diverſos modos expreſſæ.

Pag. 18. Cor. 2. Sint m, n, r, s, &c. quicunque dati integri affirmativi numeri; & erit quantitas $1 - \left(m . \frac{m-1}{2} + n . \frac{n-1}{2} + r . \frac{r-1}{2} + s . \frac{s-1}{2} + \&c.\right) + \left(m . (m-1) . \frac{m-2}{3} + n . (n-1) . \frac{n-2}{3} + \&c.\right) - \left(m . (m-1) . (m-2) . \frac{m-3}{4} + n . (n-1) . (n-2) . \frac{n-3}{4} + \&c.\right) + \frac{1}{2}\left(m . \frac{m-1}{2} \times (m-2) \times \frac{m-3}{2} + n . \frac{n-1}{2} . (n-2) . \frac{n-3}{2} + 2m . \frac{m-1}{2} . n . \frac{n-1}{2} + \&c.\right) \pm \&c. = \pm 1 . 2 . 3 .. (m-1) \times 1 . 2 . 3 .. (n-1) \times 1 . 2 . 3 .. (r-1) \times 1 . 2 . 3 .. (s-1) \times \&c.$ i. e. ſint termini formulæ $m . (m-1) .. (m-x+2) \times \frac{m-x+1}{x} \times (m-x)$;

$(m-x-1)$

ADDENDA.

$$(m-x-1)\ldots(m-x-x'+2)\times\frac{m-x-x+1}{x'}\times(m-x-x')$$

$$\times(m-x-x-1)\ldots(m-x-x'-x''+2)\times\frac{m-x-x'-x''+1}{x''}$$

$$\times \&c.;\ n.(n-1)..(n-z+2).\frac{n-z+1}{z}\times(n-z).(n-z-1)$$

$$..(n-z-z'+2).\frac{n-z-z'+1}{z'}\times(n-z-z')\times(n-z-z'-1)$$

$$..(n-z-z'-z''+2).\frac{n-z-z'-z''+1}{z''}\times \&c.,\ r.(r-1)..$$

$$(r-w+2).\frac{r-w+1}{w}\times(r-w).(r-w-1)..(r-w-w'+2)$$

$$.\frac{r-w-w'+1}{w'}\times \&c.,\ s.(s-1).(s-v+2).\frac{s-v+1}{v}\times \&c.,$$

&c. ubi x, x', x'', &c.; z, z', z'', &c.; w, w', w'', &c.; v, v', &c., &c. ſint quicunque integri affirmativi numeri: tum erunt aggregata e ſingulis hiſce quantitatibus, & e ſingulis earum productis & contentis (ſignis vero affixis cuicunque quantitati, producto vel contento exiſtentibus + vel −, prout $(x-1)+(x'-1)+(x''-1)+ \&c. + (z-1)+(z'-1)+(z''-1)+ \&c. + (w-1) + \&c.$ ſit par vel impar numerus, in quo contento continentur integri numeri x, x', x'', &c.; z, z', z'', &c.; w, &c.; &c.); erunt $= \pm 1.2.3..(m-1)\times 1.2.3..(n-1)\times 1.2.3..(r-1)\times 1.2.3..(s-1)\times \&c.$

Hoc facile deduci poteſt e prob.

Pag. 118. lin. ult. Cor. Erit $m.\frac{m+1}{2}.\frac{m+2}{3}.\frac{m+3}{4}\ldots\frac{n}{n-m+1}$

$-m$

ADDENDA.

$-m\times m.\frac{m+1}{2}.\frac{m+2}{3}\dots\frac{n-1}{n-m}+m.\frac{m-1}{2}\times m.\frac{m+1}{2}.\frac{m+2}{3}$
$\dots\frac{n-2}{n-m-1}-m.\frac{m-1}{2}.\frac{m-2}{3}\times m.\frac{m+1}{2}.\frac{m+2}{3}\dots\frac{n-3}{n-m-2}$
$+m.\frac{m-1}{2}.\frac{m-2}{3}.\frac{m-3}{4}\times m.\frac{m+1}{2}.\frac{m+2}{3}\dots\frac{n-4}{n-m-3}-$
$\dots\pm m.\frac{m-1}{2}.\frac{m-2}{3}\dots\frac{m-r+1}{r}\times m.\frac{m+1}{2}.\frac{m+2}{3}\dots$
$\frac{n-r}{n-m-r+1}=\pm(n-m-r+1).\frac{n-m-r+2}{2}.\frac{n-m-r+3}{3}$
$.\frac{n-m-r+4}{4}\dots\frac{n-m}{r}\times(m-r).\frac{m-r+1}{2}.\frac{m-r+2}{3}\dots$
$\frac{n-r}{n-m+1}.$

1. Erit $n.\frac{n-1}{2}.\frac{n-2}{3}..\frac{n-r+1}{r}+m\times n.\frac{n-1}{2}.\frac{n-2}{3}..$
$\frac{n-r+2}{r-1}+m.\frac{m-1}{2}\times n\times\frac{n-1}{2}\times\frac{n-2}{3}..\frac{n-r+3}{r-2}+m.\frac{m-1}{2}$
$.\frac{m-2}{3}.n.\frac{n-1}{2}.\frac{n-2}{3}\dots\frac{n-r+4}{r-3}+m.\frac{m-1}{2}.\frac{m-2}{3}.\frac{m-3}{4}$
$.n.\frac{n-1}{2}.\frac{n-2}{3}..\frac{n-r+5}{r-4}+\dots+m.\frac{m-1}{2}.\frac{m-2}{3}.\frac{m-3}{4}$
$\dots\frac{m-r+1}{r}=(n+m)\times\frac{n+m-1}{2}.\frac{n+m-2}{3}.\frac{n+m-3}{4}\dots$
$\frac{n+m-r+1}{r}.$

2. Erit $n.\frac{n-1}{2}.\frac{n-2}{3}..\frac{n-r+1}{r}-m\times n.\frac{n-1}{2}.\frac{n-2}{3}\dots$
$\frac{n-r+2}{r-1}+m.\frac{m+1}{2}.n.\frac{n-1}{2}.\frac{n-2}{3}\dots\frac{n-r+3}{r-2}-m.\frac{m+1}{2}$
$.\frac{m+2}{3}.n.\frac{n-1}{2}.\frac{n-2}{3}\dots\frac{n-r+4}{r-3}+m.\frac{m+1}{2}.\frac{m+2}{3}.\frac{m+3}{4}$

$n\times$

A D D E N D A.

$$\times n . \frac{n-1}{2} . \frac{n-2}{3} .. \frac{n-r+5}{r-4} - \ldots\ldots m . \frac{m+1}{2} . \frac{m+2}{3} . \frac{m+3}{4}$$

$$\ldots \frac{m+r-1}{r} = (n-m) . \frac{n-m-1}{2} . \frac{n-m-2}{3} \ldots \frac{n-m-r+1}{r} .$$

3. Erit $n . \frac{n-1}{2} . \frac{n-2}{3} .. \frac{n-r+1}{r} - n . n . \frac{n-1}{2} . \frac{n-2}{3} . \frac{n-3}{4}$

$$.. \frac{n-r+2}{r-1} + n . \frac{n-1}{2} . n . \frac{n-1}{2} . \frac{n-2}{3} \ldots \frac{n-r+3}{r-2} - n . \frac{n-1}{2}$$

$$. \frac{n-2}{3} . n . \frac{n-1}{2} . \frac{n-2}{3} \ldots \frac{n-r+4}{r-3} + n . \frac{n-1}{2} . \frac{n-2}{3} . \frac{n-3}{4} .$$

$$n . \frac{n-1}{2} . \frac{n-2}{3} \ldots \frac{n-r+5}{r-4} - \ldots\ldots n . \frac{n-1}{2} . \frac{n-2}{3} . \frac{n-3}{4} \ldots$$

$\frac{n-r+1}{r} = \pm n . \frac{n-1}{2} . \frac{n-2}{3} . \frac{n-3}{4} \ldots \frac{n-\frac{r}{2}+1}{\frac{1}{2}r}$: erit $+$, ſi $r = 4a$; ſin $r = 4a+2$, ubi a eſt integer numerus, erit $-$; ſi r ſit impar, tum erit prædictum contentum $= o$.

Pag. 119. lin. penult. Cor. Si pro $a, b, c, d,$ &c. in cor. problem. tertii ſcribantur $\frac{1-r^m}{1-r}, \frac{1-r^{2m}}{1-r^2}, \frac{1-r^{3m}}{1-r^3}, \frac{1-r^{4m}}{1-r^4},$ &c. reſpective, tum evadet ſeries $\overline{\frac{1-r^m}{1-r}}\Big|^{n} - n . \frac{n-1}{2} \overline{\frac{1-r^m}{1-r}}\Big|^{n-2} \times \frac{1-r^{2m}}{1-r^2} + n .$ $(n-1) . \frac{n-2}{3} \overline{\frac{1-r^m}{1-r}}\Big|^{n-3} \times \frac{1-r^{3m}}{1-r^3} - \frac{\&c.}{\&c.} = 1 . 2 . 3 . 4 .. n \times$ $\frac{1-r^m}{1-r} \times \frac{1-r^{m-1}}{1-r^2} \times \frac{1-r^{m-2}}{1-r^3} \ldots \frac{1-r^{m-n+1}}{1-r^n} \times r^{n . \frac{n-1}{2}}$.

Pag. 128. Cor. 2. Sint $\alpha, \beta, \gamma, \delta,$ &c., (n) radices æquationis $x^n - 1 = o$, etiamque ſint quæcunque (m) quantitates $a, b, c, d, e,$ &c.; tum erit aggregatum (π) ſingulorum contentorum ſub $(b \times n)$ quantitatibus ſubſequentibus $\alpha a, \beta a, \gamma a, \delta a,$ &c.; $\alpha b, \beta b, \gamma b, \delta b,$ &c.; $\alpha c,$ ſin-

ſingulorum contentorum ſub (b) quantitatibus ſubſequentibus a^{n}, b^{n}, c^{n}, d^{n}, &c.

Sint enim $a^{n}, b^{n}, c^{n}, d^{n}$, &c. radices æquationis $x^{m} - px^{m-1} + qx^{m-2} - rx^{m-3} + \pm Px^{m-b} + \&c. = 0$; pro x ſcribatur v^{n} & reſultet æquatio $v^{mn} - pv^{mn-n} + qv^{mn-2n} \ldots \pm Pv^{mn-bn} + \&c. = 0$, cujus radices erunt $\alpha a, \beta a, \gamma a$, &c.; $\alpha b, \beta b, \gamma b$, &c.; $\alpha c, \beta c, \gamma c$, &c.; &c.; $\alpha d, \beta d, \gamma d, \delta d$, &c., &c.; duo aggregata quæſita $(\pm \pi \;\&\; \pm \rho)$ erunt reſpective quantitati P, & conſequenter inter ſe æqualia: utraque aggregata idem habebunt ſignum, ſi ambo numeri b & bn ſint pares vel impares; ſi vero alter ſit par, alter vero impar, tum diverſa habebunt ſigna duo aggregata prædicta.

Pag. 163. Cor. 2. Sint $\alpha, \beta, \gamma, \delta, \varepsilon$, &c. radices æquationis $x^{n} - px^{n-1} + qx^{n-2} - rx^{n-3} + sx^{n-4} - \&c. = 0$, tum erit

$$\frac{1}{x^{n} - px^{n-1} + qx^{n-2} - \&c.} = \frac{a}{x-\alpha} + \frac{b}{x-\beta} + \frac{c}{x-\gamma} + \frac{d}{x-\delta} + \frac{e}{x-\varepsilon} + \&c.,$$

ubi $a = \frac{1}{(\alpha-\beta)\times(a-\gamma)\times(\alpha-\delta)\times\&c.}$, $b = \frac{1}{(\beta-\alpha)\times(\beta-\gamma)\times(\beta-\delta)\times\&c.}$, $c = \frac{1}{(\gamma-\alpha)\times(\gamma-\beta)\times(\gamma-\delta)\times\&c.}$, &c. at

$$\frac{1}{x^{n} - px^{n-1} + qx^{n-2} - \&c.} = x^{-n} + px^{-n-1} + \overline{p^{2} - q}\,x^{-n-2} + \overline{p^{3} - 2pq + r}\,x^{-n-3} + \overline{p^{4} - 3p^{2}q + 2pr - s + q^{2}}\,x^{-n-4} + \&c. = 1 \times x^{-n} + \overline{\alpha + \beta + \gamma + \delta + \&c.}\,x^{-n-1} + ((\alpha^{2} + \beta^{2} + \gamma^{2} + \delta^{2} + \&c.) + (\alpha\beta + \alpha\gamma + \beta\gamma + \alpha\delta + \beta\delta + \&c.))\,x^{-n-2} + ((\alpha^{3} + \beta^{3} + \gamma^{3} + \delta^{3} + \&c.) + (\alpha^{2}\beta + \alpha^{2}\gamma + \beta^{2}\alpha + \beta^{2}\gamma + \gamma^{2}\alpha + \gamma^{2}\beta + \&c.) + (\alpha\beta\gamma + \alpha\beta\delta + \&c.))\,x^{-n-3};$$

& in genere erit $+ ((\alpha^{m} + \beta^{m} + \gamma^{m} + \&c.) + (\alpha^{m-1}\beta + \beta^{m-1}\alpha + \alpha^{m-1}\gamma + \beta^{m-1}\gamma + \gamma^{m-1}\alpha + \gamma^{m-1}\beta + \&c.) + (\alpha^{m-2} \times (\beta^{2} + \beta\gamma + \gamma^{2} + \&c.) + \beta^{m-2} \times (\alpha^{2} + \alpha\gamma + \gamma^{2} + \&c.) + \gamma^{m-2} \times (\alpha^{2} + \alpha\beta + \beta^{2} + \&c.) + \&c.) + (\alpha^{m-3} \times (\beta^{3} + \beta^{2}\gamma + \beta\gamma^{2} + \gamma^{3} + \beta^{2}\delta + \beta\delta^{2} + \&c.) + \beta^{m-3} \times (\alpha^{3} + \beta^{2}\gamma + \gamma^{2}\alpha + \alpha^{2}\delta + \delta^{2}\alpha + \&c.) + \gamma^{m-3} \times (\alpha^{2}\beta + \&c.)) + (\alpha^{m-4} \times (\beta^{4} + \beta^{3}\gamma + \beta^{2}\gamma^{2} +$

$\beta^{2}\gamma\delta$

$\beta^2\gamma\delta + \beta\gamma\delta\epsilon + \&c.) + \&c.) + \&c.) + \&c.)\, x^{-n-m} + \&c. = P.$

Hìc dimenſiones ſingulorum terminorum eidem poteſtati quantitatis (x) annectendorum, eædem erunt; quæ erunt m, ſi modo $-n-m$ ſit poteſtas quantitatis x, cui annectuntur prædicti termini.

Singuli termini adjungendi ſunt, quorum dimenſiones erunt m.

Coefficiens ſingulorum terminorum erit unitas.

Erit $\frac{a}{x-\alpha} + \frac{b}{x-\beta} + \frac{c}{x-\gamma} + \frac{d}{x-\delta} + \&c. = (a+b+c+d + \&c.)\, x^{-1} + (a\alpha + b\beta + c\gamma + d\delta + \&c.)\, x^{-2} + (a\alpha^2 + b\beta^2 + c\gamma^2 + d\delta^2 + \&c.)\, x^{-3} \ldots\ldots + (a\alpha^{n-1} + b\beta^{n-1} + c\gamma^{n-1} + d\delta^{n-1} + \&c.)\, x^{-n} + (a\alpha^n + b\beta^n + c\gamma^n + d\delta^n + \&c.)\, x^{-n-1} + (a\alpha^{n+1} + b\beta^{n+1} + c\gamma^{n+1} + d\delta^{n+1} + \&c.)\, x^{-n-2} \ldots\ldots + (a\alpha^{n+m-1} + b\beta^{n+m-1} + c\gamma^{n+m-1} + d\delta^{n+m-1} + \&c.)\, x^{-n-m} + \&c. = Q$: fiant ſinguli correſpondentes termini, i. e. termini eidem poteſtati literæ x annexi quantitatum P & Q inter ſe æquales; tum reſultabunt $a+b+c+d+\&c. = \frac{1}{(\alpha-\beta)\times(\alpha-\gamma)\times(\alpha-\delta)\times\&c.} + \frac{1}{(\beta-\alpha)\times(\beta-\gamma)\times(\beta-\delta)\times\&c.} + \frac{1}{(\gamma-\alpha)\times(\gamma-\beta)\times(\gamma-\delta)\times\&c.} + \&c. = 0$, $\frac{\alpha}{(\alpha-\beta)\times(\alpha-\gamma)\times(\alpha-\delta)\times\&c.} + \frac{\beta}{(\beta-\alpha)\times(\beta-\gamma)\times(\beta-\delta)\times\&c.} + \frac{\gamma}{(\gamma-\alpha)\times(\gamma-\beta)\times(\gamma-\delta)\times\&c.} + \&c. = 0, \ldots\ldots \frac{\alpha^r}{(\alpha-\beta)\times(\alpha-\gamma)\times(\alpha-\delta)\times\&c.} + \frac{\beta^r}{(\beta-\alpha)\times(\beta-\gamma)\times(\beta-\delta)\times\&c.} + \frac{\gamma^r}{(\gamma-\alpha)\times(\gamma-\beta)\times(\gamma-\delta)\times\&c.} + \frac{\delta^r}{(\delta-\alpha)\times(\delta-\beta)\times(\delta-\gamma)\times\&c.} + \&c. = S$; ſi $r = n-1$, tum erit $S = 1$; ſi $r = n$, erit $S = \alpha + \beta + \gamma + \delta + \&c.$; ſi $r = n+1$, tum erit $S = (\alpha^2 + \beta^2 + \gamma^2 + \&c.) + (\alpha\beta + \alpha\gamma + \beta\gamma + \alpha\delta + \beta\delta + \&c.)$; in genere ſi $r = n+m-1$, tum erit $S = (\alpha^m + \beta^m + \gamma^m + \&c.) + (\alpha^{m-1}\beta + \beta^{m-1}\alpha + \alpha^{m-1}\gamma + \&c.) + \&c.$ poteſtati x^{-n-m} in quantitate P annexæ.

2. Sint

ADDENDA.

2. Sint $\alpha, \beta, \gamma, \delta$, &c. radices æquationis $p - qx + rx^2 - sx^3 + tx^4 - \&c. + x^n = 0$, tum erit $\frac{a}{\alpha - x} + \frac{b}{\beta - x} + \frac{c}{\gamma - x} + \frac{d}{\delta - x} + \&c. = \pm \frac{1}{p - qx + rx^2 - sx^3 + tx^4 - \&c.}$; ubi $a = \frac{1}{(\alpha - \beta) \times (\alpha - \gamma) \times (\alpha - \delta) \times \&c.}$, $b = \frac{1}{(\beta - \alpha) \times (\beta - \gamma) \times (\beta - \delta) \times \&c.}$, $c = \frac{1}{(\gamma - \alpha) \times (\gamma - \beta) \times (\gamma - \delta) \times \&c.}$, $d = \frac{1}{(\delta - \alpha) \times (\delta - \beta) \times (\delta - \gamma) \times \&c.}$, &c.

Erit $\pm \frac{1}{p - qx + rx^2 - sx^3 + tx^4 - \&c.} = \pm \left(\frac{1}{p} + \frac{q}{p^2}x + \frac{q^2 - rp}{p^3}x^2 + \frac{q^3 - 2qrp + sp^2}{p^4}x^3 + \&c.\right) = \frac{1}{\alpha\beta\gamma\delta\,\&c.} \times \left(1 + \left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} + \&c.\right)x + \left(\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} + \frac{1}{\delta^2} + \&c. + \frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\beta\gamma} + \frac{1}{\alpha\delta} + \frac{1}{\beta\delta} + \frac{1}{\gamma\delta} + \&c.\right)x^2 + \left(\frac{1}{\alpha^3} + \frac{1}{\beta^3} + \frac{1}{\gamma^3} + \frac{1}{\delta^3} + \&c. + \frac{1}{\alpha^2\beta} + \frac{1}{\beta^2\alpha} + \frac{1}{\alpha^2\gamma} + \frac{1}{\gamma^2\alpha} + \frac{1}{\beta^2\gamma} + \&c. + \frac{1}{\alpha\beta\gamma} + \frac{1}{\alpha\beta\delta} + \frac{1}{\alpha\beta\epsilon} + \frac{1}{\beta\gamma\delta} + \&c.\right)x^3 + \&c.\right) = P.$

Hæc ſeries eadem evadet ac ſeries in caſu primo tradita, ſi modo in eâ pro $\alpha, \beta, \gamma, \delta$, &c. ſcribantur reſpective $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\delta}$, &c. & pro x^m ſcribatur x^{-n-m}.

Erit $\frac{a}{\alpha - x} + \frac{b}{\beta - x} + \frac{c}{\gamma - x} + \&c. = \frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} + \&c. + \left(\frac{a}{\alpha^2} + \frac{b}{\beta^2} + \frac{c}{\gamma^2} + \&c.\right)x + \left(\frac{a}{\alpha^3} + \frac{b}{\beta^3} + \frac{c}{\gamma^3} + \&c.\right)x^2 + \&c. = Q$; fiant correſpondentes termini quantitatum P & Q reſpective inter ſe

æqua-

æquales, tum evadent $\frac{1}{\alpha} \times \frac{1}{(\alpha-\beta)\times(\alpha-\gamma)\times(\alpha-\delta)\times \&c.} + \frac{1}{\beta} \times \frac{1}{(\beta-\alpha)\times(\beta-\gamma)\times(\beta-\delta)\times \&c.} + \frac{1}{\gamma} \times \frac{1}{(\gamma-\alpha)\times(\gamma-\beta)\times(\gamma-\delta)\times \&c.} + \&c. = \frac{\pm 1}{\alpha\beta\gamma\delta \&c.}$; $\frac{1}{\alpha^2} \times \frac{1}{(\alpha-\beta)\times(\alpha-\gamma)\times(\alpha-\delta)\times \&c.} + \frac{1}{\beta^2} \times \frac{1}{(\beta-\alpha).(\beta-\gamma).(\beta-\delta)\,\&c.} + \frac{1}{\gamma^2} \times \frac{1}{(\gamma-\alpha).(\gamma-\beta).(\gamma-\delta)\,\&c.} + \&c. = \frac{\pm 1}{\alpha\beta\gamma\delta \&c.}\left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} + \frac{1}{\delta} + \&c.\right)$; & ſic deinceps.

Pag. 189. lin. 21. Aliter: ducantur datæ æquationes $(P)\ ax + by + cz + dv + \&c. = 0$, $(Q)\ a'x + b'y + c'z + \&c. = 0$, $(R)\ a''x + b''y + c''z + \&c. = 0$, &c. in quantitates $1, \alpha, \beta, \gamma$, &c. reſpective; & in æquatione reſultante $P + \alpha Q + \beta R + \gamma S + \&c. = 0$ ita aſſumantur coefficientes, ut exterminentur $(n-1)$ incognitæ quantitates $(y, z, v, \&c.)$; tum reſultant $(n-1)$ æquationes $b + b'\alpha + b''\beta + b'''\gamma + \&c. = 0$, $c + c'\alpha + c''\beta + c'''\gamma + \&c. = 0$, $d + d'\alpha + d''\beta + d'''\gamma + \&c. = 0$, &c.; quæ $(n-1)$ æquationes in $(n-2)$ æquationes per conſimilem methodum deprimi poſſunt; & ſic deinceps; & exinde regrediendo valorem quæſitæ quantitatis (x) acquirere liceat.

Pag. 293. lin. 9. Pro $A^2 - B^2$, lege $A^2 - B^2 = (A+B) \times (A-B) = \overline{e + \sqrt{(f)}} \times \overline{e - \sqrt{(f)}}$.

Pag. 354. Cor. 5. Si numerus P nec admittat diviſorem, qui ſit quadratus (a^2), nec 3; & uno modo ſolummodo reſolvi poſſit in formulam $p^2 + pq + q^2$, tum erit P primus numerus: etiamque ex iiſdem principiis, ſi $A = a^2 + mb^2$ non recipiat diviſorem, qui eſt quadratus (a^2), nec (ma^2); & unum ſolummodo valorem prædictæ formulæ $a^2 + mb^2$ recipiat, tum erit A primus numerus.

A D D E N D A.

Pag. 382. lin. ult. Cor. Sint $1+2+3+4\ldots+r-3=A$, $1+2+3+4\ldots r-4=A'$, $1+2+3+4\ldots+r-5=A''$, $1+2+3+4+\ldots+r-6=A'''$, &c.: ſint B, B', B'', B''', &c. ſumma rectangulorum e quibuſque duobus numeris $1, 2, 3, \ldots r-3$; $1, 2, 3, \ldots r-4$; $1, 2, 3, \ldots r-5$; &c. reſpective; etiamque C, C', C'', C''', &c.; D, D', D'', D''', &c.; E, E', E'', &c.; &c.; reſpective ſummæ contentorum e quibuſque tribus, quatuor, quinque, &c. prædictis numeris. Sit $S=(x+1)^2\times x^2\times((x-1)\times(x-2)\ldots(x-r+3)+H\times(x-1)\times(x-2)\ldots(x-r+4)+I\times(x-1)\times(x-2)\ldots(x-r+5)+K\times\&c.\ldots+l\times(x-1)\times(x-2)+m\times(x-1)+n))$, cujus incrementum $=S-x^2\times(x-1)\times((x-1).(x-2)\ldots(x-r+2)+H\times(x-1)\times(x-2)\ldots(x-r+3)+I\times(x-1).(x-2)\ldots(x-r+4)\ldots+l\times(x-1).(x-2).(x-3)+m\times(x-1)\times(x-2)+n\times(x-1))$: fiant $(\overline{r+1}A+\overline{r-3})=r\times H$, $(\overline{r+1}B+\overline{r-3}A)-(rHA'+\overline{r-4}H)=-\overline{r-1}I$, $(\overline{r+1}C+\overline{r-3}B)-(rHB'+\overline{r-4}HA')+(\overline{r-1}IA''+\overline{r-5}\times I)=\overline{r-2}K$, $(\overline{r+1}D+\overline{r-3}C)-(rHC'+\overline{r-4}HB')+(\overline{r-1}IB''+\overline{r-5}\times IA'')-(\overline{r-2}KA'''+\overline{r-6}K)=L$, &c., tum erit $\overline{r-3}.1.2.3.4\ldots\overline{r-3}-\overline{r-4}\times H\times 1.2.3.4\ldots\overline{r-4}+\overline{r-5}\times I\times 1.2.3.4\ldots\overline{r-5}-\overline{r-6}\times K\times 1.2.3.4\ldots\overline{r-6}+\overline{r-7}\times L\times 1.2.3.4\ldots\overline{r-7}-\ldots\pm 3\times l\times 1.2.3\mp 2\times m\times 1.2\pm n=0$; ſi modo r ſit impar numerus.

Pag. 383. lin. 3. Cor. Sint $1+2+3\ldots+r-2=A$, $1+2+3\ldots r-3=A'$, $1+2+3\ldots r-4=A''$, &c., ſint B, B', B'', &c., ſumma rectangulorum e quibuſque duobus numeris $1, 2, 3, \ldots r-2$; $1, 2, 3, \ldots r-3$; $1, 2, 3, \ldots r-4$; &c.; etiamque C, C', C'', &c., D, D', D'', &c.; E, E', E'', &c.; reſpective ſummæ con-

tentorum

A D D E N D A.

tentorum e quibuſque tribus, quatuor, quinque, &c. prædictis numeris: ſit $S = (2x+1) \times (x+1) \times (x) \times ((x-1) . (x-2) . (x-3) \ldots (x-r+2) + H \times (x-1) \times (x-2) \times (x-3) \ldots (x-r+3) + I \times (x-1) \times (x-2) \ldots (x-r+4) + K \times (x-1) . (x-2) \ldots (x-r+5) \ldots + l \times (x-1) \times (x-2) + m \times (x-1) + n))$: fiant $(2 \times (r+1) A + r - 2) = 2rH$, $(2\overline{r+1}\,B + \overline{r-2}\,A) - (2rHA' + \overline{r-3}\,H) = -2 \times (r-1) I$, $(2\overline{r+1}\,C + \overline{r-2}\,B) - (2rHB' + \overline{r-3}\,HA') + (2 \times (r-1) IA'' + \overline{r-4}\,I) = 2 \times (r-2) K$, $(2\overline{r+1}\,D + \overline{r-2}\,C) - (2rHC' + \overline{r-3}\,HB') + (2\overline{r-1}\,IB'' + \overline{r-4}\,IA'') - (2 \times (r-2) KA''' + \overline{r-5}\,K) = -2 \times (r-3) L$, &c.; tum erit $(r-2) \times 1 . 2 . 3 . 4 \ldots (r-2) - (r-3) \times H \times 1 . 2 . 3 . 4 \ldots (r-3) + (r-4) \times I \times 1 . 2 . 3 \ldots (r-4) - (r-5) \times K \times 1 . 2 . 3 \ldots (r-5) \ldots \pm 3 \times l \times 1 . 2 . 3 \mp 2 \times m \times 1 . 2 \pm n = 0$, ſi modo r ſit par numerus.

Pag. 387. lin. 4. Cor. Si requiratur, ut quantitas detegenda ſit maxime ſimplex; tum pro H, K, L, &c. aſſumatur invariabilis quantitas; & pro A, B, C, &c. aſſumatur functio quantitatum $(x, y, z$, &c.$)$ hujuſce formulæ $ex + fy + gz +$ &c. $+ d$; ubi literæ e, f, g, &c. d invariabiles quantitates reſpective denotant.